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VECTORIAL MECHANICS

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BY

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THIS BOOK IS PRODUCED IN
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TO

PROFESSOR SYDNEY CHAPMAN

M.A., D.Sc., F.R.S.

Sedleian Professor of Natural Philosophy
in the University of Oxford

and

Past Fellow of Trinity College, Cambridge

THIS VOLUME IS GRATEFULLY DEDICATED

BY HIS FORMER PUPIL

PREFACE

The writing of this volume is wholly due to the inspiration of Professor Sydney Chapman, F.R.S., my former teacher. Its genesis dates, however, from a time (about 1924) when I had ceased to be his pupil, but when his views continued to exercise, as they have always exercised, a profound influence on me. It was he who first expounded to me the view that vectors were not merely a pretty toy, suitable only for elegant proofs of general theorems, but were a powerful weapon of workaday mathematical investigation, both in research and in solving problems of the types set in English examinations.

I did not at first believe him; I had been brought up in the idea that, in the words of a distinguished applied mathematician, vectors were like a pocket-rule, which needs to be unfolded before it can be applied and used; similarly I thought (with most others at that time) that vector expressions were a mere shorthand for sets of Cartesian expressions, and that before they could be interpreted they always needed to be translated into Cartesians. Professor Chapman's reply was: 'Try for yourself!' I had faith in him to do so, and rapidly convinced myself, in spite of my previous beliefs, that he was right. I found further, again under his inspiration, that one could soon learn to think and work vectorially; that problems, dull or difficult by conventional methods, gained when treated by vector methods an interest, an ease and a delight which they previously lacked; and that, when a problem was formulated and solved vectorially, the vector solution provided a kinematic picture of the motion in question that gave far more insight into the phenomenon than the corresponding Cartesian solution. The Cartesian solution tells *where a system is*; the vector solution, with one less integration, tells *how it is moving*.

I began in consequence to use vector methods directly in my own researches, and I began to follow Chapman in lecturing systematically to students through the medium of vectors. The credit for the discovery of the *practical* advantages of vectors is, however, entirely due to Chapman.

About 1926, Professor Chapman and I made plans for a joint book on Mechanics, using vector methods. We discussed the structure of the proposed work, and I submitted to him a draft of the first few chapters, which he revised. Subsequently we became more occupied with our respective researches, and for many years no progress was made with our plan. Eventually I completed a draft. This was ready to be prepared for press by 1938, but war-work delayed the final revision. My first task since my release from the Ministry of Supply has been to re-write the MS. in final revision.

In spirit, therefore, the book is a joint work by Professor Chapman and myself, but I alone am responsible for its faults. Many of Professor Chapman's ideas have been incorporated without further acknowledgment, as I have always hoped that his name would precede mine on the title-page. But as I have developed my own usages, I expect that it deviates in many instances from what is his present habit of treatment. But I must repeat that without his initial stimulus, without his insight, the book would never have been undertaken.

The scope of the work is mechanics by elementary methods, treated throughout by means of vectors. The methods of *analytical* dynamics (Lagrange, Hamilton) have been deliberately excluded, so as to maintain the elementary character of the work. At the same time quite hard problems are shown to be soluble by these elementary methods. In one or two places I have been tempted to go outside the limits of strict dynamics, to illustrate an essentially dynamical idea in a wider context.

The work begins with an account of vector algebra containing all the results needed for reading the whole volume. This includes a sufficient account of tensors and dyadics. There is an additional chapter on integral theorems in the vector calculus, the results of which are not actually employed in this volume, but it was thought that it would be a convenience to many readers to make the section on vector algebra complete. This part of the work is followed by a systematic account of line vectors, which takes conventional three-dimensional *statics* in its stride. But it is emphasized throughout that *forces* are not the only kind of line vectors —another important example of a system of line vectors is the *momentum* of a system of particles—and the theory of line vectors is written so as to be immediately available in other contexts. Part III, though entitled *Dynamics*, opens with a systematic account of *kinematics*. Here the angular velocity of a rigid body is introduced at an early stage, and used to derive many results properly belonging to the kinematics of a particle. The sections on *particle dynamics* include some topics more elementary than the general average of the book, but they are given chiefly to accustom the student to vector solutions of vector differential equations. Vector methods reach their climax in discussing the harder problems of *rigid dynamics*, which occupy several of the later chapters. In particular, an attempt has been made to give a new interest to the often-considered dull subject of the calculation of moments and products of inertia, by showing how to calculate inertia tensors outright. The book concludes with a chapter on impulsive motion.

The standard aimed at is that of second or third year honours work at a university. With omissions dependent on the taste of the student or his instructor, the ground can be covered in a single year.

No claim is made for originality of results at any point, though some of the theorems in the tensor calculus may be novel. But, generally

speaking, the theorems proved are so standard and classical that it has not been thought necessary to insert references.

The claim is, however, made that this volume exhibits for the first time the tremendous power contained in the operation of *vector multiplication*, a power which is largely lost if suffixes are used and which hardly emerges in Hamilton's quaternions. The theme of the book is in fact the vector product, $\mathbf{A}\wedge\mathbf{B}$, and its applications. It has been thought desirable to devote a whole chapter to the vector product, and the *modus operandi* of the solution of most of the problems consists in a judicious employment of vector multiplication. There is only one formula in the book which it is essential to remember and to be able to apply readily, namely the formula for the continued vector product,

$$(\mathbf{A}\wedge\mathbf{B})\wedge\mathbf{C}=-\mathbf{A}(\mathbf{B}.\mathbf{C})+\mathbf{B}(\mathbf{A}.\mathbf{C}).$$

If the theme of the book is the vector product, the accompaniment of worked examples is also considered important. It was considered necessary to demonstrate in detail how to set about working examples. It has not been intended to invent new examples specially suitable for the applications of vectors ; instead, examples have been chosen which were originally intended to be done by Cartesian methods. Few unworked examples in dynamics are given. The student should be encouraged to attempt by vector methods many of the examples contained in standard current textbooks.

A word may be said about notation. *Brackets* are throughout used in their ordinary algebraic sense of grouping symbols together, or of indicating functional dependence on a variable—with the single exception that a line vector $\mathbf{P}$ is sometimes indicated by $(\mathbf{P})$ when it is desirable to distinguish it from the free vector $\mathbf{P}$. The scalar product of $\mathbf{P}$ and $\mathbf{Q}$ is indicated by a dot, thus $\mathbf{P}.\mathbf{Q}$; the vector product of $\mathbf{P}$ and $\mathbf{Q}$ is indicated by the sign $\wedge$, thus $\mathbf{P}\wedge\mathbf{Q}$. But brackets are nowhere used to indicate scalar or vector products as such ; the manipulations appearing in numerous examples would be impossible without the freedom to use round and square brackets in their ordinary sense. The dyadic product of $\mathbf{P}$ and $\mathbf{Q}$ is written just as $\mathbf{PQ}$. Transition is readily made to the suffix notation when this is expedient. A dot always stands for the suppression of two *adjacent* dummy suffixes. Thus $\mathbf{P}.\mathbf{Q}$ means $P_\mu Q_\mu$, where the summation convention is understood ; $\mathbf{T}.\mathbf{P}$ has for its μ-component $T_{\mu\nu}P_\nu$ and similarly $\mathbf{P}.\mathbf{T}$ has for its μ-component $P_\nu T_{\nu\mu}$. The notation $\mathbf{T}{:}\mathbf{AB}$ is used to denote $(\mathbf{T}.\mathbf{A}).\mathbf{B}$ or what in the suffix notation would be written $T_{\mu\nu}A_\nu B_\mu$; similarly $\mathbf{AB}{:}\mathbf{T}$ means $A_\mu B_\nu T_{\nu\mu}$, which has the same value ; whilst $\mathbf{A}.\mathbf{T}.\mathbf{B}$ means $A_\mu T_{\mu\nu}B_\nu$.

For the idem tensor which is such that $\mathbf{U}.\mathbf{A}=\mathbf{A}$ for all $\mathbf{A}$, the symbol $\mathbf{U}$ is used, in order to reserve $\mathbf{I}$ for the inertia tensor. The symbol $A_{\alpha\beta\gamma}$ is used to denote the *alternate* tensor, customarily denoted by $E_{\alpha\beta\gamma}$.

Use is made from time to time of Gibbs's cross-products of tensors and vectors, here denoted by $\mathbf{T}\wedge\mathbf{A}$ and $\mathbf{A}\wedge\mathbf{T}$, where $\mathbf{T}$ is a tensor, $\mathbf{A}$ a vector. These are not particularly familiar to applied mathematicians, but they are essential to the rounding off of vector algebra. When $\mathbf{T}$ is a dyad $\mathbf{PQ}$, just as $(\mathbf{PQ}).\mathbf{X}$ is equal to $\mathbf{P}(\mathbf{Q}.\mathbf{X})$, so $(\mathbf{PQ})\wedge\mathbf{X}$ is equal to the dyad $\mathbf{P}(\mathbf{Q}\wedge\mathbf{X})$. The most substantial application here made of these products is to the relation between the rate of change of a tensor $d\mathbf{T}/dt$ and its apparent rate of change in a frame of reference moving with angular velocity $\mathbf{\Omega}$: if the symbol $\partial/\partial t$ denotes the apparent rate of change, then just as we have for a vector P

$$\frac{d\mathbf{P}}{dt}=\frac{\partial\mathbf{P}}{\partial t}+\mathbf{\Omega}\wedge\mathbf{P},$$

so

$$\frac{d\mathbf{T}}{dt}=\frac{\partial\mathbf{T}}{\partial t}+\mathbf{\Omega}\wedge\mathbf{T}-\mathbf{T}\wedge\mathbf{\Omega}.$$

The latter is useful in connexion with the inertia tensor. I have not come across this formula elsewhere, but it is unlikely to be novel.

The above notation for the vector calculus is the result of much thought and discussion, and it is hoped that the book may help towards standardizing vector notation. I have at times been criticized as ' clumsy ' in my employment of vectors. I will not defend *myself* against this accusation, but I must defend the notation, which I believe to be concise and convenient. It is incomparably less clumsy and more insight-giving, *for the purposes of dynamics*, than the suffix notation, which for all its supposed terseness cannot readily describe vector products or lend itself to vector multiplication as an operation. In certain contexts (in Chapter IV), I have deliberately used the suffix notation to secure increased generality, but the results expressed in this notation are not attractive.

In the text, vector and tensor symbols without suffixes have been printed in clarendon type, but the student will find it unnecessary in manuscript work to indicate such symbols in any special manner. It is only very occasionally that the same letter is used to indicate both a scalar and a vector, *e.g.* g and **g** in the section on motion over the rotating earth ; in such a case the student may have recourse to underlining the vector symbol in manuscript work, but in general underlining is unnecessary.

On the level on which this work is written, it has not been found necessary to distinguish between co-variant and contra-variant vectors and tensors. The student who has made himself familiar with the idea of a tensor will have no difficulty in subsequently refining this idea ; but on a first introduction, where the underlying metric is just $(dx)^2+(dy)^2+(dz)^2$, to introduce a distinction which has no present consequences is pointless.

Lastly, it is hoped that the book will do something towards restoring an interest in formal applied mathematics, which in the last half-century has been steadily waning. Most students are carried away by enthusiasm on their first introduction to rigorous *pure* mathematics ; formal teaching in statics and dynamics has suffered by comparison. But if the student is enabled simultaneously to master and delight in a new technique, the technique of vector manipulation, his interest in statics and dynamics will revive. To emphasize that this is a book about applied mathematics, all discussions as to rigour of the kind investigated in pure mathematics have been omitted, but where rigour is needed to establish a result of genuine *applied* mathematics, as in the case of the existence of an angular velocity vector $\mathbf{\Omega}$ for a rigid body in motion, it has been attended to in detail.

My debt to my teachers in statics and dynamics will be manifest. I would mention in particular the late E. G. Gallop of Caius College, Cambridge; the late T. J. I'A. Bromwich of St. John's College, Cambridge; and to Mr. E. Cunningham, also of St. John's College, Cambridge, and Mr. E. Harrison of Clare College, Cambridge. Unlike Professor Chapman I was never an actual pupil of the late Horace Lamb. But I would wish to express what I owe to his textbooks, more especially as I was one of his successors in the Mathematics Department of the University of Manchester.

E. A. MILNE

August 1945

I wish to acknowledge with thanks the kind permission of the Syndics of the Cambridge University Press to use numerous examples from Lamb's *Higher Mechanics* to illustrate vector methods.

I also wish to express my thanks to my former pupil, Dr G. L. Camm, now of the University of Manchester, for his kindness in reading the proof-sheets.

E. A. M.

November 1946

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Part I. Vector Algebra

CHAPTER I

DEFINITION OF A VECTOR AND THE SCALAR PRODUCT

1. *The free vector. Definition.* Let A, B be two given points in three-dimensional space, specified with respect to a set of reference marks fixed in a given rigid body. Let A′, B′ be two other points whose position is specified with respect to the same rigid body, and such that:

(1) A′B′ is parallel to AB ;
(2) the length A′B′ is equal to the length AB ;
(3) the sense A′→B′ is the same as the sense A→B.

Here the words 'parallel' and 'length' are used in the ordinary sense of Euclidean geometry. The word 'length' assumes the existence of a transportable rigid measuring rod. The word 'sense' in connexion with the symbols A→B and A′→B′ requires a little explanation. We say that the sense A′→B′ is the same as the sense A→B if AA′ is parallel to BB′, unless AB and A′B′ are in the same straight line. If A′B′ and AB are in the same straight line then we say that the sense A′→B′ is the same as the sense A→B if a moving point moving in the direction from A to B and starting sufficiently far from A on the side remote from B encounters A′ before B′.

Then the class of all pairs of points A′, B′ corresponding to the given pair A, B (including the pair A, B itself) is said to constitute the *free vector* $\overrightarrow{AB}$, or, more briefly, the vector $\overrightarrow{AB}$. A vector will always be taken to be a free vector unless otherwise stated. The free vector $\overrightarrow{AB}$ may be described by a single symbol **P**. Any pair of points A′, B′ satisfying the conditions (1), (2), (3) is said to be a *representation* of **P**. Thus **P** is the class of all its representations, and any one representation of a vector determines the vector. To exhibit the dependence of **P** on the initially given pair of points A, B we may write

$$\mathbf{P}=\mathbf{P}(A, B).$$

We can also write this in the form

$$\mathbf{P}=\overrightarrow{AB}.$$

Since we might equally well have started with any other representation A′, B′ we have

$$\mathbf{P}(A, B) = \mathbf{P}(A', B')$$

or

$$\overrightarrow{AB} = \overrightarrow{A'B'}.$$

Usually we can omit explicit mention of a representation, and use simply **P**.

Two vectors are said to be equal if they permit of the same representation. It follows that if **P** = **Q**, and A, B and C, D are representations of **P** and **Q** respectively, then AB is parallel and equal in length to CD and has the same sense.

Nul vector. When B coincides with A, the corresponding vector **P** is said to be nul, and we write **P** = o. The representation of a nul vector **P** is any pair of coincident points A, A.

2. *Position vector. Definition.* Let O be a fixed point, P any arbitrary point. Then the pair O, P defines a free vector, say **P**. **P** is said to be the position vector of the point P with respect to O. More briefly, the position vector of a point P is the vector $\overrightarrow{OP}$, where O is understood to be a given fixed point. No confusion arises if the same symbol **P** is used for the free vector and the position vector, provided an 'origin' O is understood in the latter case. The position vector **P** is, as it were, anchored to the fixed point O, whilst the free vector **P** is not anchored. If the position vectors **P**, **Q** of two points are equal, then the two points P and Q coincide.

3. *Line vector. Definition.* Let l be a given straight line, **P** a vector represented by a pair of points A, B, lying in l. Let A′, B′ be any other pair of points in l giving a representation of **P**. Then the line vector (**P**) is defined as the class of all pairs A′, B′ lying in l. Again, no confusion arises if the same symbol **P** is used for the line vector associated with l and **P**. The line vector (**P**) is, as it were, anchored in the given line l. It may be pictured as a free vector constrained to slip along l (*vecteur glissant*).

4. Vector algebra is the calculus which is the study of the properties of free vectors. Its use is fundamental in discussing systems of position vectors and systems of line vectors. Cartesian co-ordinate geometry may be considered as the study of systems of position vectors having a common reference point or origin O. The branch of mechanics called abstract statics is the study of systems of line vectors ; it studies the conditions under which systems of line vectors are *equivalent* to one another, in a sense to be defined later. Systems of line vectors also arise in other branches of mechanics, as we shall see.

We proceed with the study of vector algebra.

5. *The angle between two vectors.* The angle between two vectors **P**, **Q** is defined to be the angle described when a line moves from a position

AB to a position AC in the plane BAC, where A, B and A, C are representations of $\mathbf{P}$ and $\mathbf{Q}$, the sense of rotation being chosen so that the angle described does not exceed π. It is denoted by $\widehat{PQ}$. When $\widehat{PQ}=\frac{1}{2}\pi$, the vectors $\mathbf{P}$, $\mathbf{Q}$ are said to be perpendicular. When $\widehat{PQ}=0$, they are said to be parallel; when $\widehat{PQ}=\pi$, they are said to be antiparallel. The angle $\widehat{PQ}$ is essentially positive, and $\widehat{PQ}=\widehat{QP}$.

6. *Coplanar vectors.* Three or more vectors $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, ... are said to be coplanar if, when O, A; O, B; O, C; ... are chosen as their respective representations, the points A, B, C, ... lie on a plane passing through O.

If O, A; O, B are representations of two vectors $\mathbf{P}$, $\mathbf{Q}$, *the plane of* $\mathbf{P}$ *and* $\mathbf{Q}$ is defined to be any plane parallel to the plane OAB.

7. *The sum of two vectors.* Let A, B be any representation of a given vector $\mathbf{P}$. Let B, C be a representation of another given vector $\mathbf{Q}$. Then the vector of which A, C is a representation is defined to be the *sum* of the vectors $\mathbf{P}$, $\mathbf{Q}$ in this order. If this vector is denoted by $\mathbf{R}$, the definition is expressed by

$$\mathbf{R}=\mathbf{P}+\mathbf{Q}.$$

By the elementary properties of parallels it is readily shown that the sum $\mathbf{R}$ of two vectors $\mathbf{P}$ and $\mathbf{Q}$ is independent of the representations of $\mathbf{P}$ and $\mathbf{Q}$ used to define it.

8. *Commutative law.*

Theorem: $\mathbf{P}+\mathbf{Q}=\mathbf{Q}+\mathbf{P}$.

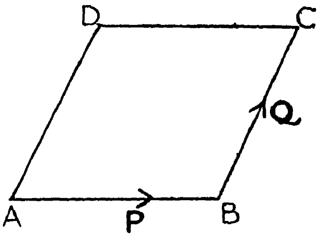

Fig. 1

For let A, B (Fig. 1) be a representation of $\mathbf{P}$; B, C a representation of $\mathbf{Q}$. Then A, C is a representation of $\mathbf{P}+\mathbf{Q}$. Let A, D be a representation of $\mathbf{Q}$. Then AD is parallel to, equal to and in the same sense as BC; hence AB is parallel to, equal to and in the same sense as DC. Hence DC is a representation of $\mathbf{P}$. Hence A, C is a representation of $\mathbf{Q}+\mathbf{P}$.

9. *Associative law.*

Theorem: $(\mathbf{P}+\mathbf{Q})+\mathbf{R}=\mathbf{P}+(\mathbf{Q}+\mathbf{R})$.

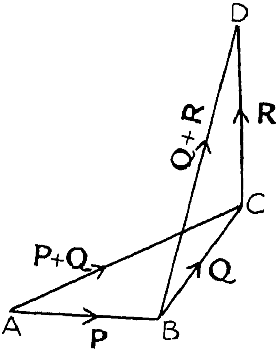

Fig. 2

For let A, B; B, C; C, D be representations of $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ respectively (Fig. 2). Then A, C is a representation of $(\mathbf{P}+\mathbf{Q})$ and accordingly A, D is a representation of $(\mathbf{P}+\mathbf{Q})+\mathbf{R}$. But B, D is a representation of $(\mathbf{Q}+\mathbf{R})$, and hence A, D is a representation of $\mathbf{P}+(\mathbf{Q}+\mathbf{R})$.

10. If A, B is a representation of a vector $\mathbf{P}$, then the vector of which B, A is a representation is defined as $-\mathbf{P}$. Clearly $\mathbf{P}+(-\mathbf{P})=0$. If $\mathbf{P}+\mathbf{Q}=\mathbf{R}$, it follows by adding $-\mathbf{Q}$ to each

Let A, B be a representation of $\mathbf{P}$. Let A, M and M, N (Fig. 4) be representations of $\mathbf{Q_P}$ and $\mathbf{R_P}$. Then A, N is a representation of $(\mathbf{Q_P}+\mathbf{R_P})$, and so of $(\mathbf{Q}+\mathbf{R})_\mathbf{P}$, by a theorem of § 14, Measure lengths positively in the sense A→B and let y, z denote the signed lengths of AM, MN. Then by the distributive theorem of algebra,

$$|\mathbf{P}|(y+z)=|\mathbf{P}|y+|\mathbf{P}|z.$$

But $$|\mathbf{P}|y=\mathbf{P}.\overrightarrow{AM}=\mathbf{P}.\mathbf{Q_P},$$

$$|\mathbf{P}|z=\mathbf{P}.\overrightarrow{MN}=\mathbf{P}.\mathbf{R_P},$$

$$|\mathbf{P}|(y+z)=\mathbf{P}.\overrightarrow{AN}=\mathbf{P}.(\mathbf{Q}+\mathbf{R})_\mathbf{P}.$$

Hence $$\mathbf{P}.(\mathbf{Q}+\mathbf{R})_\mathbf{P}=\mathbf{P}.\mathbf{Q_P}+\mathbf{P}.\mathbf{R_P}.$$

Hence by the preceding theorem,

$$\mathbf{P}.(\mathbf{Q}+\mathbf{R})=\mathbf{P}.\mathbf{Q}+\mathbf{P}.\mathbf{R}.$$

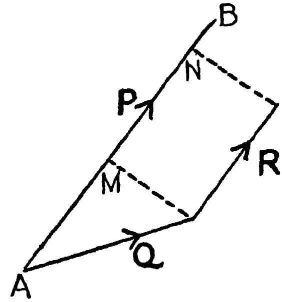

Fig. 4

It follows that $(\mathbf{Q}+\mathbf{R}).\mathbf{P}=\mathbf{Q}.\mathbf{P}+\mathbf{R}.\mathbf{P}$,

whence it follows that we can operate with scalar products as in ordinary algebra.

Example (1). $$(\mathbf{A}+\mathbf{B})^2=\mathbf{A}^2+2\mathbf{A}.\mathbf{B}+\mathbf{B}^2.$$

For $$(\mathbf{A}+\mathbf{B})^2=(\mathbf{A}+\mathbf{B}).(\mathbf{A}+\mathbf{B})=(\mathbf{A}+\mathbf{B}).\mathbf{A}+(\mathbf{A}+\mathbf{B}).\mathbf{B}$$
$$=\mathbf{A}^2+\mathbf{B}.\mathbf{A}+\mathbf{A}.\mathbf{B}+\mathbf{B}^2$$
$$=\mathbf{A}^2+2\mathbf{A}.\mathbf{B}+\mathbf{B}^2.$$

Corollary (1). $$|\mathbf{A}+\mathbf{B}|=(\mathbf{A}^2+2\mathbf{A}.\mathbf{B}+\mathbf{B}^2)^{\frac{1}{2}}.$$

Corollary (2). If $|\mathbf{B}|/|\mathbf{A}|$ is small compared with unity,

$$|\mathbf{A}+\mathbf{B}|=|\mathbf{A}|\left[1+2\frac{\mathbf{A}.\mathbf{B}}{\mathbf{A}^2}+\frac{\mathbf{B}^2}{\mathbf{A}^2}\right]^{\frac{1}{2}}$$
$$=|\mathbf{A}|\left[1+\frac{\mathbf{A}.\mathbf{B}}{\mathbf{A}^2}+o\left(\frac{\mathbf{B}^2}{\mathbf{A}^2}\right)\right].$$

Hence approximately $$|\mathbf{A}+\mathbf{B}|=|\mathbf{A}|+\frac{\mathbf{A}.\mathbf{B}}{|\mathbf{A}|}.$$

Example (2). $$(\mathbf{A}-\mathbf{B}).(\mathbf{A}+\mathbf{B})=\mathbf{A}^2-\mathbf{B}^2.$$

Example (3). Solve for $\mathbf{X}$ the vector equation

$$\alpha\mathbf{X}+\mathbf{A}(\mathbf{X}.\mathbf{B})=\mathbf{C}, \qquad (\alpha\neq 0) \quad (1)$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are given vectors and α is a given scalar.

To solve a vector equation in $\mathbf{X}$ means to find all the vectors $\mathbf{X}$ for which the equation is true. Suppose that a solution $\mathbf{X}$ of (1) exists. Multiply (1) scalarly by $\mathbf{B}$. Then

$$\alpha(\mathbf{X}.\mathbf{B})+(\mathbf{A}.\mathbf{B})(\mathbf{X}.\mathbf{B})=\mathbf{C}.\mathbf{B}.$$

This is a linear equation for the scalar $\mathbf{X.B}$. Its solution is

$$\mathbf{X.B}=\frac{\mathbf{C.B}}{\alpha+\mathbf{A.B}},$$

provided $$\alpha+\mathbf{A.B}\neq 0.$$

Assuming this condition to be satisfied, (1) now gives

$$\mathbf{X}=\frac{\mathbf{C}}{\alpha}-\mathbf{A}\frac{\mathbf{C.B}}{\alpha(\alpha+\mathbf{A.B})}. \tag{2}$$

By substitution in (1) we see that (2) is an actual solution. Hence if $\alpha+\mathbf{A.B}\neq 0$, the solution is uniquely given by (2).

If $\alpha+\mathbf{A.B}=0$, we see that we must have also $\mathbf{C.B}=0$, which is therefore necessary for the self-consistency of (1). Now if a solution $\mathbf{X}$ of (1) exists, it is a linear function of $\mathbf{A}$ and $\mathbf{C}$. Put then

$$\mathbf{X}=\lambda\mathbf{A}+\mu\mathbf{C},$$

and insert this in (1). Using $\alpha=-\mathbf{A.B}$ we get

$$-\mathbf{A.B}(\lambda\mathbf{A}+\mu\mathbf{C})+\lambda\mathbf{A.B}(\mathbf{A})=\mathbf{C},$$

whence $$\mu=-1/\mathbf{A.B},$$

provided $\mathbf{A.B}\neq 0$. With this value of μ, the equation is satisfied whatever the value of λ, and the solution is therefore

$$\mathbf{X}=\lambda\mathbf{A}-\frac{\mathbf{C}}{\mathbf{A.B}}, \qquad (\mathbf{C.B}=0,\ \alpha=-\mathbf{A.B})$$

where λ is arbitrary. Thus when $\alpha=-\mathbf{A.B}$ and $\mathbf{C.B}=0$, the solution is not unique. If in addition $\mathbf{A.B}=0$, then for consistency we must have $\mathbf{C}=0$, the equation reduces to $\mathbf{A(X.B)}=0$ and is satisfied by any vector perpendicular to $\mathbf{B}$.

The uniqueness question may also be discussed by the following method. Suppose that $\mathbf{X}_1$, $\mathbf{X}_2$ are two distinct solutions of (1). Then

$$\alpha\mathbf{X}_1+\mathbf{A}(\mathbf{X}_1.\mathbf{B})=\mathbf{C}, \qquad \alpha\mathbf{X}_2+\mathbf{A}(\mathbf{X}_2.\mathbf{B})=\mathbf{C},$$

whence $$\alpha(\mathbf{X}_1-\mathbf{X}_2)+\mathbf{A}[(\mathbf{X}_1-\mathbf{X}_2).\mathbf{B}]=0.$$

Multiply scalarly by $\mathbf{B}$. Then either $\alpha+\mathbf{A.B}=0$ or $(\mathbf{X}_1-\mathbf{X}_2).\mathbf{B}=0$. If $(\mathbf{X}_1-\mathbf{X}_2).\mathbf{B}=0$, the equation for $\mathbf{X}_1-\mathbf{X}_2$ gives at once $\mathbf{X}_1-\mathbf{X}_2=0$, and the solution is unique, provided $\alpha\neq 0$. If $\alpha+\mathbf{A.B}=0$, the solution is not unique, as we have seen above. If $\alpha=0$, the original equation requires that $\mathbf{C}$ and $\mathbf{A}$ be parallel, say $\mathbf{C}=s\mathbf{A}$, in which case $\mathbf{X.B}=s$, which defines only the component of $\mathbf{X}$ along $\mathbf{B}$. The solution is not then unique.

Example (4). Solve for $\mathbf{X}$ and $\mathbf{Y}$ the simultaneous vector equations

$$\alpha_1\mathbf{X}+\beta_1\mathbf{A}(\mathbf{Y.B})=\mathbf{C}_1,$$
$$\beta_2\mathbf{A}(\mathbf{X.B})+\alpha_2\mathbf{Y}=\mathbf{C}_2,$$

obtaining the condition that the solution is unique.

17. *Conditions for the vanishing of a vector in terms of scalar products.* Theorem: If **A, B, C** are three non-coplanar, non-zero vectors, and **P** is a vector satisfying

$$\mathbf{P.A}=0, \qquad \mathbf{P.B}=0, \qquad \mathbf{P.C}=0,$$

then

$$\mathbf{P}=0.$$

For since $\mathbf{P.A}=0$, either $\mathbf{P}=0$ or **P** is perpendicular to **A**. Since $\mathbf{P.B}=0$, either $\mathbf{P}=0$ or **P** is perpendicular to **B**. Hence if $\mathbf{P}\neq 0$, **P** being perpendicular to **A** and to **B** must be perpendicular to the plane of **A** and **B**. In that case, since $\mathbf{P.C}=0$, **C** being perpendicular to **P** must lie in the plane of **A** and **B**. This is contrary to the hypothesis that **A, B, C** are non-coplanar, and so $\mathbf{P}=0$.

Corollary. If **X, Y** are vectors such that

$$\mathbf{X.A}=\mathbf{Y.A}, \qquad \mathbf{X.B}=\mathbf{Y.B}, \qquad \mathbf{X.C}=\mathbf{Y.C},$$

then $\mathbf{X}=\mathbf{Y}$. For $\mathbf{X}-\mathbf{Y}$ satisfies the conditions of the theorem, and is therefore zero.

It follows from the theorem that one method of establishing a vector identity is to establish the three scalar identities obtained by multiplying the two sides of the vector identity scalarly by three non-coplanar vectors. There is usually some flexibility available in the choice of the appropriate multipliers, and judgment is required as to the most appropriate ones to use.

18. *Differentiation of a vector with respect to a scalar parameter.* If a vector **P** is defined for a set of values of a parameter t, **P** is said to be a vector function of t and may be written **P**(t). Any scalar derived from **P** is similarly a scalar function of t. The derivative or differential coefficient of a vector function of t with respect to t is defined by

$$\frac{d\mathbf{P}}{dt}=\lim_{\Delta t\to 0}\frac{\mathbf{P}(t+\Delta t)-\mathbf{P}(t)}{\Delta t}.$$

and is clearly also a vector. Successive differential coefficients may be derived similarly. It should be noted that a vector and its differential coefficient are not in general parallel.

By a procedure similar to that used in obtaining the derivative of a product in the algebra of a real variable, it can be shown that

$$\frac{d(\mathbf{P.Q})}{dt}=\frac{d\mathbf{P}}{dt}.\mathbf{Q}+\mathbf{P}.\frac{d\mathbf{Q}}{dt}.$$

As a particular case, take $\mathbf{Q}=\mathbf{P}$. Then

$$\frac{d(\mathbf{P.P})}{dt}=2\,\mathbf{P}.\frac{d\mathbf{P}}{dt}.$$

But $\mathbf{P}.\mathbf{P}=\mathbf{P}^2=|\mathbf{P}|^2$, and $|\mathbf{P}|^2$, being the square of a scalar function, has for its derivative

$$2\,|\mathbf{P}|\,\frac{d|\mathbf{P}|}{dt}.$$

It follows that $$\frac{d|\mathbf{P}|}{dt}=\frac{\mathbf{P}}{|\mathbf{P}|}.\frac{d\mathbf{P}}{dt},$$

a formula which is often useful. The vector $\mathbf{P}/|\mathbf{P}|$ is a unit vector, whence it follows that $d|\mathbf{P}|/dt$ has the numerical value of the component of $d\mathbf{P}/dt$ along $\mathbf{P}$.

19. *Differentiation of a scalar function of a vector with respect to that vector.* It may happen that a scalar φ is given whenever a vector variable $\mathbf{r}$ is given. Examples are $\varphi=\mathbf{r}^2$, $\varphi=\mathbf{r}.\mathbf{A}$. The question arises whether, when φ changes continuously with change of $\mathbf{r}$, it is possible to define something which will play the part of 'the derivative of φ with respect to $\mathbf{r}$,' if such can be given a meaning.

Let φ correspond to $\mathbf{r}$, $\varphi+\Delta\varphi$ to $\Delta\mathbf{r}+\mathbf{r}$, $\Delta\mathbf{r}$ being an arbitrary increment in $\mathbf{r}$. Then we define *the derivative of φ with respect to $\mathbf{r}$* as the vector $d\varphi/d\mathbf{r}$, if it exists, such that

$$\Delta\varphi=\frac{d\varphi}{d\mathbf{r}}.\Delta\mathbf{r}+o(\Delta\mathbf{r}^2),$$

for all sufficiently small values of $|\Delta\mathbf{r}|$. It should be remembered that $\Delta\mathbf{r}$ is arbitrary in direction as well as in absolute magnitude. It should be noted also that $d\varphi/d\mathbf{r}$ is not $\lim \Delta\varphi/\Delta\mathbf{r}$, which is meaningless. The vector $d\varphi/d\mathbf{r}$ so defined has many of the properties of a differential coefficient. For example, at a maximum or minimum of φ, $\Delta\varphi$ must have a sign independent of $\Delta\mathbf{r}$, and so $d\varphi/d\mathbf{r}$ must be zero. Again, it is readily proved from the above definition that

$$\frac{dF(\varphi)}{d\mathbf{r}}=F'(\varphi)\frac{d\varphi}{d\mathbf{r}},$$

$$\frac{d(\varphi\psi)}{d\mathbf{r}}=\varphi\frac{d\psi}{d\mathbf{r}}+\frac{d\varphi}{d\mathbf{r}}\psi.$$

Again, $d\varphi/d\mathbf{r}$, if it exists, is unique. For if two vectors φ'_1, φ'_2 both satisfied the conditions for a derivative, we should have

$$(\varphi'_1-\varphi'_2).\Delta\mathbf{r}+o(\Delta\mathbf{r}^2)=0,$$

for all sufficiently small values of $|\Delta\mathbf{r}|$. If $\varphi'_1-\varphi'_2$ had a determinate direction, it would always be possible to choose a value for $\Delta\mathbf{r}$ violating this condition. Hence $\varphi'_1-\varphi'_2$ can have no determinate direction, and so must be a nul vector.

Example (1). If $\varphi=\mathbf{r}.\mathbf{A}$, $\Delta\varphi=\mathbf{A}.\Delta\mathbf{r}$

and so $$\frac{d(\mathbf{r}.\mathbf{A})}{d\mathbf{r}}=\mathbf{A}.$$

Example (2). If $\varphi = \mathbf{r}^2$, $\Delta\varphi = 2\mathbf{r}.\Delta\mathbf{r} + \Delta\mathbf{r}^2$

and so
$$\frac{d(\mathbf{r}^2)}{d\mathbf{r}} = 2\mathbf{r}.$$

Example (3). Since $\mathbf{r}^2 = |\mathbf{r}|^2$ and $\dfrac{d|\mathbf{r}|^2}{d\mathbf{r}} = 2|\mathbf{r}|\dfrac{d|\mathbf{r}|}{d\mathbf{r}}$,

it follows that
$$\frac{d|\mathbf{r}|}{d\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|}.$$

Example (4). Prove that
$$\frac{df|\mathbf{r}|}{dt} = \frac{df|\mathbf{r}|}{d\mathbf{r}}.\frac{d\mathbf{r}}{dt},$$
where f is any differentiable scalar function of $\mathbf{r}$.

20. The question suggests itself at this stage: Can we define something which can play the part of the derivative of a vector with regard to another vector of which it is a function? By analogy with what precedes, we should seek to give a meaning to a symbol $\mathbf{T}$, which is such that if $\mathbf{F}(\mathbf{r})$ is a vector function of a vector $\mathbf{r}$, then
$$\Delta\mathbf{F} = \mathbf{T}.\Delta\mathbf{r} + o(\Delta\mathbf{r}^2).$$
Then $\mathbf{T}.\Delta\mathbf{r}$ would have the significance of a vector. Symbols $\mathbf{T}$ possessing this significance will be known as *tensors*. We shall deal with them formally in due course.

CHAPTER II

THE VECTOR PRODUCT

21. *Continuous deformation of a triad.* Take any four non-coplanar points O, A, B, C, attention being paid to the sequence of the points. The triad OABC is defined as consisting of the three lines OA, OB, OC considered as forming a rigid body ; the position of A on OA is immaterial provided it is maintained on the same side of O, and similarly for B and C. Rotate OB about O in the plane AOB so that the angle AOB becomes $\frac{1}{2}\pi$, the direction of rotation of OB being such that OB moves through an angle less than $\frac{1}{2}\pi$. Next, rotate OC about the line in AOB to which it is perpendicular, until it becomes perpendicular to the plane AOB, in such a way that OC moves through an angle less than $\frac{1}{2}\pi$. Writing now A′ for A, and calling the new position of the triad OA′B′C′, we say that OA′B′C′ is derived from OABC by continuous deformation. The new triad OA′B′C′ is said to be a mutually perpendicular or orthogonal triad ; the angles BOA, COA, COB are all right angles.

22. *Superposability.* Consider the three triads OABC, OBCA, OCAB, and obtain by continuous deformation the corresponding orthogonal triads OA′B′C′, OB″C″A″, OC‴A‴B‴. Then these triads can be superposed on one another. It is sufficient to show that if OPQR is an orthogonal triad, then it may be superposed on OQRP. Rotate the triad OPQR (Fig. 5) as a rigid body round OR so that OP comes into coincidence with OQ. Then OQ comes into coincidence with PO produced, and OR is unaltered. Let OP′Q′R denote the new position of the triad. Next, rotate the triad OP′Q′R about OP′ so that OR comes into coincidence with OP. Then OQ′ comes into coincidence with OR. The new position OP′Q″R″ is now superposed on OQRP.

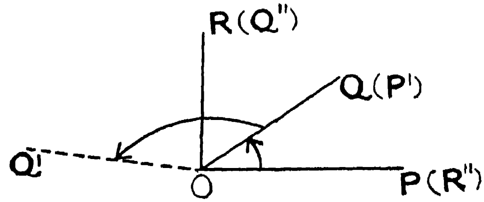

Fig. 5

It follows that in any orthogonal triad OABC, cyclical interchange of the letters ABC gives rise to another orthogonal triad superposable on the original one.

Given an orthogonal triad OABC, another triad OA′BC may be

derived by producing AO to A′. Clearly the two triads OABC, OA′BC cannot be superposed. They are said to be opposite in sense.

All orthogonal triads can be superposed either on a given orthogonal triad OABC or on its opposite OA′BC. For if a triad OPQR is given, and it is superposed on OABC in such a way that OQ falls on OB and OR on OC, then OP must fall either on OA or OA′.

One of the two triads OABC, OA′BC is defined as being a positive triad, and used as a standard. The other is then defined as negative. It is immaterial which is chosen as positive. Usually the so-called right-handed convention is adopted to select which triad is to be called positive, i.e. that triad is positive for which the direction of rotation from OA to OB propels a right-handed screw in the direction OC ; it then follows from the superposition properties that the direction of rotation from OB to OC propels the same screw in the direction OA, etc. The only way, however, of defining a right-handed screw is to exhibit one.

Thus there are just two essentially distinct types of triad. This is an essential property of three-dimensional space. By continuous deformation, *any* triad can be labelled as positive or negative.

It now follows readily that two orthogonal triads OABC, OBAC have opposite signs. The proof is left to the reader.

Note.—The existence of just two types of orthogonal triad in three-dimensional space is associated with the fact that there are just two sides, or two oppositely directed normals, to a given plane. It is also associated with the fact that there are just two distinct orders for three symbols x, y, z when cyclical interchange is permitted. In four dimensions there are six different non-superposable arrangements of the four symbols x, y, z, u. In n dimensions the number is $(n-1)!$ The overwhelmingly important rôle of the vector product of two vectors in three dimensions (now to be defined) distinguishes the vector analysis of three dimensions from that of more than three dimensions.

23. *The vector product. Definition.* Let **P**, **Q** be any two vectors. Then the vector product of **P** with **Q** in this order is defined to be the vector **R**, of modulus $|\mathbf{P}|\ |\mathbf{Q}| \sin \widehat{PQ}$ perpendicular to the plane of **P** and **Q** in such a sense that representations OA, OB, OC of **P**, **Q**, **R** form a *positive* triad OABC. The definition is expressed by

$$\mathbf{R} = \mathbf{P} \wedge \mathbf{Q},$$

where $\wedge$ is the sign of vector multiplication.*

24. *Properties of the vector product.*

Theorem : $$\mathbf{Q} \wedge \mathbf{P} = -\mathbf{P} \wedge \mathbf{Q}.$$

For if $\mathbf{R} = \mathbf{P} \wedge \mathbf{Q}$ and $\mathbf{R}' = \mathbf{Q} \wedge \mathbf{P}$, and if **P**, **Q**, **R**, **R′** are represented by

* Some writers use the symbol [**P.Q**] for the vector product. This has the objection that it employs square brackets in a special sense, when they are often needed in the ordinary algebraic sense of grouping. Other writers use the symbol $\mathbf{P} \times \mathbf{Q}$. But the special sign × is apt to be confused with X in manuscript work, and is to be avoided.

OA, OB, OC, OC′, then OC′ and OC are equal and have opposite senses.

Theorem: If $\mathbf{P}\wedge\mathbf{Q}=0$, then either $\mathbf{P}=0$, or $\mathbf{Q}=0$ or $\mathbf{P}$ and $\mathbf{Q}$ are parallel or antiparallel. For either $|\mathbf{P}|=0$, or $|\mathbf{Q}|=0$ or $\sin\widehat{PQ}=0$, i.e. $\widehat{PQ}=0$ or π.

Corollary. If $\mathbf{P}\wedge\mathbf{Q}=0$, then $\mathbf{Q}=\lambda\mathbf{P}$, where λ may be positive (case of parallelism), negative (antiparallelism) or zero.

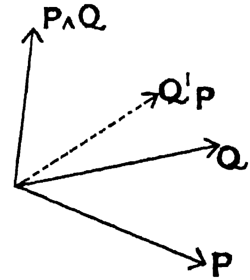

Fig. 6

Theorem: $\mathbf{P}\wedge\mathbf{Q}=\mathbf{P}\wedge\mathbf{Q}'_{\mathbf{P}}$.

For (Fig. 6) $|\mathbf{P}\wedge\mathbf{Q}|=|\mathbf{P}|\ |\mathbf{Q}|\sin\widehat{PQ}$

and $|\mathbf{P}\wedge\mathbf{Q}'_{\mathbf{P}}|=|\mathbf{P}|\ |\mathbf{Q}'_{\mathbf{P}}|\sin\tfrac{1}{2}\pi$

$=|\mathbf{P}|\ |\mathbf{Q}|\sin\widehat{PQ}$.

Further, the three vectors $\mathbf{P}$, $\mathbf{Q}'_{\mathbf{P}}$, $\mathbf{P}\wedge\mathbf{Q}'_{\mathbf{P}}$, which form a positive triad, can be continuously deformed into the triad formed by $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{P}\wedge\mathbf{Q}$, by rotations of $\mathbf{Q}'_{\mathbf{P}}$ into the position $\mathbf{Q}$ through an angle less than $\tfrac{1}{2}\pi$ (Fig. 6). The vector $\mathbf{P}\wedge\mathbf{Q}'_{\mathbf{P}}$ therefore has the same sense as $\mathbf{P}\wedge\mathbf{Q}$.

25. *The distributive law for vector products.*

Theorem: $\mathbf{P}\wedge(\mathbf{Q}+\mathbf{R})=\mathbf{P}\wedge\mathbf{Q}+\mathbf{P}\wedge\mathbf{R}$.

Let OA be a representation of $\mathbf{P}$. Take OB and BC as representations of $\mathbf{Q}'_{\mathbf{P}}$ and $\mathbf{R}'_{\mathbf{P}}$. Then OB and BC are perpendicular to OA, and OBC is a plane perpendicular to OA. Rotate the triangle OBC as a rigid body about OA through a right angle (Fig. 7), and increase the sides in the ratio $|\mathbf{P}|:1$. Denote the new position of OBC by OB′C′. Let the sense of rotation be such that OBB′A is a positive triad; it is, of course, an orthogonal triad. Then OCC′A is also a positive orthogonal triad, for it is superposable on OBB′A. Hence OABB′ and OACC′ are positive orthogonal triads. Hence OB′ is a representation of $\mathbf{P}\wedge\mathbf{Q}'_{\mathbf{P}}$. Similarly B′C′ is a representation of $\mathbf{P}\wedge\mathbf{R}'_{\mathbf{P}}$. Now OC is a representation of $\mathbf{Q}'_{\mathbf{P}}+\mathbf{R}'_{\mathbf{P}}$, and OC′ accordingly a representation of $\mathbf{P}\wedge(\mathbf{Q}'_{\mathbf{P}}+\mathbf{R}'_{\mathbf{P}})$. But OC′ represents the sum of the vectors represented by OB′ and B′C′. Hence

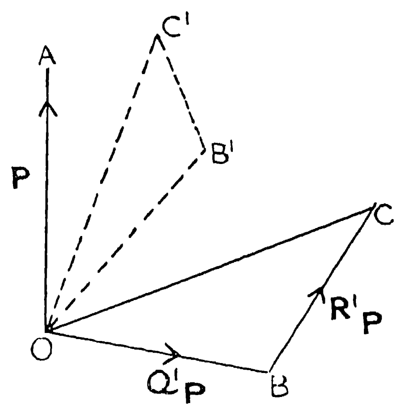

Fig. 7

$$\mathbf{P}\wedge(\mathbf{Q}'_{\mathbf{P}}+\mathbf{R}'_{\mathbf{P}})=\mathbf{P}\wedge\mathbf{Q}'_{\mathbf{P}}+\mathbf{P}\wedge\mathbf{R}'_{\mathbf{P}}.$$

But by the last theorem of § 24,

$$\mathbf{P}\wedge\mathbf{Q}'_{\mathbf{P}}=\mathbf{P}\wedge\mathbf{Q},\qquad \mathbf{P}\wedge\mathbf{R}'_{\mathbf{P}}=\mathbf{P}\wedge\mathbf{R},$$

and we have $$\mathbf{Q}'_{\mathbf{P}}+\mathbf{R}'_{\mathbf{P}}=(\mathbf{Q}+\mathbf{R})'_{\mathbf{P}}$$
by a theorem of § 14. Also
$$\mathbf{P}\wedge(\mathbf{Q}+\mathbf{R})'_{\mathbf{P}}=\mathbf{P}\wedge(\mathbf{Q}+\mathbf{R}).$$
The required theorem now follows.

Corollary. $$(\mathbf{Q}+\mathbf{R})\wedge\mathbf{P}=\mathbf{Q}\wedge\mathbf{P}+\mathbf{R}\wedge\mathbf{P}.$$

26. We may pause here for a moment to mention that there are two fundamentally important applications of the vector product in mechanics. One is the *moment of a force* P about a point O, which is given by $\mathbf{r}\wedge\mathbf{P}$, where $\mathbf{r}$ is the positive vector of some point on the line of action of the line vector $\mathbf{P}$. The other is the velocity of a particle $\mathbf{r}$ of a rigid body which is in motion with angular velocity $\mathbf{\Omega}$ about the point O with respect to which $\mathbf{r}$ is measured; the velocity of $\mathbf{r}$ is given by $\mathbf{\Omega}\wedge\mathbf{r}$. We shall treat formally of these in due course.

27. Why do we confine attention, in the *vector* analysis of three dimensions, to just these two products, the scalar product and the vector product? The analytical theory we give later will show that by means of these two we can handle all the scalar and vector combinations of vectors that can arise. We shall introduce later a further product denoted by symbols such as **PQ**, but this defines something beyond a scalar and vector, and is called a dyad, a constituent of a dyadic or tensor.

28. *Triple product of three vectors.* Let **P, Q, R** be three given vectors. The scalar number $(\mathbf{P}\wedge\mathbf{Q}).\mathbf{R}$ is called the *triple product* of **P, Q, R** in this order. It may also be written $\mathbf{R}.(\mathbf{P}\wedge\mathbf{Q})$. The brackets can be omitted without ambiguity, thus $\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}$ or $\mathbf{R}.\mathbf{P}\wedge\mathbf{Q}$, since the alternative ways of inserting brackets, namely $\mathbf{P}\wedge(\mathbf{Q}.\mathbf{R})$ or $(\mathbf{R}.\mathbf{P})\wedge\mathbf{Q}$, lead to meaningless expressions. It is clear that

$$\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}=-\mathbf{Q}\wedge\mathbf{P}.\mathbf{R}.$$

Theorem: $$\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}=\mathbf{Q}\wedge\mathbf{R}.\mathbf{P}=\mathbf{R}\wedge\mathbf{P}.\mathbf{Q}, \qquad \text{(i)}$$
$$\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}=\mathbf{P}.\mathbf{Q}\wedge\mathbf{R}. \qquad \text{(ii)}$$

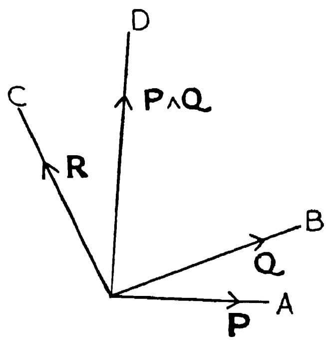

Fig. 8

Here (ii) is simply another way of writing (i). The theorem may be stated in words in the form that we may interchange the order of the vector symbols cyclically, leaving the signs of multiplication unaltered, or we may interchange the multiplication signs leaving the vector symbols unaltered; in each case the triple product is unaltered in value. To prove this theorem, let OA, OB, OC (Fig. 8) be representations of **P, Q, R.** The vector $\mathbf{P}\wedge\mathbf{Q}$ is perpendicular to the plane OAB and has for its representation OD, where OABD is a positive triad and the length OD is $|\mathbf{P}|\ |\mathbf{Q}|\sin\widehat{PQ}$, or twice the area of the triangle OAB. The triple product $(\mathbf{P}\wedge\mathbf{Q}).\mathbf{R}$ is equal to $(\mathbf{P}\wedge\mathbf{Q}).\mathbf{R}_{\mathbf{P}\wedge\mathbf{Q}}$, and is accordingly the product of the

length of $\mathbf{R}_{\mathbf{P}\wedge\mathbf{Q}}$ with the area of the triangle OAB, taken with positive or negative sign according as $\mathbf{R}_{\mathbf{P}\wedge\mathbf{Q}}$ is parallel or antiparallel to $\mathbf{P}\wedge\mathbf{Q}$. If OC can be displaced to OD by rotation through an angle less than $\frac{1}{2}\pi$, OABC is a positive triad and $\mathbf{R}_{\mathbf{P}\wedge\mathbf{Q}}$ is parallel to $\mathbf{P}\wedge\mathbf{Q}$; if the angle is greater than $\frac{1}{2}\pi$, OABC is a negative triad, and $\mathbf{R}_{\mathbf{P}\wedge\mathbf{Q}}$ is antiparallel to $\mathbf{P}\wedge\mathbf{Q}$. But the length of $\mathbf{R}_{\mathbf{P}\wedge\mathbf{Q}}$ is equal to the length of the perpendicular from C on to the plane OAB. Hence $(\mathbf{P}\wedge\mathbf{Q}).\mathbf{R}$ is equal to six times the volume of the tetrahedron OABC, taken with positive or negative sign according as OABC is a positive or a negative triad. If we permute the vectors $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ cyclically, the numerical value of the triple product is accordingly unaltered, and its sign is conserved since the sign of the triad OABC is unaltered by cyclical interchange of A, B, C. The theorem then follows.

This proof appeals to the concepts of area and volume. Independent proofs will be given later.

Theorem : If any two of $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ are parallel, or if $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ are coplanar, or if any of $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ vanish, then

$$\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}=0.$$

For if two of the vectors are parallel, their vector product vanishes. If the three vectors are coplanar, the volume of the associated tetrahedron vanishes.

Corollary. If any two members of the triple product $\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}$ are equal, the triple product vanishes.

Theorem : If $\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}=0$, then $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ are coplanar. For $\mathbf{R}$ must be perpendicular to $\mathbf{P}\wedge\mathbf{Q}$, and therefore must lie in the plane of $\mathbf{P}$ and $\mathbf{Q}$.

Conversely, if $\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}\neq 0$, $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ cannot be coplanar. Again, if $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ are not coplanar, and no two of them are parallel, $\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}\neq 0$.

29. *Linearly independent vectors.* If $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ are three vectors, and if no values of λ, μ, ν exist (save $\lambda=\mu=\nu=0$) for which $\lambda\mathbf{P}+\mu\mathbf{Q}+\nu\mathbf{R}=0$, then $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ are said to be linearly independent.

Theorem : Three non-coplanar vectors are linearly independent. For, if possible, suppose that a relation $\lambda\mathbf{P}+\mu\mathbf{Q}+\nu\mathbf{R}=0$ exists between three non-coplanar vectors $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, where λ, μ, ν are not all zero. Suppose $\lambda\neq 0$. Multiply both sides of the relation scalarly by $\mathbf{Q}\wedge\mathbf{R}$. Then since $\mathbf{Q}.\mathbf{Q}\wedge\mathbf{R}=0$ and $\mathbf{R}.\mathbf{Q}\wedge\mathbf{R}=0$, we have $\lambda\mathbf{P}.\mathbf{Q}\wedge\mathbf{R}=0$. Hence since $\lambda\neq 0$, $\mathbf{P}.\mathbf{Q}\wedge\mathbf{R}=0$ and so the three vectors are coplanar, contradicting the hypothesis.

Conversely, if $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ are linearly independent, then the triple product $\mathbf{P}.\mathbf{Q}\wedge\mathbf{R}\neq 0$.

Theorem : If $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ are three linearly independent vectors, any fourth vector $\mathbf{X}$ can be expressed in the form

$$\mathbf{X}=\lambda\mathbf{P}+\mu\mathbf{Q}+\nu\mathbf{R}.$$

For, assume first that values of λ, μ, ν exist which make this relation true. Multiply scalarly by $\mathbf{Q}\wedge\mathbf{R}$. Then

$$\mathbf{X}.\mathbf{Q}\wedge\mathbf{R}=\lambda\mathbf{P}.\mathbf{Q}\wedge\mathbf{R}.$$

But
$$\mathbf{P}.\mathbf{Q}\wedge\mathbf{R}\neq 0.$$

Hence
$$\lambda=\frac{\mathbf{X}.\mathbf{Q}\wedge\mathbf{R}}{\mathbf{P}.\mathbf{Q}\wedge\mathbf{R}}.$$

Similarly μ and ν may be determined. Hence if the decomposition of X is possible, its form is

$$\mathbf{X}=\frac{(\mathbf{X}.\mathbf{Q}\wedge\mathbf{R})\mathbf{P}+(\mathbf{X}.\mathbf{R}\wedge\mathbf{P})\mathbf{Q}+(\mathbf{X}.\mathbf{P}\wedge\mathbf{Q})\mathbf{R}}{\mathbf{P}.\mathbf{Q}\wedge\mathbf{R}}.$$

Now write
$$\mathbf{Y}=\mathbf{X}-\Sigma\frac{(\mathbf{X}.\mathbf{Q}\wedge\mathbf{R})\mathbf{P}}{\mathbf{P}.\mathbf{Q}\wedge\mathbf{R}}.$$

Actual scalar multiplication shows that

$$\mathbf{Y}.\mathbf{Q}\wedge\mathbf{R}=0, \qquad \mathbf{Y}.\mathbf{R}\wedge\mathbf{P}=0, \qquad \mathbf{Y}.\mathbf{P}\wedge\mathbf{Q}=0.$$

If we knew that $\mathbf{Q}\wedge\mathbf{R}$, $\mathbf{R}\wedge\mathbf{P}$, $\mathbf{P}\wedge\mathbf{Q}$ were linearly independent vectors we could appeal to the theorem of § 17 and infer $\mathbf{Y}=0$. That these three vectors are linearly independent if $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ are linearly independent will be proved simply later. We can, however, construct a direct proof that $\mathbf{Y}=0$ as follows. Since $\mathbf{Y}.\mathbf{Q}\wedge\mathbf{R}=0$, either $\mathbf{Y}=0$ or $\mathbf{Y}$ is coplanar with $\mathbf{Q}$ and $\mathbf{R}$. If $\mathbf{Y}\neq 0$, $\mathbf{Y}$ is by a similar argument coplanar with $\mathbf{P}$ and $\mathbf{R}$. By taking representations OA, OB, OC of $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ it follows that $\mathbf{Y}$ must have its representative parallel to OC. But since $\mathbf{Y}.\mathbf{P}\wedge\mathbf{Q}=0$, $\mathbf{Y}$ must be coplanar with $\mathbf{P}$ and $\mathbf{Q}$, i.e. OA, OB, OC must be coplanar. Hence $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ cannot be linearly independent. This contradicts the hypothesis. Hence $\mathbf{Y}=0$.

It follows that $\mathbf{X}$ has the above expansion. It is said to give the resolution of $\mathbf{X}$ into components along $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$.

Example (1). If a given vector $\mathbf{X}$ is coplanar with given vectors $\mathbf{P}$ and $\mathbf{Q}$, where $\mathbf{P}$ and $\mathbf{Q}$ are not parallel or antiparallel, then $\mathbf{X}$ is of the form

$$\mathbf{X}=\lambda\mathbf{P}+\mu\mathbf{Q}.$$

For, take any vector $\mathbf{R}$ forming with $\mathbf{P}$ and $\mathbf{Q}$ a linearly independent set, and expand $\mathbf{X}$ in terms of $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$. Then since $\mathbf{X}$, $\mathbf{P}$, $\mathbf{Q}$ are coplanar, $\mathbf{X}.\mathbf{P}\wedge\mathbf{Q}=0$, and so the coefficient of $\mathbf{R}$ in the expansion is zero. Hence $\mathbf{X}$ is of the above form.

Clearly the values of λ and μ must be independent of the choice of $\mathbf{R}$. To determine the values of λ and μ, multiply the above expansion by $\mathbf{P}$ and $\mathbf{Q}$ scalarly. We get

$$\mathbf{X}.\mathbf{P}=\lambda\mathbf{P}^2+\mu\mathbf{P}.\mathbf{Q}$$

$$\mathbf{X}.\mathbf{Q}=\lambda\mathbf{P}.\mathbf{Q}+\mu\mathbf{Q}^2.$$

These can be solved for λ and μ provided

$$\begin{vmatrix} \mathbf{P}^2 & \mathbf{P.Q} \\ \mathbf{P.Q} & \mathbf{Q}^2 \end{vmatrix} \neq 0$$

i.e. provided $$\mathbf{P}^2\mathbf{Q}^2-(\mathbf{P.Q})^2\neq 0$$

i.e. provided $$\sin^2 \widehat{PQ}\neq 0$$

i.e. provided $\widehat{PQ}$ is neither 0 nor π.

This can be seen alternatively by multiplying the given expansion vectorially by $\mathbf{P}$ and $\mathbf{Q}$ in turn.

We get $$\mathbf{X}\wedge\mathbf{P}=\mu(\mathbf{Q}\wedge\mathbf{P}),\qquad \mathbf{X}\wedge\mathbf{Q}=\lambda(\mathbf{P}\wedge\mathbf{Q}).$$

These determine values for μ and λ provided $|\mathbf{P}\wedge\mathbf{Q}|\neq 0$, i.e. provided $\widehat{PQ}\neq 0,\ \pi$.

Example (2). If
$$\begin{aligned} a\mathbf{P}+b\mathbf{Q}+c\mathbf{R}&=0,\\ a'\mathbf{P}+b'\mathbf{Q}+c'\mathbf{R}&=0,\end{aligned}$$

and if no two of $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ are parallel, then
$$a/a'=b/b'=c/c'.$$

Example (3). If
$$\begin{aligned} a\mathbf{P}+b\mathbf{Q}+c\mathbf{R}+d\mathbf{S}&=0,\\ a'\mathbf{P}+b'\mathbf{Q}+c'\mathbf{R}+d'\mathbf{S}&=0,\end{aligned}$$

and if $a/a'\neq b/b'\neq c/c'\neq d/d'$, then $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{S}$ are coplanar.

30. *Components of a vector with respect to an orthogonal triad of unit vectors.* By an orthogonal triad of unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is meant a set of three vectors, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ each of unit modulus, whose directions form a positive orthogonal triad. From this definition,

$$\mathbf{i}^2=1,\qquad \mathbf{j}^2=1,\qquad \mathbf{k}^2=1,$$

and $$|\mathbf{j}\wedge\mathbf{k}|=1,\qquad |\mathbf{k}\wedge\mathbf{i}|=1,\qquad |\mathbf{i}\wedge\mathbf{j}|=1.$$

Moreover, since $\mathbf{j}\wedge\mathbf{k}$ is parallel to and in the same sense as $\mathbf{i}$, by definition of the vector product (since $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form a *positive* triad) it follows that

$$\mathbf{j}\wedge\mathbf{k}=\mathbf{i}$$

and similarly $$\mathbf{k}\wedge\mathbf{i}=\mathbf{j},\qquad \mathbf{i}\wedge\mathbf{j}=\mathbf{k}.$$

Further $$\mathbf{j.k}=0,\qquad \mathbf{k.i}=0,\qquad \mathbf{i.j}=0,$$

and $$\mathbf{j}\wedge\mathbf{k.i}=\mathbf{i.i}=+1.$$

Theorem: Any vector $\mathbf{X}$ may be put in the form

$$\mathbf{X}=(\mathbf{X.i})\mathbf{i}+(\mathbf{X.j})\mathbf{j}+(\mathbf{X.k})\mathbf{k}.$$

This follows on using the expansion found in the theorem of § 29, on taking $\mathbf{i}, \mathbf{j}, \mathbf{k}$ for the three vectors $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$. It also readily follows independently, by assuming an expansion $\mathbf{X}=\lambda\mathbf{i}+\mu\mathbf{j}+\nu\mathbf{k}$, multiplying scalarly by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in turn, and showing that the vector $\mathbf{X}-\Sigma(\mathbf{X.i})\mathbf{i}$ vanishes identically.

Now let two vectors **P**, **Q** be expressed in the form

$$\mathbf{P}=p_1\mathbf{i}+p_2\mathbf{j}+p_3\mathbf{k},$$
$$\mathbf{Q}=q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}.$$

Then by actual scalar multiplication we have

$$\mathbf{P}.\mathbf{Q}=(p_1\mathbf{i}+p_2\mathbf{j}+p_3\mathbf{k}).(q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k})$$
$$=p_1q_1+p_2q_2+p_3q_3.$$

It follows that the scalar Σp_1q_1 is independent of the orthogonal unit triad selected. In other form,

$$(\mathbf{P}.\mathbf{i})(\mathbf{Q}.\mathbf{i})+(\mathbf{P}.\mathbf{j})(\mathbf{Q}.\mathbf{j})+(\mathbf{P}.\mathbf{k})(\mathbf{Q}.\mathbf{k})$$

is an *invariant*, independent of the set **i j k** chosen.

Similarly, by actual vector multiplication,

$$\mathbf{P}\wedge\mathbf{Q}=(p_1\mathbf{i}+p_2\mathbf{j}+p_3\mathbf{k})\wedge(q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k})$$
$$=\Sigma(p_2q_3\mathbf{j}\wedge\mathbf{k}+p_3q_2\mathbf{k}\wedge\mathbf{j})$$
$$=(p_2q_3-p_3q_2)\mathbf{i}+(p_3q_1-p_1q_3)\mathbf{j}+(p_1q_2-p_2q_1)\mathbf{k}.$$

Lastly, if $$\mathbf{R}=r_1\mathbf{i}+r_2\mathbf{j}+r_3\mathbf{k},$$

then $$\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}=\Sigma(p_2q_3-p_3q_2)r_1,$$

or $$\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}=\begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix}$$

The last formula affords another proof of the theorem of § 28, since the value of a three-rowed determinant is unaltered by cyclical interchange of the rows.

31. *The continued vector product.* Let **P**, **Q**, **R** be any three vectors. The product $(\mathbf{P}\wedge\mathbf{Q})\wedge\mathbf{R}$ is called the continued vector product of the three vectors, in this order. Since $\mathbf{P}\wedge\mathbf{Q}$ is perpendicular to the plane of the vectors **P** and **Q**, and since $(\mathbf{P}\wedge\mathbf{Q})\wedge\mathbf{R}$ is perpendicular to the vector $\mathbf{P}\wedge\mathbf{Q}$, the vector $(\mathbf{P}\wedge\mathbf{Q})\wedge\mathbf{R}$ must lie in the plane of the vectors **P** and **Q**, and so is of the form $\lambda\mathbf{P}+\mu\mathbf{Q}$. The coefficients λ and μ are given by the following theorem :

$$(\mathbf{P}\wedge\mathbf{Q})\wedge\mathbf{R}=-\mathbf{P}(\mathbf{Q}.\mathbf{R})+\mathbf{Q}(\mathbf{P}.\mathbf{R}).$$

This is the most deep-going theorem of three-dimensional vector analysis. It includes a great number of other theorems arising in different connexions. It is extremely useful for solving vector equations, and reducing vector expressions. The student will find it essential to know this formula by heart, and to be able to quote it readily. By noting that $\mathbf{P}\wedge(\mathbf{Q}\wedge\mathbf{R})=-(\mathbf{Q}\wedge\mathbf{R})\wedge\mathbf{P}$ and applying the theorem to the latter continued product, we have the equivalent formula

$$\mathbf{P}\wedge(\mathbf{Q}\wedge\mathbf{R})=\mathbf{Q}(\mathbf{P}.\mathbf{R})-\mathbf{R}(\mathbf{P}.\mathbf{Q}).$$

In practice it is only necessary to remember *one* of these formulæ.

Owing to its fundamental character we give several different proofs of the theorem. The different types of proof illustrate different possible types of mathematical procedure. Another proof will be given when we consider tensor analysis.

Proof (i). A direct verification of the theorem is obtained by taking an arbitrary orthogonal triad of unit vectors **i, j, k,** and decomposing **P, Q, R** with respect to them. Using the notation of § 30, we have

$$(\mathbf{P}\wedge\mathbf{Q})\wedge\mathbf{R}=\Big[\sum_{1,2,3}(p_2q_3-p_3q_2)\mathbf{i}\Big]\wedge\Big[\sum_{1,2,3}r_1\mathbf{i}\Big].$$

Multiplying out the right-hand side and using $\mathbf{i}\wedge\mathbf{i}=0$, $\mathbf{j}\wedge\mathbf{k}=\mathbf{i}$, etc., we have

$$\begin{aligned}(\mathbf{P}\wedge\mathbf{Q})\wedge\mathbf{R}&=\Sigma[(p_3q_1-p_1q_3)r_3-(p_1q_2-p_2q_1)r_2]\mathbf{i}\\&=\Sigma[(p_1r_1+p_2r_2+p_3r_3)q_1-(q_1r_1+q_2r_2+q_3r_3)p_1]\mathbf{i}\\&=(\Sigma p_1r_1)\mathbf{Q}-(\Sigma q_1r_1)\mathbf{P}\\&=-\mathbf{P}(\mathbf{Q}.\mathbf{R})+\mathbf{Q}(\mathbf{P}.\mathbf{R}).\end{aligned}$$

The objection to this verification is that the rearrangement of the coefficients of **i, j, k** so as to introduce the expressions Σp_1r_1, Σq_1r_1 is artificial and forced, and the procedure gives no insight into why these scalar products should appear. A more natural line of investigation is the following.

Proof (ii). We have seen that

$$(\mathbf{P}\wedge\mathbf{Q})\wedge\mathbf{R}=\lambda\mathbf{P}+\mu\mathbf{Q},$$

where λ, μ are numbers which may *a priori* be any functions of **P, Q, R.** Multiplying this equation scalarly by **R,** the triple product on the left-hand side vanishes, and we get

$$0=\lambda(\mathbf{P}.\mathbf{R})+\mu(\mathbf{Q}.\mathbf{R}).$$

Hence we may put

$$\frac{\lambda}{-\mathbf{Q}.\mathbf{R}}=\frac{\mu}{\mathbf{P}.\mathbf{R}}=k(\mathbf{P},\mathbf{Q};\mathbf{R}).$$

Hence $$(\mathbf{P}\wedge\mathbf{Q})\wedge\mathbf{R}=k(\mathbf{P},\mathbf{Q};\mathbf{R})[-\mathbf{P}(\mathbf{Q}.\mathbf{R})+\mathbf{Q}(\mathbf{P}.\mathbf{R})]. \tag{1}$$

By interchanging **P** and **Q** and using $\mathbf{P}\wedge\mathbf{Q}=-(\mathbf{Q}\wedge\mathbf{P})$, we find that k is symmetrical in **P** and **Q.** Now let $\mathbf{R}_1$, $\mathbf{R}_2$ be two arbitrary vectors. Apply relation (1) in turn to the vectors $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_1+\mathbf{R}_2$ in place of **R,** and add the first two resulting equations and subtract the third. For brevity put

$$\mathbf{X}=k(\mathbf{P},\mathbf{Q};\mathbf{R}_1)\mathbf{R}_1+k(\mathbf{P},\mathbf{Q};\mathbf{R}_2)\mathbf{R}_2-k(\mathbf{P},\mathbf{Q};\mathbf{R}_1+\mathbf{R}_2)(\mathbf{R}_1+\mathbf{R}_2).$$

Then the resulting equation is

$$(\mathbf{X}.\mathbf{Q})\mathbf{P}=(\mathbf{X}.\mathbf{P})\mathbf{Q}.$$

But in general **P** and **Q** are not parallel or zero. Hence $\mathbf{X}.\mathbf{P}=0$, $\mathbf{X}.\mathbf{Q}=0$. Hence either $\mathbf{X}=0$, or **X** is perpendicular to both **P** and **Q.** But **X** is a

linear combination of the arbitrary vectors $\mathbf{R}_1$ and $\mathbf{R}_2$; hence it cannot be in general normal to $\mathbf{P}$ and $\mathbf{Q}$. Hence $\mathbf{X}=0$. Hence

$$[k(\mathbf{P}, \mathbf{Q}; \mathbf{R}_1)-k(\mathbf{P}, \mathbf{Q}; \mathbf{R}_1+\mathbf{R}_2)]\mathbf{R}_1=-[k(\mathbf{P}, \mathbf{Q}; \mathbf{R}_2)-k(\mathbf{P}, \mathbf{Q}; \mathbf{R}_1+\mathbf{R}_2)]\mathbf{R}_2.$$

But $\mathbf{R}_1$ and $\mathbf{R}_2$ are in general in different directions. Hence the coefficients of $\mathbf{R}_1$ and $\mathbf{R}_2$ must be zero. Hence

$$k(\mathbf{P}, \mathbf{Q}; \mathbf{R}_1)=k(\mathbf{P}, \mathbf{Q}; \mathbf{R}_1+\mathbf{R}_2).$$

Hence k is independent of its argument $\mathbf{R}$. We can now write it $k(\mathbf{P}, \mathbf{Q})$.

Now proceed as before with (1) applied to the three vectors $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_1+\mathbf{P}_2$ in place of $\mathbf{P}$. Writing

$$\mathbf{Y}=k(\mathbf{P}_1, \mathbf{Q})\mathbf{P}_1+k(\mathbf{P}_2, \mathbf{Q})\mathbf{P}_2-k(\mathbf{P}_1+\mathbf{P}_2, \mathbf{Q})(\mathbf{P}_1+\mathbf{P}_2),$$

we can express the resulting equality in the form

$$(\mathbf{Q}.\mathbf{R})\mathbf{Y}=\mathbf{Q}(\mathbf{Y}.\mathbf{R}).$$

But $\mathbf{Y}$ is in the plane of $\mathbf{P}_1$ and $\mathbf{P}_2$, and so cannot in general be parallel to $\mathbf{Q}$. Hence we must have $\mathbf{Y}=0$. It follows as before that

$$k(\mathbf{P}_1, \mathbf{Q})=k(\mathbf{P}_1+\mathbf{P}_2, \mathbf{Q}).$$

Hence k is independent of its argument $\mathbf{P}$, and so also, by the symmetry already proved, of its argument $\mathbf{Q}$. Hence k is a constant. Take the particular case $\mathbf{P}=\mathbf{i}$, $\mathbf{Q}=\mathbf{j}$, $\mathbf{R}=\mathbf{j}$. Then

$$(\mathbf{i}\wedge\mathbf{j})\wedge\mathbf{j}=k[-\mathbf{i}(\mathbf{j}.\mathbf{j})+\mathbf{j}(\mathbf{i}.\mathbf{j})].$$

The left-hand side reduces to $\mathbf{k}\wedge\mathbf{j}$, i.e. to $-\mathbf{i}$. The right-hand side reduces to $-k\mathbf{i}$. Hence $k=1$. This establishes the theorem.

The above proof depends only on the *linear* (i.e. additive) properties of vectors. For a proof depending on the properties of the triple product, see *Math. Gaz.*, 23, 37, 1939.

Proof (iii). The following proof depends only on the properties of triads and the definition of the vector product.

Take a unit vector $\mathbf{k}$ perpendicular to the plane of $\mathbf{P}$ and $\mathbf{Q}$. Then $\mathbf{k}$ is parallel to $\mathbf{P}\wedge\mathbf{Q}$ and so

$$\mathbf{P}\wedge\mathbf{Q}=\mu\mathbf{k},$$

where μ is determined by multiplying this relation scalarly by $\mathbf{k}$ in the form

$$\mathbf{P}\wedge\mathbf{Q}.\mathbf{k}=\mu.$$

The vectors $\mathbf{k}\wedge\mathbf{P}$ and $\mathbf{k}\wedge\mathbf{Q}$ are in the plane of $\mathbf{P}$ and $\mathbf{Q}$, whilst $\mathbf{k}$ itself is perpendicular to the plane of $\mathbf{P}$ and $\mathbf{Q}$. Hence the three vectors $\mathbf{k}\wedge\mathbf{P}$, $\mathbf{k}\wedge\mathbf{Q}$, $\mathbf{k}$ are linearly independent. Hence any vector $\mathbf{R}$ may be expressed in the form

$$\mathbf{R}=\alpha(\mathbf{k}\wedge\mathbf{P})+\beta(\mathbf{k}\wedge\mathbf{Q})+\gamma\mathbf{k}.$$

Then $$(\mathbf{P}\wedge\mathbf{Q})\wedge\mathbf{R}=\mu\mathbf{k}\wedge[\alpha(\mathbf{k}\wedge\mathbf{P})+\beta(\mathbf{k}\wedge\mathbf{Q})+\gamma\mathbf{k}].$$

But by definition $\mathbf{k}$, $\mathbf{P}$, $\mathbf{k}\wedge\mathbf{P}$ form a positive triad. Hence $\mathbf{k}$, $\mathbf{k}\wedge\mathbf{P}$, $\mathbf{P}$ form a negative triad. But $\mathbf{k}$, $\mathbf{k}\wedge\mathbf{P}$ are perpendicular, and their moduli

are unity and $|\mathbf{P}|$. Hence the vector product $\mathbf{k}\wedge(\mathbf{k}\wedge\mathbf{P})$ is equal to $-\mathbf{P}$. Similarly, $\mathbf{k}\wedge(\mathbf{k}\wedge\mathbf{Q})$ is equal to $-\mathbf{Q}$. Hence

$$(\mathbf{P}\wedge\mathbf{Q})\wedge\mathbf{R}=-\mu\alpha\mathbf{P}-\mu\beta\mathbf{Q}.$$

But *

$$\mathbf{Q}.\mathbf{R}=\alpha(\mathbf{Q}.\mathbf{k}\wedge\mathbf{P})=\alpha(\mathbf{P}\wedge\mathbf{Q}.\mathbf{k})=\alpha\mu,$$

$$\mathbf{P}.\mathbf{R}=\beta(\mathbf{P}.\mathbf{k}\wedge\mathbf{Q})=\beta(\mathbf{Q}\wedge\mathbf{P}.\mathbf{k})=-\beta\mu.$$

Hence

$$(\mathbf{P}\wedge\mathbf{Q})\wedge\mathbf{R}=-(\mathbf{Q}.\mathbf{R})\mathbf{P}+(\mathbf{P}.\mathbf{R})\mathbf{Q}.$$

The idea of the foregoing proof is that we resolve **R** into components perpendicular to **P**, **Q** and to the plane of **P** and **Q**. If we attempt to proceed by resolving **R** into components *along* **P** and **Q**, we encounter difficulties.

Example (1). $$(\mathbf{P}\wedge\mathbf{Q})^2=\mathbf{P}^2\mathbf{Q}^2-(\mathbf{P}.\mathbf{Q})^2.$$

This result is equivalent to the statement

$$\sin^2 \widehat{PQ}=1-\cos^2 \widehat{PQ},$$

but it follows also from the continued vector product theorem, thus

$$(\mathbf{P}\wedge\mathbf{Q})^2=(\mathbf{P}\wedge\mathbf{Q}).(\mathbf{P}\wedge\mathbf{Q}),$$

or using the triple product property, on regarding this as the triple product of **P**, **Q** and $\mathbf{P}\wedge\mathbf{Q}$,

$$\begin{aligned}(\mathbf{P}\wedge\mathbf{Q})^2&=[(\mathbf{P}\wedge\mathbf{Q})\wedge\mathbf{P}].\mathbf{Q}\\&=[-\mathbf{P}(\mathbf{Q}.\mathbf{P})+\mathbf{Q}(\mathbf{P}^2)].\mathbf{Q}\\&=-(\mathbf{P}.\mathbf{Q})^2+\mathbf{P}^2\mathbf{Q}^2.\end{aligned}$$

By writing $\mathbf{P}=p_1\mathbf{i}+p_2\mathbf{j}+p_3\mathbf{k}$, $Q=q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}$, we have Lagrange's identity,

$$\Sigma(p_2q_3-p_3q_2)^2=(\Sigma p_1{}^2)(\Sigma q_1{}^2)-(\Sigma p_1q_1)^2$$

where Σ denotes $\sum\limits_{1,2,3}$.

Example (2).

$$(\mathbf{P}\wedge\mathbf{Q}).(\mathbf{P}\wedge\mathbf{R})=\mathbf{P}^2(\mathbf{Q}.\mathbf{R})-(\mathbf{P}.\mathbf{Q})(\mathbf{P}.\mathbf{R}).$$

Example (3).

$$(\mathbf{P}\wedge\mathbf{Q}).(\mathbf{R}\wedge\mathbf{S})=(\mathbf{P}.\mathbf{R})(\mathbf{Q}.\mathbf{S})-(\mathbf{Q}.\mathbf{R})(\mathbf{P}.\mathbf{S}).$$

(Examples (2) and (3) yield generalizations of Lagrange's identity.)

Example (4).

$$(\mathbf{Q}\wedge\mathbf{R})\wedge\mathbf{P}+(\mathbf{R}\wedge\mathbf{P})\wedge\mathbf{Q}+(\mathbf{P}\wedge\mathbf{Q})\wedge\mathbf{R}=0.$$

This follows at once from the continued vector product theorem. Alternatively, if the left-hand side is denoted by **X**, then $\mathbf{X}.\mathbf{P}=0$, $\mathbf{X}.\mathbf{Q}=0$, $\mathbf{X}.\mathbf{R}=0$; whence $\mathbf{X}=0$.

Example (5). By the continued vector product theorem,

$$(\mathbf{P}\wedge\mathbf{Q})\wedge(\mathbf{R}\wedge\mathbf{A})=-\mathbf{P}(\mathbf{Q}.\mathbf{R}\wedge\mathbf{A})+\mathbf{Q}(\mathbf{P}.\mathbf{R}\wedge\mathbf{A}).$$

* We do here appeal to a property of the triple product.

But also $$(\mathbf{P}\wedge\mathbf{Q})\wedge(\mathbf{R}\wedge\mathbf{A}) = -(\mathbf{R}\wedge\mathbf{A})\wedge(\mathbf{P}\wedge\mathbf{Q})$$
$$= \mathbf{R}(\mathbf{A}.\mathbf{P}\wedge\mathbf{Q}) - \mathbf{A}(\mathbf{R}.\mathbf{P}\wedge\mathbf{Q}).$$

Equating the two expansions, we have if $\mathbf{R}.\mathbf{P}\wedge\mathbf{Q} \neq 0$,

$$\mathbf{A} = \frac{\mathbf{P}(\mathbf{Q}\wedge\mathbf{R}.\mathbf{A}) + \mathbf{Q}(\mathbf{R}\wedge\mathbf{P}.\mathbf{A}) + \mathbf{R}(\mathbf{P}\wedge\mathbf{Q}.\mathbf{A})}{\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}}.$$

We thus recover the expansion of § 29.

Example (6). Consider $[(\mathbf{P}\wedge\mathbf{Q})\wedge\mathbf{A}]\wedge\mathbf{R}$. We can regard this as containing a continued vector product in two ways. We have in fact

$$[(\mathbf{P}\wedge\mathbf{Q})\wedge\mathbf{A}]\wedge\mathbf{R} = [-\mathbf{P}(\mathbf{Q}.\mathbf{A}) + \mathbf{Q}(\mathbf{P}.\mathbf{A})]\wedge\mathbf{R}$$

and also $$[(\mathbf{P}\wedge\mathbf{Q})\wedge\mathbf{A}]\wedge\mathbf{R} = -(\mathbf{P}\wedge\mathbf{Q})\mathbf{A}.\mathbf{R} + \mathbf{A}(\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}).$$

Equating the two expansions, we have if $\mathbf{P}\wedge\mathbf{Q}.\mathbf{R} \neq 0$,

$$\mathbf{A} = \frac{(\mathbf{Q}\wedge\mathbf{R})\mathbf{P}.\mathbf{A} + (\mathbf{R}\wedge\mathbf{P})\mathbf{Q}.\mathbf{A} + (\mathbf{P}\wedge\mathbf{Q})\mathbf{R}\mathbf{A}}{\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}}.$$

This expands an arbitrary vector $\mathbf{A}$ as a linear function of the three vectors $\mathbf{Q}\wedge\mathbf{R}$, $\mathbf{R}\wedge\mathbf{P}$, $\mathbf{P}\wedge\mathbf{Q}$. This suggests that $\mathbf{P}\wedge\mathbf{Q}$, $\mathbf{Q}\wedge\mathbf{R}$, $\mathbf{R}\wedge\mathbf{P}$ are linearly independent vectors if $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ are linearly independent vectors. An analytical proof of this is provided by the following example.

Example (7). $[(\mathbf{Q}\wedge\mathbf{R})\wedge(\mathbf{R}\wedge\mathbf{P})].(\mathbf{P}\wedge\mathbf{Q}) = (\mathbf{P}\wedge\mathbf{Q}.\mathbf{R})^2$.

For, considering the expression in the square bracket as the continued vector product of $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{R}\wedge\mathbf{P}$ and noting that $\mathbf{R}.\mathbf{R}\wedge\mathbf{P} = 0$, we have

$$[(\mathbf{Q}\wedge\mathbf{R})\wedge(\mathbf{R}\wedge\mathbf{P})].(\mathbf{P}\wedge\mathbf{Q}) = [-\mathbf{Q}(0) + \mathbf{R}(\mathbf{Q}.\mathbf{R}\wedge\mathbf{P})].(\mathbf{P}\wedge\mathbf{Q})$$
$$= [\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}]^2.$$

If $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ are linearly independent, $\mathbf{P}\wedge\mathbf{Q}.\mathbf{R} \neq 0$, and so $\mathbf{Q}\wedge\mathbf{R}$, $\mathbf{R}\wedge\mathbf{P}$, $\mathbf{P}\wedge\mathbf{Q}$ are linearly independent. Further, since $[\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}]^2$ is essentially positive, the three vectors $\mathbf{Q}\wedge\mathbf{R}$, $\mathbf{R}\wedge\mathbf{P}$, $\mathbf{P}\wedge\mathbf{Q}$ form in this order a *positive* triad, whatever the sign of the triad $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$.

Example (8). Show that

$$(\mathbf{P}\wedge\mathbf{Q}.\mathbf{R})\Big[\sum_{\mathbf{P},\mathbf{Q},\mathbf{R}} (\mathbf{P}.\mathbf{A})(\mathbf{P}\wedge\mathbf{A})\Big] = \sum_{\mathbf{P},\mathbf{Q},\mathbf{R}} [(\mathbf{Q}.\mathbf{A})(\mathbf{P}\wedge\mathbf{Q}.\mathbf{A}) + (\mathbf{R}.\mathbf{A})(\mathbf{P}\wedge\mathbf{R}.\mathbf{A})](\mathbf{Q}\wedge\mathbf{R}).$$

(Expand the left-hand side as a linear function of $\mathbf{Q}\wedge\mathbf{R}$, $\mathbf{R}\wedge\mathbf{P}$, $\mathbf{P}\wedge\mathbf{Q}$.)

Example (9). The square $[(\mathbf{P}\wedge\mathbf{Q})\wedge\mathbf{R}]^2$ is equal to $[-\mathbf{P}(\mathbf{Q}.\mathbf{R}) + \mathbf{Q}(\mathbf{P}.\mathbf{R})]^2$, which is equal to

$$\mathbf{P}^2(\mathbf{Q}.\mathbf{R})^2 + \mathbf{Q}^2(\mathbf{P}.\mathbf{R})^2 - 2(\mathbf{P}.\mathbf{Q})(\mathbf{Q}.\mathbf{R})(\mathbf{P}.\mathbf{R}).$$

But if we put $\mathbf{X} = \mathbf{P}\wedge\mathbf{Q}$, the same square is of the form $[\mathbf{X}\wedge\mathbf{R}]^2$, which by example (1) is equal to $\mathbf{X}^2\mathbf{R}^2 - (\mathbf{X}.\mathbf{R})^2$, i.e. to

$$[\mathbf{P}^2\mathbf{Q}^2 - (\mathbf{P}.\mathbf{Q})^2]\mathbf{R}^2 - (\mathbf{P}\wedge\mathbf{Q}.\mathbf{R})^2.$$

Equating the two expansions we have

$$(\mathbf{P}\wedge\mathbf{Q}.\mathbf{R})^2 = \mathbf{P}^2\mathbf{Q}^2\mathbf{R}^2 - (\mathbf{Q}.\mathbf{R})^2\mathbf{P}^2 - (\mathbf{R}.\mathbf{P})^2\mathbf{Q}^2 - (\mathbf{P}.\mathbf{Q})^2\mathbf{R}^2 + 2(\mathbf{P}.\mathbf{Q})(\mathbf{Q}.\mathbf{R})(\mathbf{R}.\mathbf{P}).$$

It does not appear possible to expand $[\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}]^2$ by a more direct method * ; the verification by means of the components of **P**, **Q**, **R** with respect to an orthogonal unit triad **i j k** is tedious and uninformative.

This example gives an immediate verification of the theorem that the triple product $\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}$ is unaltered by cyclical interchange of **P**, **Q**, **R**. For, first, its modulus is clearly unaltered in value, by the example. Further, $\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}$ is reversed in sign if **P** and **Q** are interchanged. Hence

$$|\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}| = |\mathbf{Q}\wedge\mathbf{R}.\mathbf{P}| = |\mathbf{R}\wedge\mathbf{P}.\mathbf{Q}|$$

whence $$\mathbf{P}\wedge\mathbf{Q}.\mathbf{R} = \pm\mathbf{Q}\wedge\mathbf{R}.\mathbf{P} = \pm\mathbf{R}\wedge\mathbf{P}.\mathbf{Q}.$$

If we had $$\mathbf{P}\wedge\mathbf{Q}.\mathbf{R} = -\mathbf{Q}\wedge\mathbf{R}.\mathbf{P}$$

then by cyclical interchanges

$$-\mathbf{Q}\wedge\mathbf{R}.\mathbf{P} = \mathbf{R}\wedge\mathbf{P}.\mathbf{Q}$$

and $$\mathbf{R}\wedge\mathbf{P}.\mathbf{Q} = -\mathbf{P}\wedge\mathbf{Q}.\mathbf{R},$$

giving a contradiction. This establishes the triple product theorem as a consequence of the continued vector product theorem.

The formula of this example has a well-known geometrical interpretation. For $|\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}|$ is the volume of the parallelepiped defined by coterminal representations of **P**, **Q**, **R**. If these vectors have lengths p, q, r and if the angles between them are α, β, γ, then the example asserts that the volume of the parallelepiped in question is

$$pqr[1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma]^{\frac{1}{2}}.$$

This illustrates the fact that the continued vector product theorem is equivalent to much solid geometry.

Example (10). $\Sigma[(\mathbf{R}\wedge\mathbf{P})\wedge(\mathbf{P}\wedge\mathbf{Q})]\wedge(\mathbf{Q}\wedge\mathbf{R}) = 0.$

Example (11). If **P** is any vector in the plane of two vectors **x** and **y**, **k** a unit vector perpendicular to their plane, prove that

$$(\mathbf{x}\wedge\mathbf{y})\wedge\mathbf{P} = (\mathbf{x}\wedge\mathbf{y}.\mathbf{k})(\mathbf{k}\wedge\mathbf{P}).$$

Example (12). Show that $\mathbf{A}\wedge(\mathbf{B}\wedge\mathbf{C})$ may be expressed as a linear function of $\mathbf{A}\wedge\mathbf{B}$ and $\mathbf{A}\wedge\mathbf{C}$.

(Expand $\mathbf{A}\wedge(\mathbf{B}\wedge\mathbf{C})$ as a linear function of $\mathbf{B}\wedge\mathbf{C}$, $\mathbf{C}\wedge\mathbf{A}$ and $\mathbf{A}\wedge\mathbf{B}$, and show that the coefficient of $\mathbf{B}\wedge\mathbf{C}$ is zero ; or use representations.)

Example (13). If **i**, **j**, **k** are three mutually perpendicular unit vectors, then for any fourth vector **P**,

$$(\mathbf{i}.\mathbf{P})(\mathbf{i}\wedge\mathbf{P})+(\mathbf{j}.\mathbf{P})(\mathbf{j}\wedge\mathbf{P})+(\mathbf{k}.\mathbf{P})(\mathbf{k}\wedge\mathbf{P}) = 0.$$

Example (14). Solve for **X** the vector equation

$$\alpha\mathbf{X}+\mathbf{X}\wedge\mathbf{A} = \mathbf{B}. \qquad (\alpha \neq 0)$$

* An alternative method is to expand $\mathbf{P}\wedge\mathbf{Q}$ in the form $\lambda\mathbf{P}+\mu\mathbf{Q}+\nu\mathbf{R}$, determining λ, μ, ν by multiplying scalarly in turn by $\mathbf{Q}\wedge\mathbf{R}$, $\mathbf{R}\wedge\mathbf{P}$ and $\mathbf{P}\wedge\mathbf{Q}$. Forming then the scalar product $(\mathbf{P}\wedge\mathbf{Q}).\mathbf{R}$ we obtain the desired expansion.

Assume a solution **X** exists. Multiply the two sides of the equation vectorially by **A**. Then

$$\alpha(\mathbf{X}\wedge\mathbf{A})+[-\mathbf{X}\mathbf{A}^2+\mathbf{A}(\mathbf{X}.\mathbf{A})]=\mathbf{B}\wedge\mathbf{A}.$$

To find **X.A** multiply the given equation scalarly by **A**. Then

$$\alpha\mathbf{X}.\mathbf{A}=\mathbf{B}.\mathbf{A}.$$

Hence
$$\alpha(\mathbf{X}\wedge\mathbf{A})-\mathbf{X}\mathbf{A}^2=\mathbf{B}\wedge\mathbf{A}-\mathbf{A}\frac{\mathbf{B}.\mathbf{A}}{\alpha}.$$

The last equation together with the given equation form a pair of linear simultaneous equations in the vectors **X** and **X∧A**. Eliminating **X∧A** we find

$$\mathbf{X}=\frac{\alpha^2\,\mathbf{B}-\alpha(\mathbf{B}\wedge\mathbf{A})+\mathbf{A}(\mathbf{B}.\mathbf{A})}{\alpha(\alpha^2+\mathbf{A}^2)}.$$

By substitution this is found to be an actual solution. The solution is moreover unique.

Example (15). Solve for **X** and **Y** the simultaneous vector equations

$$\alpha\mathbf{X}+\mathbf{Y}\wedge\mathbf{P}=\mathbf{A},$$
$$\beta\mathbf{Y}+\mathbf{X}\wedge\mathbf{P}=\mathbf{B}.$$

Example (16). Solve for **X** the equations

$$\mathbf{X}\wedge\mathbf{A}=\mathbf{B},\qquad \mathbf{X}.\mathbf{C}=\alpha.\qquad (\mathbf{B}\text{ perpendicular to }\mathbf{A})$$

Assume a solution **X** exists. Multiply both sides of the first equation vectorially by **C**, and use the second. We find

$$-\mathbf{X}(\mathbf{A}.\mathbf{C})+\alpha\mathbf{A}=\mathbf{B}\wedge\mathbf{C}$$

whence
$$\mathbf{X}=\frac{\alpha\mathbf{A}-\mathbf{B}\wedge\mathbf{C}}{\mathbf{A}.\mathbf{C}},$$

provided $\mathbf{A}.\mathbf{C}\neq 0$.

To see the position when $\mathbf{A}.\mathbf{C}=0$, let **X**, **Y** be any two solutions. Then $(\mathbf{X}-\mathbf{Y})\wedge\mathbf{A}=0$, $(\mathbf{X}-\mathbf{Y}).\mathbf{C}=0$, so that either $\mathbf{X}=\mathbf{Y}$ or $\mathbf{X}-\mathbf{Y}$ is parallel to **A** and perpendicular to **C**. Hence there is only one solution unless **A** is perpendicular to **C**. Thus, if $\mathbf{A}.\mathbf{C}=0$, we may expect a multiplicity of solutions. Let $\mathbf{X}_0$ be any one solution. Then $(\mathbf{X}-\mathbf{X}_0)\wedge\mathbf{A}=0$, whence $\mathbf{X}=\mathbf{X}_0+\lambda\mathbf{A}$, where, if $\mathbf{A}.\mathbf{C}=0$, λ is arbitrary. It is therefore sufficient to find a single solution $\mathbf{X}_0$. Impose the condition $\mathbf{X}_0.\mathbf{A}=0$. Multiplying the original equation vectorially by **A**, we find $-\mathbf{X}_0\mathbf{A}^2=\mathbf{B}\wedge\mathbf{A}$. Thus if $\mathbf{A}.\mathbf{C}=0$, the general solution is

$$\mathbf{X}=\frac{\mathbf{A}\wedge\mathbf{B}}{\mathbf{A}^2}+\lambda\mathbf{A},$$

and the satisfaction of the relation $\mathbf{X}.\mathbf{C}=\alpha$ requires the consistency condition

$$\alpha\mathbf{A}^2=\mathbf{A}\wedge\mathbf{B}.\mathbf{C}.$$

This solution differs in form from the previous one. Can we express the two solutions in such forms that they pass into one another as $\mathbf{A.C}\to 0$? To examine this, seek a solution of the form

$$\mathbf{X}=\lambda\mathbf{A}+\mu\mathbf{A}\wedge\mathbf{B}+\nu\mathbf{A}\wedge\mathbf{C}.$$

Such an expansion of $\mathbf{X}$ will be possible if $\mathbf{A}$, $\mathbf{A}\wedge\mathbf{B}$ and $\mathbf{A}\wedge\mathbf{C}$ are linearly independent. Now

$$[\mathbf{A}\wedge(\mathbf{A}\wedge\mathbf{B})].(\mathbf{A}\wedge\mathbf{C})=[-\mathbf{B}\mathbf{A}^2+\mathbf{A}(\mathbf{A.B})].(\mathbf{A}\wedge\mathbf{C})=\mathbf{A}^2(\mathbf{A}\wedge\mathbf{B.C}),$$

so that $\mathbf{A}$, $\mathbf{A}\wedge\mathbf{B}$ and $\mathbf{A}\wedge\mathbf{C}$ are linearly independent if $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are linearly independent. Assume that they are. Then inserting this expansion for $\mathbf{X}$ in the given equation $\mathbf{X}\wedge\mathbf{A}=\mathbf{B}$, we find

$$\mu[-\mathbf{A}(\mathbf{B.A})+\mathbf{B}\mathbf{A}^2]+\nu[-\mathbf{A}(\mathbf{C.A})+\mathbf{C}\mathbf{A}^2]=\mathbf{B}.$$

This is a linear relation between the three vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and hence the coefficients of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ must vanish separately. Hence

$$-\mu(\mathbf{B.A})-\nu(\mathbf{C.A})=0, \qquad \mu\mathbf{A}^2=1, \qquad \nu=0.$$

The first of these is now automatically satisfied, since $\mathbf{A.B}=0$. Hence

$$\mathbf{X}=\lambda\mathbf{A}+\frac{\mathbf{A}\wedge\mathbf{B}}{\mathbf{A}^2},$$

and the relation $\mathbf{X.C}=\alpha$ then determines λ. We find

$$\mathbf{X}=\left(\alpha-\frac{\mathbf{A}\wedge\mathbf{B.C}}{\mathbf{A}^2}\right)\frac{\mathbf{A}}{\mathbf{A.C}}+\frac{\mathbf{A}\wedge\mathbf{B}}{\mathbf{A}^2}.$$

The identification of this solution with the solution first obtained follows if we can establish the identity

$$\mathbf{B}\wedge\mathbf{C}=\frac{1}{\mathbf{A}^2}[\mathbf{A}(\mathbf{A}\wedge\mathbf{B.C})-(\mathbf{A}\wedge\mathbf{B})(\mathbf{A.C})],$$

given $\mathbf{A.B}=0$. This is left to the reader.

It is now clear that if $\mathbf{A.C}\to 0$, for a finite solution we must have $(\mathbf{A}\wedge\mathbf{B.C})/\mathbf{A}^2\to\alpha$, and the coefficient of $\mathbf{A}$ is then indeterminate.

Example (17). The most general solution of the equation

$$\mathbf{X}\wedge\mathbf{A}=\mathbf{B}, \qquad (\mathbf{A.B}=0)$$

an equation of frequent occurrence, is most simply expressed as follows. If $\mathbf{X}_0$ is any one solution, $\mathbf{X}=\mathbf{X}_0+\lambda\mathbf{A}$ is the most general solution. Choose $\mathbf{X}_0$ to satisfy $\mathbf{X}_0.\mathbf{A}=0$, if possible. Multiplying the given equation vectorially by $\mathbf{A}$, we have then $-\mathbf{X}_0\mathbf{A}^2=\mathbf{B}\wedge\mathbf{A}$, whence the most general solution is

$$\mathbf{X}=\frac{\mathbf{A}\wedge\mathbf{B}}{\mathbf{A}^2}+\lambda\mathbf{A}.$$

It will be seen that $\mathbf{X}_0$ is the component of the most general solution perpendicular to $\mathbf{A}$.

Example (18). Show that the necessary and sufficient condition that the equations

$$\mathbf{X}\wedge\mathbf{A}=\mathbf{B}, \qquad (\mathbf{A}.\mathbf{B}=0)$$
$$\mathbf{X}\wedge\mathbf{C}=\mathbf{D}, \qquad (\mathbf{C}.\mathbf{D}=0)$$

may have a common solution $\mathbf{X}$ is

$$\mathbf{B}.\mathbf{C}+\mathbf{A}.\mathbf{D}=0.$$

Example (19). Show that the equations

$$\alpha\mathbf{X}+\beta\mathbf{Y}=\mathbf{A}, \qquad \mathbf{X}\wedge\mathbf{Y}=\mathbf{B} \qquad (\mathbf{A}.\mathbf{B}=0)$$

have a one parameter system of solutions $\mathbf{X}$, $\mathbf{Y}$, and obtain them.

Example (20). Solve the equation

$$\mathbf{X}\wedge\mathbf{A}+(\mathbf{X}.\mathbf{B})\mathbf{C}=\mathbf{D}.$$

Example (21). Solve the equation

$$\alpha\mathbf{X}+\mathbf{X}\wedge\mathbf{A}+(\mathbf{X}.\mathbf{B})\mathbf{C}=\mathbf{D}.$$

Example (22). If $\mathbf{A}$, $\mathbf{B}$ are given non-parallel vectors, and $\mathbf{X}$ and $\mathbf{Y}$ vectors satisfying $\mathbf{X}\wedge\mathbf{A}=\mathbf{Y}\wedge\mathbf{B}$, show that $\mathbf{X}$ and $\mathbf{Y}$ are linear functions of $\mathbf{A}$ and $\mathbf{B}$, and obtain their most general forms.

(Expand $\mathbf{X}$ and $\mathbf{Y}$ as linear functions of the three linearly independent vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{A}\wedge\mathbf{B}$.)

Example (23). Prove that

$$(\mathbf{X}-\mathbf{P})\wedge(\mathbf{X}-\mathbf{Q}).(\mathbf{X}-\mathbf{R})=\mathbf{X}.[\Sigma\mathbf{P}\wedge\mathbf{Q}]-\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}.$$

Deduce that if $\mathbf{X}-\mathbf{P}$, $\mathbf{X}-\mathbf{Q}$, $\mathbf{X}-\mathbf{R}$ are linearly dependent vectors, then $\mathbf{X}$ is of the form $\alpha\mathbf{P}+\beta\mathbf{Q}+\gamma\mathbf{R}$, where $\alpha+\beta+\gamma=1$, provided $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ are linearly independent.

Example (24). Find the general solution of the equations in $\mathbf{X}$ and $\mathbf{Y}$

$$\alpha\mathbf{X}+\beta\mathbf{Y}=\mathbf{A}, \qquad \mathbf{X}.\mathbf{Y}=\rho.$$

Example (25). Obtain a first integral of the linear vector differential equation

$$\frac{d^2\mathbf{X}}{dt^2}+\alpha\left(\frac{d\mathbf{X}}{dt}\wedge\mathbf{i}\right)+\beta\mathbf{X}=0,$$

where α, β are constants, $\mathbf{i}$ a constant unit vector and $\mathbf{X}.\mathbf{i}=0$.

Try a solution $$\frac{d\mathbf{X}}{dt}=p(\mathbf{X}\wedge\mathbf{i}).$$

Then $$\frac{d\mathbf{X}}{dt}\wedge\mathbf{i}=-p\mathbf{X},$$

and $$\frac{d^2\mathbf{X}}{dt^2}=p[p(\mathbf{X}\wedge\mathbf{i})\wedge\mathbf{i}]=-p^2\mathbf{X}.$$

Introducing these in the given equation, we have

$$(-p^2-\alpha p+\beta)\mathbf{X}=0.$$

Hence we find two first integrals,

$$\frac{d\mathbf{X}}{dt} = p_1(\mathbf{X}\wedge\mathbf{i}), \qquad \frac{d\mathbf{X}}{dt} = p_2(\mathbf{X}\wedge\mathbf{i})$$

where p_1, p_2 are roots of the quadratic

$$p^2 + \alpha p - \beta = 0.$$

This procedure has important dynamical applications. The kinematic interpretation of the solution will appear later.

32. *Formulæ of spherical trigonometry.* The establishment of the formula for the continued vector product carries with it the establishment of the main formulæ of spherical trigonometry, with the minimum of further spatial appeals.

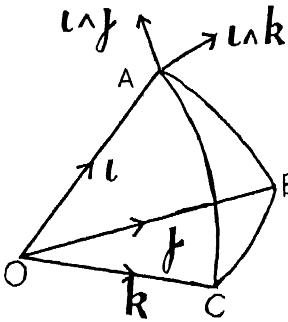

Fig. 9

Take a spherical triangle ABC on the surface of a sphere of unit radius, centre O (Fig. 9). Let **i, j, k** denote the three vectors OA, OB, OC, not necessarily mutually perpendicular. The vector $\mathbf{i}\wedge\mathbf{j}$ is normal to the plane of OA and OB, and thus has a representation normal to the plane OAB at A. Similarly $\mathbf{i}\wedge\mathbf{k}$ has a representation normal to the plane OAC at A. The modulus $|\mathbf{i}\wedge\mathbf{j}|$ is sin $\hat{AB}$, or sin c ; the modulus $|\mathbf{i}\wedge\mathbf{k}|$ is sin $\hat{AC}$, or sin b. The angle between the representations of $\mathbf{i}\wedge\mathbf{j}$ and $\mathbf{i}\wedge\mathbf{k}$ is equal to the angle between the planes OAB and OAC, which is just A. Hence, evaluating the scalar product of $(\mathbf{i}\wedge\mathbf{j})$ and $(\mathbf{i}\wedge\mathbf{k})$, we have

$$(\mathbf{i}\wedge\mathbf{j}).(\mathbf{i}\wedge\mathbf{k}) = \sin c \sin b \cos A.$$

But

$$\begin{aligned}(\mathbf{i}\wedge\mathbf{j}).(\mathbf{i}\wedge\mathbf{k}) &= \mathbf{i}.[\mathbf{j}\wedge(\mathbf{i}\wedge\mathbf{k})] \\ &= \mathbf{i}.[-\mathbf{k}(\mathbf{i}.\mathbf{j}) + \mathbf{i}(\mathbf{j}.\mathbf{k})] \\ &= -(\mathbf{i}.\mathbf{k})(\mathbf{i}.\mathbf{j}) + (\mathbf{j}.\mathbf{k}) \\ &= -\cos c \cos b + \cos a.\end{aligned}$$

Hence

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Again, by the definition of a vector product,

$$\sin A = \frac{|(\mathbf{i}\wedge\mathbf{j})\wedge(\mathbf{i}\wedge\mathbf{k})|}{|\mathbf{i}\wedge\mathbf{j}|\,|\mathbf{i}\wedge\mathbf{k}|} = \frac{|-\mathbf{i}(\mathbf{j}.\mathbf{i}\wedge\mathbf{k}) + \mathbf{j}(\mathbf{i}.\mathbf{i}\wedge\mathbf{k})|}{\sin b \sin c} = \frac{|\mathbf{i}\wedge\mathbf{j}.\mathbf{k}|}{\sin b \sin c}.$$

Hence

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{6 \text{ vol (OABC)}}{\sin a \sin b \sin c}.$$

33. *Procedures for decomposing vectors.* The following useful rules have already been partly illustrated in foregoing examples.

(1) If $\mathbf{X}$ is to be discussed in relation to *three* given vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, write either

$$\mathbf{X}=\lambda\mathbf{A}+\mu\mathbf{B}+\nu\mathbf{C}$$

or

$$\mathbf{X}=\lambda\mathbf{B}\wedge\mathbf{C}+\mu\mathbf{C}\wedge\mathbf{A}+\nu\mathbf{A}\wedge\mathbf{B}.$$

(2) If $\mathbf{X}$ is to be discussed in relation to *two* given vectors $\mathbf{A}$, $\mathbf{B}$, write

$$\mathbf{X}=\lambda\mathbf{A}+\mu\mathbf{B}+\nu\mathbf{A}\wedge\mathbf{B}.$$

This is always possible, since $\mathbf{A}$, $\mathbf{B}$, $\mathbf{A}\wedge\mathbf{B}$ are linearly independent.

(3) If $\mathbf{X}$ is to be discussed in relation to *one* given vector $\mathbf{A}$, write

$$\mathbf{X}=\lambda\mathbf{A}+\mathbf{A}\wedge\mathbf{Y},$$

where

$$\mathbf{A}.\mathbf{Y}=0.$$

This is equivalent to decomposing $\mathbf{X}$ into a component along $\mathbf{A}$ and one perpendicular to $\mathbf{A}$. To show that the decomposition is always possible, we observe that λ may be chosen so that $\mathbf{X}-\lambda\mathbf{A}$ is perpendicular to $\mathbf{A}$; for the value of λ is then given by $\mathbf{X}.\mathbf{A}=\lambda\mathbf{A}^2$. Then $\mathbf{X}-\lambda\mathbf{A}$ is of the form $\mathbf{A}\wedge\mathbf{Y}$, and as the component of $\mathbf{Y}$ parallel to $\mathbf{A}$ is irrelevant, we may choose $\mathbf{Y}$ so that it is perpendicular to $\mathbf{A}$, i.e. so that $\mathbf{Y}.\mathbf{A}=0$.

To determine $\mathbf{Y}$, multiply vectorially by $\mathbf{A}$. We find

$$\mathbf{X}\wedge\mathbf{A}=\mathbf{Y}\mathbf{A}^2,$$

which determines $\mathbf{Y}$. The solution is now the identity,

$$\mathbf{X}=\frac{\mathbf{X}.\mathbf{A}}{\mathbf{A}^2}\mathbf{A}+\frac{\mathbf{A}\wedge(\mathbf{X}\wedge\mathbf{A})}{\mathbf{A}^2}.$$

34. *Reciprocal vectors.* We have seen that given any vector $\mathbf{X}$, and any three linearly independent vectors $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, the vector $\mathbf{X}$ may be put in the form

$$\mathbf{X}=\frac{\mathbf{P}(\mathbf{Q}\wedge\mathbf{R}.\mathbf{X})+\mathbf{Q}(\mathbf{R}\wedge\mathbf{P}.\mathbf{X})+\mathbf{R}(\mathbf{P}\wedge\mathbf{Q}.\mathbf{X})}{\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}}.$$

The vectors $\mathbf{P}'$, $\mathbf{Q}'$, $\mathbf{R}'$ given by

$$\mathbf{P}'=\frac{\mathbf{Q}\wedge\mathbf{R}}{\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}},\qquad \mathbf{Q}'=\frac{\mathbf{R}\wedge\mathbf{P}}{\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}},\qquad \mathbf{R}'=\frac{\mathbf{P}\wedge\mathbf{Q}}{\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}},$$

are said to be *reciprocal* to $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$. It is clear that we have

$$\mathbf{X}=(\mathbf{X}.\mathbf{P}')\mathbf{P}+(\mathbf{X}.\mathbf{Q}')\mathbf{Q}+(\mathbf{X}.\mathbf{R}')\mathbf{R}.$$

The reason for the use of the word *reciprocal* may be seen as follows. We have

$$\mathbf{P}'\wedge\mathbf{Q}'.\mathbf{R}'=\frac{(\mathbf{Q}\wedge\mathbf{R})\wedge(\mathbf{R}\wedge\mathbf{P}).(\mathbf{P}\wedge\mathbf{Q})}{(\mathbf{P}\wedge\mathbf{Q}.\mathbf{R})^3}=\frac{1}{\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}}.$$

Hence if $\mathbf{P}''$, $\mathbf{Q}''$, $\mathbf{R}''$ are the vectors reciprocal to $\mathbf{P}'$, $\mathbf{Q}'$, $\mathbf{R}'$,

$$\mathbf{P}''=\frac{\mathbf{Q}'\wedge\mathbf{R}'}{\mathbf{P}'\wedge\mathbf{Q}'.\mathbf{R}'}=\frac{(\mathbf{R}\wedge\mathbf{P})\wedge(\mathbf{P}\wedge\mathbf{Q})}{\mathbf{P}\wedge\mathbf{Q}\,\mathbf{R}}=\mathbf{P}.$$

Hence the relation of $\mathbf{P}'$, $\mathbf{Q}'$, $\mathbf{R}'$ to $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ is a reciprocal one. Further, $\mathbf{P}'$, $\mathbf{Q}'$, $\mathbf{R}'$ are linearly independent, and form a triad of the same sign as the triad $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$.

35. *Self-reciprocal sets of vectors.* Let us find the condition that $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ may be self-reciprocal, i.e. the conditions that $\mathbf{P}'=\mathbf{P}$, $\mathbf{Q}'=\mathbf{Q}$, $\mathbf{R}'=\mathbf{R}$. The condition

$$\mathbf{P}=\frac{\mathbf{Q}\wedge\mathbf{R}}{\mathbf{P}\wedge\mathbf{Q}.\mathbf{R}}$$

gives at once $\mathbf{P}^2=1$; similarly $\mathbf{Q}^2=1$, $\mathbf{R}^2=1$. Again $\mathbf{P}.\mathbf{Q}=0$, $\mathbf{P}.\mathbf{R}=0$ and similarly $\mathbf{Q}.\mathbf{R}=0$. Hence $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ must be an orthogonal set of unit vectors. Conversely, any orthogonal positive triad of unit vectors is self-reciprocal. For if $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ form such a triad, $\mathbf{i}\wedge\mathbf{j}.\mathbf{k}=1$, and so $\mathbf{i}'=\mathbf{j}\wedge\mathbf{k}=\mathbf{i}$. It is to this property that orthogonal triads of unit vectors owe their importance.

Example. Prove that if $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are mutually perpendicular unit vectors such that $\mathbf{j}\wedge\mathbf{k}=-\mathbf{i}$, $\mathbf{k}\wedge\mathbf{i}=-\mathbf{j}$, $\mathbf{i}\wedge\mathbf{j}=-\mathbf{k}$, then they also form a self-reciprocal set.

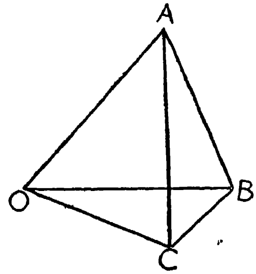

Fig. 10

36. *Tetrahedron properties involving vector products.* (1) Vectors along the outward perpendiculars to the faces of a tetrahedron and proportional to the areas of the corresponding faces have a vector sum zero. For, if OABC (Fig. 10) is the tetrahedron, the vectors are proportional to $\overrightarrow{OA}\wedge\overrightarrow{OB}$, $\overrightarrow{OB}\wedge\overrightarrow{OC}$, $\overrightarrow{OC}\wedge\overrightarrow{OA}$, and $\overrightarrow{AC}\wedge\overrightarrow{AB}$. Their sum is

$$\overrightarrow{OA}\wedge\overrightarrow{OB}+\overrightarrow{OB}\wedge\overrightarrow{OC}+\overrightarrow{OC}\wedge\overrightarrow{OA}+(\overrightarrow{OC}-\overrightarrow{OA})\wedge(\overrightarrow{OB}-\overrightarrow{OA})$$

which reduces identically to zero.

(2) Conversely, if four vectors perpendicular to the faces of a tetrahedron have the sum zero, their magnitudes are proportional to the areas of the corresponding faces. For, let the vectors be $p_1\overrightarrow{OB}\wedge\overrightarrow{OC}$, $p_2\overrightarrow{OC}\wedge\overrightarrow{OA}$, $p_3\overrightarrow{OA}\wedge\overrightarrow{OB}$, $p_4\overrightarrow{AC}\wedge\overrightarrow{AB}$. Then by hypothesis $p_1\overrightarrow{OB}\wedge\overrightarrow{OC}+p_2\overrightarrow{OC}\wedge\overrightarrow{OA}+p_3\overrightarrow{OA}\wedge\overrightarrow{OB}+p_4(\overrightarrow{OC}-\overrightarrow{OA})\wedge(\overrightarrow{OB}-\overrightarrow{OA})=0$. Multiply scalarly by $\overrightarrow{OA}$. Then since $\overrightarrow{OA}\wedge\overrightarrow{OB}.\overrightarrow{OC}\neq 0$, we find at once $p_1-p_4=0$. Similarly $p_2=p_3=p_4$.

37. *Planimeter theory.* The area of a triangle OPP′ is equal to the modulus of the vector $\frac{1}{2}(\overrightarrow{OP}\wedge\overrightarrow{PP'})$. We may say that the latter vector *represents* the area of the triangle. If the point P describes a closed plane curve (Fig. 11) surrounding O, and if $\mathbf{P}$ denotes the position vector $\overrightarrow{OP}$ of P with respect to O,

then by addition of elementary triangles it follows that the area of the curve is represented by

$$\tfrac{1}{2}\int \mathbf{P}\wedge d\mathbf{P},$$

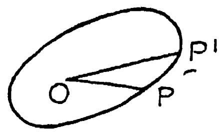

Fig. 11

where the integral is taken round the curve. If we take any other origin O_1 (inside or outside the curve), and write $\overrightarrow{O_1P}=\mathbf{P}_1$, $\overrightarrow{OO_1}=\mathbf{a}$, then $\mathbf{P}=\mathbf{P}_1-\mathbf{a}$, $d\mathbf{P}=d\mathbf{P}_1$, and so the above integral is equal to

$$\tfrac{1}{2}\int (\mathbf{P}_1-\mathbf{a})\wedge d\mathbf{P}_1.$$

but $\int \mathbf{a}\wedge d\mathbf{P}_1=\mathbf{a}\wedge\int d\mathbf{P}_1=0$, since the curve is closed. Hence for *any* origin O, the area is represented by

$$\tfrac{1}{2}\int \mathbf{P}_1\wedge d\mathbf{P}_1.$$

Amsler's planimeter. A planimeter is a mechanism for determining the area enclosed by an arbitrary closed curve. The essential property of Amsler's planimeter may be established as follows. Let the vector $\overrightarrow{P'P}$ be of constant length, and let $\mathbf{P}'$ move to a neighbouring point $\mathbf{P}'+d\mathbf{P}'$, $\mathbf{P}$ to $\mathbf{P}+d\mathbf{P}$ (Fig. 12). The vector may be pictured as a rigid rod, of length, say, l. The area swept out by the rod in a small displacement is represented by a vector $d\Sigma$, where

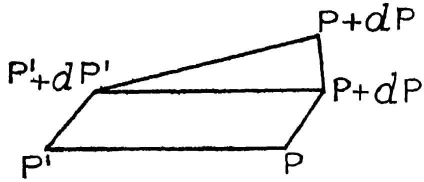

Fig. 12

$$\begin{aligned} d\Sigma &= (\text{area of parallelogram})+(\text{area of triangle}) \\ &= (\mathbf{P}-\mathbf{P}')\wedge d\mathbf{P}'+\tfrac{1}{2}(\mathbf{P}-\mathbf{P}')\wedge[(\mathbf{P}+d\mathbf{P})-(\mathbf{P}'+d\mathbf{P}')] \\ &= (\mathbf{P}-\mathbf{P}')\wedge d\mathbf{P}'+\tfrac{1}{2}(\mathbf{P}-\mathbf{P}')\wedge(d\mathbf{P}-d\mathbf{P}') \qquad (1) \\ &= \tfrac{1}{2}(\mathbf{P}-\mathbf{P}')\wedge(d\mathbf{P}+d\mathbf{P}'), \end{aligned}$$

or

$$d\Sigma=\tfrac{1}{2}\mathbf{P}\wedge d\mathbf{P}-\tfrac{1}{2}\mathbf{P}'\wedge d\mathbf{P}'+\tfrac{1}{2}d(\mathbf{P}\wedge\mathbf{P}'). \qquad (2)$$

Now let the rod move in a plane so that P and P′ describe closed curves. Equation (2), integrated round the circuit, gives the result that the vector Σ representing the area swept out by the rod is the difference of the vectors representing the areas A, A′ of the curves described by P and P′, since $\int d(\mathbf{P}\wedge\mathbf{P}')=0$. Hence the obvious result, $\Sigma=A-A'$. Now integrate equation (1) round the circuit. The integral $\tfrac{1}{2}\int(\mathbf{P}-\mathbf{P}')\wedge d(\mathbf{P}-\mathbf{P}')$ represents the area swept out by a 'radius vector' of fixed length l parallel to the rod, one of the ends of the radius vector being fixed; this area is therefore $n\pi l^2\mathbf{v}$, where n is the number of complete turns of the rod, $\mathbf{v}$ a unit vector perpendicular to the plane. Also

$$(\mathbf{P}-\mathbf{P}')\wedge d\mathbf{P}'=\mathbf{v}l\,dx,$$

where dx is the travel of the end P′ of the rod in a direction normal to the rod.

Hence

$$\int(\mathbf{P}-\mathbf{P}')\wedge d\mathbf{P}'=\mathbf{v}lx,$$

where x is the integrated travel of P′ normal to the rod. It follows that

$$\Sigma = \mathbf{A} - \mathbf{A}' = \mathbf{v}(lx + n\pi l^2).$$

Now let P′ be connected by a link of fixed length r to an origin O (Fig. 13). Then in any motion, P′ describes an arc of a circle centre O, radius r. If, whilst P turns n times round P′ in the positive direction, P′ describes the circle round O m times (m may be zero), then

$$|\mathbf{A}| = lx + n\pi l^2 + m\pi r^2.$$

Amsler's planimeter carries at P′ a drum, rolling on the plane, with its axis along P′P ; its total rotation thus measures x. Hence the area $|\mathbf{A}|$ of any closed curve may be determined by choosing any fixed point O and passing P round the curve.

A verification of the formula is afforded by taking for the curve described by P a circle round O of radius r+l. Then $x = 2\pi r$, $m = 1$, $n = 1$, and $|\mathbf{A}| = \pi(2lr + l^2 + r^2) = \pi(r+l)^2$, which is correct.

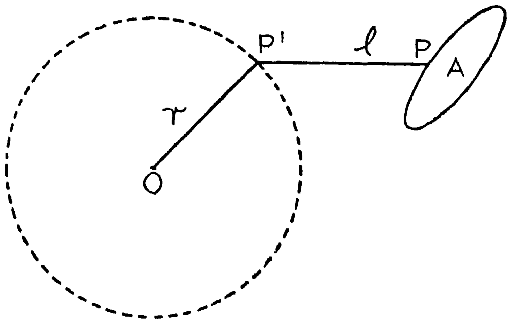

Fig. 13

Another well-known planimeter consists of a rod provided with a vertical pin at P and a sharp knife-edge at P′ with edge parallel to P′P. Then as P travels round a curve, the motion of P′ is always parallel to P′P, so that $(\mathbf{P} - \mathbf{P}') \wedge d\mathbf{P}' = 0$. Hence by (1)

$$d\Sigma = \tfrac{1}{2}(\mathbf{P} - \mathbf{P}') \wedge d(\mathbf{P} - \mathbf{P}')$$

and hence, using (2), if P describes a closed curve,

$$\tfrac{1}{2}\int \mathbf{P} \wedge d\mathbf{P} = \tfrac{1}{2}\int \mathbf{P}' \wedge d\mathbf{P}' - \tfrac{1}{2}[\mathbf{P} \wedge \mathbf{P}'] + \tfrac{1}{2}\int (\mathbf{P} - \mathbf{P}') \wedge d(\mathbf{P} - \mathbf{P}'),$$

where $[\mathbf{P} \wedge \mathbf{P}']$ denotes the difference between the final and initial values. Choose for origin O the initial or final position of P. Then the initial and final values of P are zero, and

$$\mathrm{A} = \tfrac{1}{2}\int \mathbf{P} \wedge d\mathbf{P} = \tfrac{1}{2}\int \mathbf{P}' \wedge d\mathbf{P}' + \tfrac{1}{2}\int (\mathbf{P} - \mathbf{P}') \wedge d(\mathbf{P} - \mathbf{P}').$$

The last integral is numerically equal to the triangular area subtended at the origin O by the initial and final positions of P′, i.e. the area defined by the initial and final positions of the rod. The area $\tfrac{1}{2}\int \mathbf{P}' \wedge d\mathbf{P}'$ of the locus of $\mathbf{P}'$ is in general small, in practice. Hence we get an approximate evaluation of A.

CHAPTER III

ELEMENTARY TENSOR ANALYSIS

38. *Preliminary considerations.* Take an orthogonal positive triad of unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Then we have seen that any vector $\mathbf{P}$ may be put in the form

$$\mathbf{P}=\mathrm{p}_1\mathbf{i}+\mathrm{p}_2\mathbf{j}+\mathrm{p}_3\mathbf{k}. \qquad (\mathrm{p}_1=\mathbf{P}.\mathbf{i}, \text{ etc.})$$

We can write this more concisely as

$$\mathbf{P}=\sum_\alpha \mathrm{p}_\alpha \mathbf{i}_\alpha, \qquad (\mathrm{p}_\alpha=\mathbf{P}.\mathbf{i}_\alpha)$$

where α takes the values 1, 2, 3 and $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ stand for $\mathbf{i}, \mathbf{j}, \mathbf{k}$. More concisely still, following Einstein, we can write this in the form

$$\mathbf{P}=\mathrm{p}_\alpha \mathbf{i}_\alpha,$$

where the occurrence of the suffix α *twice* is held to imply summation over the values $\alpha=1, 2, 3$.

We shall adopt this so-called 'summation convention' throughout unless the contrary is expressly stated. A suffix when repeated is called a 'dummy' suffix; it may be replaced by any other suffix-symbol not already present.

Consider now any other set $\mathbf{i}'_1$, $\mathbf{i}'_2$, $\mathbf{i}'_3$ of unit vectors forming a positive orthogonal triad. They will be described collectively as the set $\mathbf{i}'_\alpha$. Applying the above formula for any vector $\mathbf{P}$ to the case in which $\mathbf{P}$ is simply $\mathbf{i}'_1$, we have

$$\mathbf{i}'_1=\mathrm{l}_{11}\mathbf{i}_1+\mathrm{l}_{12}\mathbf{i}_2+\mathrm{l}_{13}\mathbf{i}_3=\mathrm{l}_{1\alpha}\mathbf{i}_\alpha,$$

say, where

$$\mathrm{l}_{11}=\mathbf{i}'_1.\mathbf{i}_1, \qquad \mathrm{l}_{12}=\mathbf{i}'_1.\mathbf{i}_2, \qquad \mathrm{l}_{13}=\mathbf{i}'_1.\mathbf{i}_3.$$

Applying the same procedure to the vectors $\mathbf{i}'_2$ and $\mathbf{i}'_3$, we have generally

$$\mathbf{i}'_\alpha=\mathrm{l}_{\alpha\mu}\mathbf{i}_\mu$$

where $$\mathrm{l}_{\alpha\mu}=\mathbf{i}'_\alpha.\mathbf{i}_\mu.$$

The l's are simply the 'direction-cosines' of the set $\mathbf{i}'_\alpha$ with respect to the set $\mathbf{i}_\alpha$, and we have the scheme:

	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{i}'_1$	l_{11}	l_{12}	l_{13}
$\mathbf{i}'_2$	l_{21}	l_{22}	l_{33}
$\mathbf{i}'_3$	l_{31}	l_{32}	l_{33}

Now let the same vector **P**, introduced above, be expanded as a linear function of the $\mathbf{i}'_\alpha$'s, in the form

$$\mathbf{P}=p'_\alpha \mathbf{i}'_\alpha.$$

Expressing the $\mathbf{i}'_\alpha$'s in terms of the $\mathbf{i}_\alpha$'s and comparing with the former expansion of **P** we have

$$p_\alpha \mathbf{i}_\alpha = p'_\alpha l_{\alpha\mu} \mathbf{i}_\mu.$$

This is a linear relation between the three linearly independent vectors $\mathbf{i}_\alpha$. Hence its coefficients must be separately zero. Hence rewriting the relation in the form

$$(p_\alpha - p'_\nu l_{\nu\alpha})\mathbf{i}_\alpha = 0.$$

(where we have used fresh symbols for dummy suffixes), we have

$$p_\alpha = l^{\nu\alpha} p'_\nu.$$

Now $$\mathbf{P}^2 = p_\alpha p_\alpha = p'_\alpha p'_\alpha.$$

Hence $$(l_{\nu\alpha} p'_\nu)(l_{\mu\alpha} p'_\mu) = p'_\mu p'_\mu.$$

This must be an identity in the p′'s. Hence we must have

$$\begin{aligned} l_{\nu\alpha} l_{\mu\alpha} &= 1 \text{ if } \nu=\mu \\ &= 0 \text{ if } \nu \neq \mu. \end{aligned}$$

We write this relation as

$$l_{\nu\alpha} l_{\mu\alpha} = \delta_{\mu\nu},$$

where $\delta_{\mu\nu}$ is the 'Kronecker symbol,' equal to unity if the two suffixes are equal and equal to zero if they are unequal. As a coefficient $\delta_{\mu\nu}$ is called the 'substitution operator,' for the effect of multiplying by $\delta_{\mu\nu}$ an expression containing μ, and thereby summing with respect to μ, is to substitute ν for μ in the expression. Similarly for the rôles of μ and ν interchanged.

We now suppose the equations giving the p_α's as linear functions of the p'_α's, namely $p_\alpha = l_{\nu\alpha} p'_\nu$, solved for the p'_α's. Let the solution be

$$p'_\alpha = L_{\nu\alpha} p_\nu.$$

This solution is always possible, for we could equally have started with the triad $\mathbf{i}'_\alpha$. It follows that

$$p'_\alpha = L_{\nu\alpha} l_{\mu\nu} p'_\mu.$$

This must be an identity in the p′'s. Hence

$$L_{\nu\alpha} l_{\mu\nu} = \delta_{\alpha\mu}.$$

Likewise we have $$p_\alpha = l_{\nu\alpha} L_{\nu\mu} p_\mu,$$

so that $$L_{\mu\nu} l_{\nu\alpha} = \delta_{\alpha\mu}.$$

Also it follows as for the l's that

$$L_{\nu\alpha} L_{\mu\alpha} = \delta_{\mu\nu}.$$

Now, take the last relation but one, and multiply by $l_{\beta\alpha}$, carrying out the implied summation. Then since $l_{\nu\alpha}l_{\beta\alpha}=\delta_{\nu\beta}$, we get

$$L_{\mu\nu}\delta_{\nu\beta}=\delta_{\alpha\mu}l_{\beta\alpha},$$

or, using the substitution property of the δ-symbol,

$$L_{\mu\beta}=l_{\beta\mu}.$$

Thus we have shown analytically that the scheme of the L's is derived from the scheme of the l's by interchanging rows and columns.

It now follows that

$$L_{\alpha\nu}L_{\alpha\mu}=\delta_{\mu\nu},$$
$$l_{\alpha\nu}l_{\alpha\mu}=\delta_{\mu\nu}.$$

All these relations are obvious geometrically, but we have thought it worth while to derive them analytically to illustrate the power of the repeated suffix convention.

Determinants. It follows from $L_{\mu\beta}=l_{\beta\mu}$ that the product of the determinant of the l's by the determinant of the L's is equal to the square of the determinant of the l's. But by the rule for multiplication of determinants, this product is

$$\begin{vmatrix} l_{1\mu}L_{\mu 1}, & l_{1\mu}L_{\mu 2}, & l_{1\mu}L_{\mu 3} \\ l_{2\mu}L_{\mu 1} & l_{2\mu}L_{\mu 2} & l_{2\mu}L_{\mu 3} \\ l_{3\mu}L_{\mu 1} & l_{3\mu}L_{\mu 2} & l_{3\mu}L_{\mu 3} \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{vmatrix}$$

which is just unity. Hence the determinant of the l's is ± 1. When the determinant is $+1$, we can make small alterations, cumulatively, in the values of the l's, thus displacing the $\mathbf{i}'_\alpha$ triad, until the scheme reduces to

$$\begin{pmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{pmatrix}.$$

In that case we have been able to superpose the $\mathbf{i}'_\alpha$ triad on the $\mathbf{i}_\alpha$ triad, and the two triads have the same sign in the sense of Chapter II. If the determinant is -1, the two triads are not superposable, and they have then opposite signs. Since the square of the determinant of the l's is essentially positive, the determinants of the l's and L's have the same signs.

We summarize the main results of this section in the form of the two transformation formulæ we have arrived at, namely

$$p_\alpha=l_{\nu\alpha}p'_\nu \tag{1}$$

and

$$p'_\alpha=L_{\nu\alpha}p_\nu=l_{\alpha\nu}p_\nu, \tag{2}$$

where the p's and p′'s are components of a vector $\mathbf{P}$ with regard to the triads $\mathbf{i}_\alpha$ and $\mathbf{i}'_\alpha$, namely

$$\mathbf{P}=p_\alpha\mathbf{i}_\alpha=p'_\alpha\mathbf{i}'_\alpha. \tag{3}$$

Here

$$\mathbf{i}'_\alpha\cdot\mathbf{i}_\mu=l_{\alpha\mu}=L_{\mu\alpha}. \tag{4}$$

39. The vector **P** can now be considered as equivalent to the class of all triplets of numbers p'_α correlated with a given triplet of numbers p_α by relations of the type (2), each triplet p'_α being associated with a given triad $\mathbf{i}'_\alpha$. The vector **P** may be considered as that which is common to all the different triplets of numbers p_α. We can scarcely *define* a vector **P** by this process, since (3) introduces the notion of the primitive unit vectors $\mathbf{i}_\alpha$ and $\mathbf{i}'_\alpha$. We have preferred to introduce the notion of a vector geometrically, by means of its representations ; but as far as the expression of a vector by means of its components is concerned, we could define a vector **P** as the class of all triplets p'_α associated with triads $\mathbf{i}'_\alpha$ without prior introduction of the notion of unit vectors. But we prefer to regard (3) as the formal expression of that which is common to p_α and the class of derived triplets p'_α.

40. We now seek to generalize this notion. Let us consider what meaning could formally be attached to a 'product' of vectors **PQ**, from a purely analytic point of view. Let us see what formal properties this 'product' would have if it had to obey the rules of algebra save the commutative property of multiplication. If **P** be expressed as a sum of vectors

$$\mathbf{P}=\mathbf{A}+\mathbf{B}+\mathbf{C}+\ldots,$$

Q as a sum

$$\mathbf{Q}=\mathbf{A}'+\mathbf{B}'+\mathbf{C}'+\ldots,$$

then we should wish the product **PQ** to be developable in the form

$$\begin{aligned}\mathbf{PQ}=\mathbf{AA}'+\mathbf{AB}'+\mathbf{AC}'+\ldots\\ +\mathbf{BA}'+\mathbf{BB}'+\mathbf{BC}'+\ldots\\ +\mathbf{CA}'+\mathbf{CB}'+\mathbf{CC}'+\ldots.\end{aligned}$$

Thus any meaning we attach to a 'product' of two vectors must be attached also to a sum of 'products.' Now any vector may be expressed as a linear function of the three vectors $\mathbf{i}_\alpha$ associated with a given triad. The result of forming sums of 'products' of such linear functions will be a linear function of the nine 'products' of unit vectors

$$\begin{matrix}\mathbf{i}_1\mathbf{i}_1, & \mathbf{i}_1\mathbf{i}_2, & \mathbf{i}_1\mathbf{i}_3,\\ \mathbf{i}_2\mathbf{i}_1, & \mathbf{i}_2\mathbf{i}_2, & \mathbf{i}_2\mathbf{i}_3,\\ \mathbf{i}_3\mathbf{i}_1, & \mathbf{i}_3\mathbf{i}_2, & \mathbf{i}_3\mathbf{i}_3.\end{matrix}$$

Accordingly the most general form of meaning we can attach to products should be capable of being attached also to a formal expression of the type

$$t_{\alpha\beta}\mathbf{i}_\alpha\mathbf{i}_\beta,$$

where repetition of the suffixes α and β implies summation over the values $\alpha=1, 2, 3$; $\beta=1, 2, 3$. Denote by **T** a formal expression of this kind. Let us see how the description of **T** alters when we change the

triad of reference from $\mathbf{i}_\alpha$ to $\mathbf{i}'_\alpha$. Expanding $\mathbf{i}_\alpha$ as a linear function of the vectors $\mathbf{i}'_\alpha$ we have as usual, by (4),

$$\mathbf{i}_\alpha = (\mathbf{i}_\alpha . \mathbf{i}'_\mu)\mathbf{i}'_\mu = l_{\mu\alpha}\mathbf{i}'_\mu$$

so that
$$\mathbf{T} = t_{\alpha\beta}\mathbf{i}_\alpha\mathbf{i}_\beta = t_{\alpha\beta}l_{\mu\alpha}l_{\nu\beta}\mathbf{i}'_\mu\mathbf{i}'_\nu = t'_{\mu\nu}\mathbf{i}'_\mu\mathbf{i}'_\nu,$$

say, where
$$t'_{\mu\nu} = l_{\mu\alpha}l_{\nu\beta}t_{\alpha\beta}.$$

Comparing this with the formula (2) of § 38 for the transformation of the components of a vector, a formula which we rewrite with a change of suffix in the form

$$p'_\mu = l_{\mu\alpha}p_\alpha$$

we see that the transformation formula for deriving the nine numbers $t'_{\mu\nu}$ from the nine numbers $t_{\alpha\beta}$ is a simple generalization of the formula for deriving the three numbers p'_α from the three numbers p_α, products of l's replacing the simple l's themselves. The set of nine numbers $t'_{\mu\nu}$ describes $\mathbf{T}$ just as adequately with respect to the triad $\mathbf{i}'_\alpha$ as does the set of nine numbers $t_{\mu\nu}$ with respect to the triad $\mathbf{i}_\alpha$. This leads us to propose the following formal definition of something we shall call a *tensor.*

41. *Definition of a tensor.* With each of a set of orthogonal unit triads of vectors associate a set of nine numbers $t_{\mu\nu}$. Then the sets are said to describe a tensor $\mathbf{T}$ provided that if $t_{\mu\nu}$ is the set associated with the triad $\mathbf{i}_\alpha$, $t'_{\mu\nu}$ the set associated with $\mathbf{i}'_\alpha$, then

$$t'_{\mu\nu} = l_{\mu\alpha}l_{\nu\beta}t_{\alpha\beta},$$

where
$$l_{\rho\sigma} = \mathbf{i}'_\rho . \mathbf{i}_\sigma.$$

We express this fact by writing

$$\mathbf{T} = t_{\mu\nu}\mathbf{i}_\mu\mathbf{i}_\nu = t'_{\mu\nu}\mathbf{i}'_\mu\mathbf{i}'_\nu.$$

It will be seen that this mode of symbolism $\mathbf{T}$, without a suffix, suggests the notion of an entity which remains permanent, behind the changing façades of the triads of reference. We cannot describe a tensor (according to our definition) without reference to *some* triad ; but the tensor itself is to be distinguished from any one of its descriptions. Just as a vector $\mathbf{P}$ is the class of all its representations, so a tensor $\mathbf{T}$ is the class of all its descriptions.

42. *Transitive property.* For consistency we must show that if the transformation relation exists between the sets associated with the triads $\mathbf{i}_\alpha$ and $\mathbf{i}'_\alpha$, and between the sets associated with the triads $\mathbf{i}'_\alpha$ and $\mathbf{i}''_\alpha$, then it also exists between the sets $\mathbf{i}_\alpha$ and $\mathbf{i}''_\alpha$. Let the l's be specified by

$$\mathbf{i}'_\alpha = l^{(1)}{}_{\alpha\mu}\mathbf{i}_\mu,$$

$$\mathbf{i}''_\beta = l^{(2)}{}_{\beta\nu}\mathbf{i}'_\nu.$$

Then
$$\mathbf{i}''_\beta = l^{(2)}{}_{\beta\alpha}l^{(1)}{}_{\alpha\mu}\mathbf{i}_\mu$$

$$= \lambda_{\beta\mu}\mathbf{i}_\mu,$$

say. We are given that

$$t'_{\alpha\beta} = l^{(1)}_{\alpha\mu} l^{(1)}_{\beta\nu} t_{\mu\nu},$$

$$t''_{\rho\sigma} = l^{(2)}_{\rho\gamma} l^{(2)}_{\sigma\delta} t'_{\gamma\delta}.$$

Hence

$$t''_{\rho\sigma} = l^{(2)}_{\rho\alpha} l^{(2)}_{\sigma\beta} l^{(1)}_{\alpha\mu} l^{(1)}_{\beta\nu} t_{\mu\nu}$$
$$= \lambda_{\rho\mu}\lambda_{\sigma\nu} t_{\mu\nu}.$$

This establishes the transitive property.

43. The nine numbers $t_{\alpha\beta}$ are called the *components* of the tensor **T** with respect to the triad $\mathbf{i}_\alpha$. The definition of a tensor provides an immediate test of whether any proposed aggregate of sets of nine numbers associated with triads constitutes a tensor or not. It is particularly to be noted that no meaning attaches to asking whether a *particular* set of nine numbers constitutes a tensor or not. It is only when we are given a rule for obtaining the corresponding set in any other triad that we can compare the result of applying the rule with the definition of a tensor and so can answer the question.

44. More precisely, the definition we have given defines a *tensor of rank* 2. A vector may be considered a tensor of rank 1, a scalar as a tensor of rank zero. Tensors of ranks 3, 4, ..., are defined by using products of 3, 4, ..., l-factors in the definition.

45. *The product of two vectors. Dyads.*

Theorem : If $\mathbf{P}(=p_\alpha \mathbf{i}_\alpha)$, $\mathbf{Q}(=q_\alpha \mathbf{i}_\alpha)$ are two vectors, then the nine products $p_\alpha q_\beta$ are the components of a tensor. For, if $t_{\alpha\beta} = p_\alpha q_\beta$, then

$$t'_{\alpha\beta} = p'_\alpha q'_\beta = (l_{\alpha\mu} p_\mu)(l_{\beta\nu} q_\nu)$$
$$= l_{\alpha\mu} l_{\beta\nu} t_{\mu\nu}.$$

We call this tensor **PQ**, A tensor which is the product of two vectors is called a *dyad* (Gibbs). Particular cases of dyads are the nine dyads **ii, ij, ik, ji, jj, jk, ki, kj, kk.** Each of the symbols **ii, ij,** ... denotes a tensor with, of course, nine components with respect to the triad **i, j, k,** as well as nine components with respect to any other triad. For example, the components of the dyads **ii, ij** with respect to the triad **i, j, k** are given by the schemes

$$\mathbf{ii} = \begin{array}{c|ccc} & \beta \rightarrow & & \\ \hline \alpha & 1 & 0 & 0 \\ \downarrow & 0 & 0 & 0 \\ & 0 & 0 & 0 \end{array} \qquad \mathbf{ij} = \begin{array}{c|ccc} & \beta \rightarrow & & \\ \hline \alpha & 0 & 1 & 0 \\ \downarrow & 0 & 0 & 0 \\ & 0 & 0 & 0 \end{array}$$

The components $(\mathbf{PQ})_{\alpha\beta}$ of the dyad **PQ** with respect to the triad **i, j, k,** if (P_1, P_2, P_3), (Q_1, Q_2, Q_3) are the components of **P, Q** with respect to the same triad, are given by

$$\mathbf{PQ} = \begin{array}{c|ccc} & \beta \rightarrow & & \\ \hline \alpha & P_1Q_1 & P_1Q_2 & P_1Q_3 \\ \downarrow & P_2Q_1 & P_2Q_2 & P_2Q_3 \\ & P_3Q_1 & P_3Q_2 & P_3Q_3 \end{array}$$

It will be noticed that in this illustration we have used the same *letter* P to denote the vector **P** and the numbers P_α. This is a convenient practice. It will be noticed that the products **PQ** and **QP** denote different tensors. A tensor equation $\mathbf{T}=\mathbf{S}$ implies nine scalar equations $T_{\alpha\beta}=S_{\alpha\beta}$.

46. *The sum of two tensors.* If $T_{\alpha\beta}$, $S_{\alpha\beta}$ are the components of two tensors **T, S** with respect to the same triad $\mathbf{i}_\alpha$, then the nine sums $T_{\alpha\beta}+S_{\alpha\beta}$ are the components of another tensor with respect to $\mathbf{i}_\alpha$, and this tensor is denoted by $\mathbf{T}+\mathbf{S}$.

It is clear that any tensor **T** may be considered as the sum of nine dyads. Written out at length, the tensor **T** is given by the expression

$$\begin{aligned}\mathbf{T}=T_{11}\mathbf{ii}+T_{12}\mathbf{ij}+T_{13}\mathbf{ik}\\ +T_{21}\mathbf{ji}+T_{22}\mathbf{jj}+T_{23}\mathbf{jk}\\ +T_{31}\mathbf{ki}+T_{32}\mathbf{kj}+T_{33}\mathbf{kk}.\end{aligned}$$

47. *Conjugate tensors.* Let **T** be a tensor with components $T_{\alpha\beta}$ with respect to a triad $\mathbf{i}_\alpha$. Then if we write, with respect to any triad

$$S_{\alpha\beta}=T_{\beta\alpha},$$

we have in any other triad

$$\begin{aligned}S'_{\alpha\beta}&=T'_{\beta\alpha}\\ &=l_{\beta\mu}l_{\alpha\nu}T_{\mu\nu}\\ &=l_{\alpha\nu}l_{\beta\mu}S_{\nu\mu},\end{aligned}$$

so that the sets $S_{\alpha\beta}$ form a tensor. This is defined to be the tensor *conjugate* to **T**, and is written conj **T** or $\overline{\mathbf{T}}$. Thus $\overline{T}_{\alpha\beta}=T_{\beta\alpha}$. If **P, Q** are vectors, the conjugate of the dyad **PQ** is the dyad **QP**.

If $T_{\alpha\beta}=T_{\beta\alpha}$ for all α, β, **T** is said to be *self-conjugate*. Clearly if a tensor is self-conjugate in any one triad it is self-conjugate in all.

48. *The scalar of a tensor.* We now construct an invariant out of the components of a tensor.

Theorem: If **T** is any given tensor, $T_{\alpha\alpha}$ (or $T_{11}+T_{22}+T_{33}$) has the same value in all triads of reference. For, by the definition of a tensor,

$$T'_{\alpha\alpha}=l_{\alpha\mu}l_{\alpha\nu}T_{\mu\nu}=\delta_{\mu\nu}T_{\mu\nu}=T_{\mu\mu}.$$

i.e. $$T'_{11}+T'_{22}+T'_{33}=T_{11}+T_{22}+T_{33}.$$

This invariant $T_{\alpha\alpha}$ is called the scalar of T and is written sca **T.**

If **T** is a dyad **PQ,** sca **T** is the scalar product **P.Q.** If **T** is the dyad **PP,** sca **T** is the scalar $\mathbf{P}^2$.

49. *Contraction.* The foregoing is a particular case of the process called *contraction.* Suppose that we have a tensor of any rank, of components say $T_{\alpha\beta\gamma\delta\ldots}$. Then the expression $T_{\alpha\alpha\gamma\delta}$ denotes the sums $\sum_\alpha T_{\alpha\alpha\gamma\delta}$, summed for $\alpha=1, 2, 3$ for any fixed γ, δ, ... These sums may be written

$S_{\gamma\delta\ldots}$, and they can be shown to enumerate the components of a tensor **S**, of rank *two less than* that of the original tensor. For

$$\begin{aligned} S'_{\gamma\delta\ldots} &= T'_{\alpha\alpha\gamma\delta\ldots} = l_{\alpha\mu} l_{\alpha\nu} l_{\gamma\rho} l_{\delta\sigma} \ldots T_{\mu\nu\rho\sigma\ldots} \\ &= \delta_{\mu\nu} l_{\gamma\rho} l_{\delta\sigma} \ldots T_{\mu\nu\rho\sigma\ldots} = l_{\gamma\rho} l_{\delta\sigma} \ldots T_{\mu\mu\rho\sigma\ldots} \\ &= l_{\gamma\rho} l_{\delta\sigma} \ldots S_{\rho\sigma\ldots}, \end{aligned}$$

so that **S** obeys the tensor law for rank two lower.

50. *Product of two tensors.* If $T_{\alpha\beta\ldots}$, $S_{\alpha\beta\ldots}$ are the components of tensors of any rank, then the products

$$T_{\alpha\beta\ldots}\ S_{\rho\sigma\ldots}$$

are easily seen to be the components of a tensor of rank equal to the sum of the ranks of the given tensors. The new tensor is called the product **TS** of the given tensors in this order. In particular, we may have the products **PT** and **TP** of a vector and a tensor of rank 2, which are tensors of rank 3.

51. *Inner products of a tensor of rank 2 and a vector.* Let **T** be a tensor,* **P** a vector. Let their components in a given triad be denoted by $T_{\alpha\beta}$, P_γ. Consider the sums

$$T_{\alpha\beta} P_\beta .$$

This is a contraction of the tensor $T_{\alpha\beta} P_\gamma$ of rank 3 ; it accordingly describes a vector (of rank 1). Denoting the components of this vector by Q_α, we have, for $\alpha = 1, 2, 3$,

$$Q_\alpha = T_{\alpha\beta} P_\beta ,$$

which we write in the form $\quad \mathbf{Q} = \mathbf{T.P.}$

This use of the dot (.) is consistent with the usage in the scalar product ; it represents the presence, in the suffix notation, of two adjacent, equal and therefore dummy suffixes. In practice it is simple to pass from the dot notation to the suffix notation.

Similarly the sums $P_\alpha T_{\alpha\beta}$ represent the components R_β of a vector **R**, which we write

$$\mathbf{R} = \mathbf{P.T.}$$

We call **T.P** and **P.T** *inner products*. In general **T.P** and **P.T** are unequal. If, however, **T** is self-conjugate, we have **T.P** = **P.T**. For now $(\mathbf{T.P})_\alpha = T_{\alpha\mu} P_\mu = P_\mu T_{\mu\alpha} = (\mathbf{P.T})_\alpha$.

Written out in full, the components of **T.P** are

$$T_{11}P_1 + T_{12}P_2 + T_{13}P_3, \quad T_{21}P_1 + T_{22}P_2 + T_{23}P_3, \quad T_{31}P_1 + T_{32}P_2 + T_{33}P_3,$$

those of **P.T** are

$$P_1T_{11} + P_2T_{21} + P_3T_{31}, \quad P_1T_{12} + P_2T_{22} + P_3T_{32}, \quad P_1T_{13} + P_2T_{23} + P_3T_{33}.$$

* The rank of a tensor will in future be taken to be 2 unless otherwise stated.

Similarly, the notation **T.S** denotes the tensor which is the inner product of the tensors **T, S** in this order, namely the tensor of components $T_{\alpha\mu}S_{\mu\beta}$.

52. *Inner product of a dyad and a vector.*

Theorem :
$$(\mathbf{PQ}).\mathbf{R}=\mathbf{P}(\mathbf{Q}.\mathbf{R}),$$
$$\mathbf{P}.(\mathbf{QR})=(\mathbf{P}.\mathbf{Q})\mathbf{R}.$$

For the α-component of **(PQ).R** in any triad is $P_\alpha Q_\beta R_\beta = P_\alpha(\mathbf{Q}.\mathbf{R})$, which is the α-component of **P(Q.R)**. We note that the inner product of a dyad and a vector is itself a vector. Without ambiguity we can now omit the brackets and write **PQ.R** to denote either **P(Q.R)** or **(PQ).R.**

This is a very useful theorem, of frequent application. It affords a common method of generating tensors from vectors. For example, if we have an aggregate of pairs of vectors $\mathbf{P}_s$, $\mathbf{Q}_s$ and another vector **R,** then *

$$\sum_s \mathbf{P}_s(\mathbf{Q}_s.\mathbf{R})=\sum_s[(\mathbf{P}_s\mathbf{Q}_s).\mathbf{R}]=(\sum_s \mathbf{P}_s\mathbf{Q}_s).\mathbf{R}=\mathbf{T}.\mathbf{R}$$

where
$$\mathbf{T}=\sum_s \mathbf{P}_s\mathbf{Q}_s.$$

This is the type of context in which tensors often make their first appearance.

53. *Double inner product.* The process of contraction may be repeated. If **T, S** are two tensors, we may form first their inner product **T.S,** and then contracting again form the scalar of this, whose value is $T_{\alpha\mu}S_{\mu\alpha}$. We write this, in extension of our former convention, **T:S.** Similarly we can form the invariant $\mathbf{T}:\bar{\mathbf{S}}$. In particular, we have the invariants

$$\mathbf{T}:\mathbf{T}=\Sigma T_{11}{}^2+2\,\Sigma T_{23}T_{32},$$
$$\mathbf{T}:\bar{\mathbf{T}}=\Sigma T_{11}{}^2+\Sigma(T_{23}{}^2+T_{32}{}^2).$$

From these we can form the invariant

$$(\mathbf{T}:\bar{\mathbf{T}})-(\mathbf{T}:\mathbf{T})=\Sigma(T_{23}-T_{32})^2.$$

54. *Anti-symmetrical tensors.* A tensor **T** is said to be anti-symmetrical if its components in any triad satisfy the relation

$$T_{\beta\alpha}=-T_{\alpha\beta}.$$

If **T** is anti-symmetrical in any one triad, it is so in any other. For if $T_{\alpha\beta}=-T_{\beta\alpha}$, then

$$T'_{\alpha\beta}=l_{\alpha\mu}l_{\beta\nu}T_{\mu\nu}=-l_{\beta\nu}l_{\alpha\mu}T_{\nu\mu}=-T'_{\beta\alpha}.$$

If a tensor is anti-symmetrical, a component with two equal suffixes is zero ; for $T_{11}=-T_{11}$, so that $T_{11}=0$, and similarly $T_{22}=0$, $T_{33}=0$. Accordingly the general form of the components of an anti-symmetrical tensor is in any triad

$$\begin{array}{c|ccc} & \beta\rightarrow & & \\ \hline \alpha & 0 & h & -g \\ \downarrow & -h & 0 & f \\ & g & -f & 0 \end{array}$$

* s is not the component suffix here.

The non-zero components f, g, h, *with these signs*, form a vector. This will be proved later directly, but it is instructive to establish it by actual transformation. We put

$$f = T_{23}, \qquad g = T_{31}, \qquad h = T_{12}$$

and then
$$f' = T'_{23} = l_{2\mu} l_{3\nu} T_{\mu\nu}.$$

Using the fact that **T** is anti-symmetrical, we find that this reduces to

$$f' = (l_{22}l_{33} - l_{23}l_{32})T_{23} + (l_{23}l_{31} - l_{21}l_{33})T_{31} + (l_{21}l_{32} - l_{22}l_{31})T_{12}.$$

The coefficients of the T's are the components of the vector products of the vectors whose components in $\mathbf{i}_\alpha$ are

$$\begin{matrix} l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33}. \end{matrix}$$

The latter vectors are

$$(\mathbf{i}'_2.\mathbf{i}_1)\mathbf{i}_1 + (\mathbf{i}'_2.\mathbf{i}_2)\mathbf{i}_2 + (\mathbf{i}'_2.\mathbf{i}_3)\mathbf{i}_3,$$
$$(\mathbf{i}'_3.\mathbf{i}_1)\mathbf{i}_1 + (\mathbf{i}'_3.\mathbf{i}_2)\mathbf{i}_2 + (\mathbf{i}'_3.\mathbf{i}_3)\mathbf{i}_3,$$

i.e. $\mathbf{i}'_2$ and $\mathbf{i}'_3$. Their vector product $\mathbf{i}_2' \wedge \mathbf{i}_3'$ is just $\mathbf{i}_1'$, whose components in $\mathbf{i}_\alpha$ are (l_{11}, l_{12}, l_{13}). Thus

$$f' = l_{11}f + l_{12}g + l_{13}h.$$

Similar relations hold for g′ and h′, and the three together show that (f, g, h) is a vector.

If **T** is any tensor, the tensor whose components are $T_{\alpha\beta} + T_{\beta\alpha}$ is a symmetrical or self-conjugate tensor ; the tensor whose components are $T_{\alpha\beta} - T_{\beta\alpha}$ is an anti-symmetrical tensor. We have

$$T_{\alpha\beta} = \tfrac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha}) + \tfrac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha})$$

which we write
$$\mathbf{T} = \overset{=}{\mathbf{T}} + \overset{\times}{\mathbf{T}}.$$

Thus any tensor may be expressed as the sum of a self-conjugate tensor and an anti-symmetrical tensor. To avoid trouble in printing, it is sometimes convenient to write

$$\overset{=}{\mathbf{T}} = \text{sym } \mathbf{T}, \qquad \overset{\times}{\mathbf{T}} = \text{antisym } \mathbf{T}.$$

55. *The idem tensor.* We have seen that any vector **P** may be expressed as a linear function of the members **i, j, k** of an orthogonal triad of unit vectors in the form

$$\mathbf{P} = (\mathbf{P}.\mathbf{i})\mathbf{i} + (\mathbf{P}.\mathbf{j})\mathbf{j} + (\mathbf{P}.\mathbf{k})\mathbf{k}.$$

Using the theorem of § 52, this may be written in the form

$$\mathbf{P} = \mathbf{P}.(\mathbf{ii} + \mathbf{jj} + \mathbf{kk}).$$

It may also be written in the form

$$\mathbf{P} = (\mathbf{ii} + \mathbf{jj} + \mathbf{kk}).\mathbf{P}.$$

It follows that the tensor **U** defined by

$$\mathbf{U} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk}$$

has the property of reproducing the vector it is multiplied into to form an inner product. For this reason it is called the *idem tensor* ; it plays the part of a unit multiplier.

The components of **U** with regard to the triad **i, j, k** are clearly

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1, \end{matrix}$$

and thus the components $U_{\alpha\beta}$ with regard to the triad **i, j, k** are given by

$$U_{\alpha\beta} = \delta_{\alpha\beta}.$$

It is readily shown that the tensor has the same components in any other orthogonal triad **i′, j′, k′**. For by the tensor transformation law

$$\begin{aligned} U'_{\alpha\beta} &= l_{\alpha\mu} l_{\beta\nu} U_{\mu\nu} \\ &= l_{\alpha\mu} l_{\beta\nu} \delta_{\mu\nu} \\ &= l_{\alpha\mu} l_{\beta\mu} \\ &= \delta_{\alpha\beta}. \end{aligned}$$

Hence $$\mathbf{U} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} = \mathbf{i'i'} + \mathbf{j'j'} + \mathbf{k'k'}.$$

It follows also that if **T** is a tensor of any rank,

$$\mathbf{U.T} = \mathbf{T}, \qquad \mathbf{T.U} = \mathbf{T}.$$

56. *The quotient theorem.* The following theorem affords a convenient test for the tensor character of any proposed expression.

Theorem: If $T_{\alpha\beta}$, $T'_{\alpha\beta}$, ... are sets of nine numbers associated with triads $\mathbf{i}_\alpha$, $\mathbf{i}'_\alpha$, ..., and if, whatever vector **Q** is chosen, the expressions $T_{\alpha\beta}Q_\beta$, $T'_{\alpha\beta}Q'$, ...define the components P_α, P'_α, ... in $\mathbf{i}_\alpha$, $\mathbf{i}'_\alpha$, ... of some vector **P**, then the sets $T_{\alpha\beta}$, $T'_{\alpha\beta}$, ... describe a tensor **T**.

For, by hypothesis, we have

$$T_{\alpha\beta} Q_\beta = P_\alpha, \qquad T'_{\alpha\beta} Q'_\beta = P'_\alpha,$$

where, since **P** is a vector, $\quad P'_\alpha = l_{\alpha\mu} P_\mu.$

Hence $$T'_{\alpha\beta} Q'_\beta = l_{\alpha\mu} T_{\mu\beta} Q_\beta.$$

But since **Q** is a vector, $\quad Q_\beta = l_{\nu\beta} Q'_\nu.$

Hence, changing a dummy suffix,

$$T'_{\alpha\nu} Q'_\nu = l_{\alpha\mu} T_{\mu\beta} l_{\nu\beta} Q'_\nu,$$

or $$Q'_\nu (T'_{\alpha\nu} - l_{\alpha\mu} l_{\nu\beta} T_{\mu\beta}) = 0.$$

By the data of the problem, for any fixed α this relation holds good for all sets of three numbers Q'_ν, $(\nu = 1, 2, 3)$. Hence for any fixed α and for $\nu = 1, 2, 3$, the coefficients must vanish and so

$$T'_{\alpha\nu} = l_{\alpha\mu} l_{\nu\beta} T_{\mu\beta}.$$

But α is arbitrary $(\alpha = 1, 2, 3)$. Hence by the definition of a tensor, the sets $T'_{\alpha\beta}$, $T_{\alpha\beta}$, ... describe a tensor.

Similar results hold for sets of numbers of any rank. In particular, if $P_\alpha, P'_\alpha, \ldots$ are sets of three numbers associated with triads $\mathbf{i}_\alpha, \mathbf{i}'_\alpha, \ldots$ such that for any vector $\mathbf{Q}$, $P_\alpha Q_\alpha$, $P'_\alpha Q'_\alpha, \ldots$ form a scalar (i.e. take the same value in each $\mathbf{i}_\alpha$), then the sets P_α, $P'_\alpha, \ldots$ describe the components of a vector $\mathbf{P}$.

57. *The alternate tensor.* Define a set of twenty-seven numbers $\varepsilon_{\alpha\beta\gamma}$ by the properties:

$\varepsilon_{\alpha\beta\gamma}=0$ if any two of α, β, γ are equal;
$\varepsilon_{\alpha\beta\gamma}=+1$ if α, β, γ are all different and the number of inversions of the natural order 1, 2, 3 in the sequence $\alpha\beta\gamma$ is *even*;
$\varepsilon_{\alpha\beta\gamma}=-1$ if $\alpha\beta\gamma$ are all different and the number of inversions of the natural order in the sequence $\alpha\beta\gamma$ is *odd*.

Now take two positive triads of orthogonal vectors $\mathbf{i}_\alpha$, $\mathbf{i}'_\alpha$, and put as usual $l_{\alpha\beta}=\mathbf{i}'_\alpha.\mathbf{i}_\beta$. Then the expression

$$l_{\alpha\mu}l_{\beta\nu}l_{\gamma\sigma}\varepsilon_{\mu\nu\sigma}$$

where summation is implied with respect to μ, ν, σ, (μ, ν, $\sigma=1, 2, 3$), denotes a number depending only on α, β, γ. This number is the value of the determinant

$$\begin{vmatrix} l_{\alpha 1} & l_{\beta 1} & l_{\gamma 1} \\ l_{\alpha 2} & l_{\beta 2} & l_{\gamma 2} \\ l_{\alpha 3} & l_{\beta 3} & l_{\gamma 3} \end{vmatrix}$$

by the definition of a determinant. This determinant vanishes if any two of α, β, γ are equal; it is equal to $+1$ if the parity of the sequence α, β, γ is even, and equal to -1 if the parity of the sequence α, β, γ is odd.

To establish the latter statement, we note that since

$$\mathbf{i}'_\alpha=l_{\alpha\mu}\mathbf{i}_\mu,$$

it follows that

$$\mathbf{i}'_\alpha\wedge\mathbf{i}'_\beta.\mathbf{i}'_\gamma=(l_{\alpha\mu}\mathbf{i}_\mu)\wedge(l_{\beta\nu}\mathbf{i}_\nu).(l_{\gamma\sigma}\mathbf{i}_\sigma)=\Delta_{\alpha\beta\gamma}(\mathbf{i}_1\wedge\mathbf{i}_2.\mathbf{i}_3),$$

where $\Delta_{\alpha\beta\gamma}$ is the value of the above-written determinant. Since $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ form a positive triad, the value of $\mathbf{i}_1\wedge\mathbf{i}_2.\mathbf{i}_3$ is $+1$. And $\mathbf{i}'_\alpha\wedge\mathbf{i}'_\beta.\mathbf{i}'_\gamma$ is $+1$ or -1 according as $\mathbf{i}'_\alpha$, $\mathbf{i}'_\beta$, $\mathbf{i}'_\gamma$ form a positive or negative triad, i.e. according as the sequence α, β, γ is cyclically equivalent to 1, 2, 3 or to 1, 3, 2.

It follows that $\qquad l_{\alpha\mu}l_{\beta\nu}l_{\gamma\sigma}\varepsilon_{\mu\nu\sigma}=\varepsilon_{\alpha\beta\gamma}.$

Now associate with every triad $\mathbf{i}_\alpha$ a set of numbers $A_{\alpha\beta\gamma}$ defined by

$$A_{\alpha\beta\gamma}=\varepsilon_{\alpha\beta\gamma}.$$

Then if $A'_{\alpha\beta\gamma}$ denote the set associated with $\mathbf{i}'_\alpha$, it follows from the above that $\qquad l_{\alpha\mu}l_{\beta\nu}l_{\gamma\sigma}A_{\mu\nu\sigma}=\varepsilon_{\alpha\beta\gamma}=A'_{\alpha\beta\gamma}.$

Hence the sets $A_{\alpha\beta\gamma}$, $A'_{\alpha\beta\gamma}, \ldots$ define a tensor of the third rank. This tensor is called the *alternate tensor*, and is denoted by $\mathbf{A}$.

58. *The vector product in terms of the alternate tensor.* It follows that if **P, Q** are vectors, the expressions

$$A_{\alpha\beta\gamma}P_\beta Q_\gamma$$

for $\alpha=1, 2, 3$ constitute the components in $\mathbf{i}_\alpha$ of a vector **R**. Written out in full, these components are

$$R_1=P_2Q_3-P_3Q_2, \qquad R_2=P_3Q_1-P_1Q_3, \qquad R_3=P_1Q_2-P_2Q_1.$$

But these are just the components in $\mathbf{i}_\alpha$ of the vector $\mathbf{P}\wedge\mathbf{Q}$. It follows that

$$(\mathbf{P}\wedge\mathbf{Q})_\alpha=A_{\alpha\beta\gamma}P_\beta Q_\gamma.$$

The importance of the alternate tensor arises from the circumstance that its rank is equal to the number of dimensions of the space concerned, namely three. Expressions involving it thus refer to properties of three-dimensional space. We see that the vector product exhibits a characteristic association with the number of dimensions of our familiar physical space.

59. *An important identity.* We first establish the following arithmetical identity :

Theorem : $$\varepsilon_{\alpha\beta\gamma}\varepsilon_{\alpha\mu\nu}=\delta_{\beta\mu}\delta_{\gamma\nu}-\delta_{\beta\nu}\delta_{\gamma\mu}.$$

Each side is a function of four independent numbers, β, γ, μ, ν, which can each take the values 1, 2, 3. If $\beta=\gamma$, the left-hand side is zero, and so is the right-hand side ; the equality is therefore established if $\beta=\gamma$. Similarly if $\mu=\nu$.

Now take any set of nine numbers $t_{\mu\nu}$, and consider the expression

$$s_{\beta\gamma}=\varepsilon_{\alpha\beta\gamma}\varepsilon_{\alpha\mu\nu}t_{\mu\nu}.$$

Suppose $\beta\neq\gamma$. Let $\varkappa'$ be the value of β, $\varkappa''$ the value of γ, and $\varkappa$ the remaining member of the set 1, 2, 3 which is neither $\varkappa'$ nor $\varkappa''$. In the sum $s_{\varkappa'\varkappa''}$, the only non-zero terms arise from $\alpha=\varkappa$; and for $\alpha=\varkappa$, the only non-zero values of $\varepsilon_{\alpha\mu\nu}$ occur for $\mu=\varkappa'$, $\nu=\varkappa''$ and $\mu=\varkappa''$, $\nu=\varkappa'$. Thus

$$s_{\varkappa'\varkappa''}=\varepsilon_{\varkappa\varkappa'\varkappa''}(\varepsilon_{\varkappa\varkappa'\varkappa''}t_{\varkappa'\varkappa''}+\varepsilon_{\varkappa\varkappa''\varkappa'}t_{\varkappa''\varkappa'}) \qquad \text{(not summed).}$$

But $$\varepsilon_{\varkappa\varkappa'\varkappa''}\varepsilon_{\varkappa\varkappa'\varkappa''}=\varepsilon_{\varkappa\varkappa'\varkappa''}^2=+1 \qquad \text{(not summed)}$$

and $$\varepsilon_{\varkappa\varkappa'\varkappa''}\varepsilon_{\varkappa\varkappa''\varkappa'}=-1 \qquad \text{(not summed).}$$

Hence $$\begin{aligned} s_{\varkappa'\varkappa''}&=t_{\varkappa'\varkappa''}-t_{\varkappa''\varkappa'} \\ &=\delta_{\varkappa'\mu}t_{\mu\varkappa''}-\delta_{\varkappa''\mu}t_{\mu\varkappa'} \\ &=(\delta_{\varkappa'\mu}\delta_{\varkappa''\nu}-\delta_{\varkappa''\mu}\delta_{\varkappa'\nu})t_{\mu\nu}. \end{aligned}$$

Hence when $\beta\neq\gamma$ we have

$$[\varepsilon_{\alpha\beta\gamma}\varepsilon_{\alpha\mu\nu}-(\delta_{\beta\mu}\delta_{\gamma\nu}-\delta_{\gamma\mu}\delta_{\beta\nu})]t_{\mu\nu}=0$$

for any set of nine arbitrary numbers $t_{\mu\nu}$. Hence the coefficients of $t_{\mu\nu}$ must separately vanish. This establishes the theorem.*

* The identity can also be verified directly, by taking all possible sets of values β, γ, μ, ν. But this procedure would give no insight into the origin of the identity. The proof in the text synthesizes as well as demonstrates the theorem.

60. *The* (**A, U**) *theorem.* The following theorem now follows from the above identity:

Theorem *: In any triad the components of the tensors **A, U,** are connected by the relations

$$A_{\alpha\beta\gamma}A_{\alpha\mu\nu}=U_{\beta\mu}U_{\gamma\nu}-U_{\beta\nu}U_{\gamma\mu}.$$

For the components of **A** and **U** take the same numerical values in all triads of reference.

It should be noted that the (ε, δ) relation is one between numbers; the (**A, U**) relation is one between components of tensors in all triads of reference.

Corollary. If **T** is any tensor,

$$A_{\alpha\beta\gamma}A_{\alpha\mu\nu}T_{\mu\nu}=T_{\beta\gamma}-T_{\gamma\beta}.$$

61. *Contraction of the foregoing identities.* By contraction of the (ε, δ) relation we have

$$\varepsilon_{\alpha\beta\mu}\varepsilon_{\alpha\beta\nu}=\delta_{\beta\beta}\delta_{\mu\nu}-\delta_{\beta\nu}\delta_{\mu\beta}.$$

But $$\delta_{\beta\beta}=1+1+1=3$$

and $$\delta_{\beta\nu}\delta_{\mu\beta}=\delta_{\mu\nu}.$$

Hence $$\varepsilon_{\alpha\beta\mu}\varepsilon_{\alpha\beta\nu}=2\delta_{\mu\nu}.$$

Similarly $$A_{\alpha\beta\mu}A_{\alpha\beta\nu}=2U_{\mu\nu}.$$

62. *Applications of the* (**A, U**) *theorem.* (1) *The continued vector product.* All operations involving the equivalent of two vector product operations are readily conducted by use of the (**A, U**) theorem. To establish the continued vector product formula, let **P, Q, R** be three vectors. Then in any triad

$$\begin{aligned}[(\mathbf{P}\wedge\mathbf{Q})\wedge\mathbf{R}]_\alpha &=A_{\alpha\beta\gamma}(\mathbf{P}\wedge\mathbf{Q})_\beta R_\gamma\\ &=A_{\alpha\beta\gamma}A_{\beta\mu\nu}P_\mu Q_\nu R_\gamma\\ &=A_{\beta\gamma\alpha}A_{\beta\mu\nu}P_\mu Q_\nu R_\gamma\\ &=(U_{\gamma\mu}U_{\alpha\nu}-U_{\gamma\nu}U_{\alpha\mu})P_\mu Q_\nu R_\gamma\\ &=P_\gamma Q_\alpha R_\gamma-P_\alpha Q_\gamma R_\gamma\\ &=[-\mathbf{P}(\mathbf{Q}.\mathbf{R})+\mathbf{Q}(\mathbf{P}.\mathbf{R})]_\alpha.\end{aligned}$$

Since α is arbitrary, the vector product formula follows.

Example. If $\mathbf{P}\wedge\mathbf{X}=\mathbf{Q}\wedge\mathbf{Y}$, prove that $\mathbf{PX}-\mathbf{QY}=\mathbf{XP}-\mathbf{YQ}$.

(2). *Triple product theorem.* We have

$$(\mathbf{P}\wedge\mathbf{Q}).\mathbf{R}=A_{\alpha\beta\gamma}P_\beta Q_\gamma R_\alpha=A_{\beta\gamma\alpha}Q_\gamma R_\alpha P_\beta=(\mathbf{Q}\wedge\mathbf{R}).\mathbf{P}.$$

(3). *Determinantal relations.* If **T** is a tensor of rank 2, the set of numbers

$$A_{\alpha\beta\gamma}T_{\alpha\mu}T_{\beta\nu}T_{\gamma\sigma}$$

* This theorem is worth learning by heart, as its recollection avoids the memorizing of a host of easily deduced theorems.

are the components in the corresponding triad of a tensor of rank 3. Each non-zero component is the determinant

$$\pm \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix}$$

the upper or lower sign being taken according as the sequence μ, ν, σ is of even or odd parity. Calling this determinant det $\mathbf{T}$ we have

$$A_{\alpha\beta\gamma}T_{\alpha\mu}T_{\beta\nu}T_{\gamma\sigma}=A_{\mu\nu\sigma}(\det \mathbf{T}).$$

Similarly,
$$A_{\alpha\beta\gamma}T_{\mu\alpha}T_{\nu\beta}T_{\sigma\gamma}=A_{\mu\nu\sigma}(\det \mathbf{T}).$$

The sum $A_{\mu\nu\sigma}A_{\mu\nu\sigma}$ is equal to 6, since it contains 3×2 terms each equal to ($\pm$unity) squared. It follows that

$$A_{\alpha\beta\gamma}A_{\mu\nu\sigma}T_{\alpha\mu}T_{\beta\nu}T_{\gamma\sigma}=6(\det \mathbf{T}).$$

This shows that det $\mathbf{T}$ is an invariant.

The rules for the multiplication of determinants can be at once deduced from the above formulæ. For, if $\mathbf{T}$, $\mathbf{S}$ are any two tensors (of rank 2), then

$$A_{\mu\nu\sigma}(\det \mathbf{T})=A_{\alpha\beta\gamma}T_{\alpha\mu}T_{\beta\nu}T_{\gamma\sigma},$$
$$A_{\mu\nu\sigma}(\det \mathbf{S})=A_{\alpha'\beta'\gamma'}S_{\mu\alpha'}S_{\nu\beta'}S_{\sigma\gamma'}.$$

Multiplying, we have

$$\begin{aligned}6(\det \mathbf{T})(\det \mathbf{S})&=A_{\alpha\beta\gamma}A_{\alpha'\beta'\gamma'}(T_{\alpha\mu}S_{\mu\alpha'})(T_{\beta\nu}S_{\nu\beta'})(T_{\gamma\sigma}S_{\sigma\gamma'})\\&=A_{\alpha\beta\gamma}A_{\alpha'\beta'\gamma'}(\mathbf{T.S})_{\alpha\alpha'}(\mathbf{T.S})_{\beta\beta'}(\mathbf{T.S})_{\gamma\gamma'}\\&=6\det(\mathbf{T.S}).\end{aligned}$$

whence
$$(\det \mathbf{T})(\det \mathbf{S})=\det(\mathbf{T.S}).$$

Similarly
$$(\det \mathbf{T})(\det \mathbf{S})=\det(\mathbf{T.\bar{S}})=\det(\mathbf{\bar{T}.S}).$$

Example. If $\mathbf{T}=\mathbf{PQ}$, det $\mathbf{T}=0$.

63. *The vector of a tensor.* If $T_{\alpha\beta}$ are the components of a tensor $\mathbf{T}$ in any triad, the set of three numbers

$$A_{\mu\alpha\beta}T_{\alpha\beta}$$

are the components of a vector. We define the *vector* of the tensor $\mathbf{T}$ by the relation

$$(\text{vec }\mathbf{T})_{\mu}=\tfrac{1}{2}A_{\mu\alpha\beta}T_{\alpha\beta}.$$

The components of vec $\mathbf{T}$ are

$$\tfrac{1}{2}(T_{23}-T_{32}),\qquad \tfrac{1}{2}(T_{31}-T_{13}),\qquad \tfrac{1}{2}(T_{12}-T_{21}).$$

If $\mathbf{T}$ is a dyad $\mathbf{PQ}$, then

$$\text{vec }(\mathbf{PQ})=\tfrac{1}{2}(\mathbf{P}\wedge\mathbf{Q}).$$

If
$$\mathbf{T}=\mathbf{AX}+\mathbf{BY}+\mathbf{CZ}+\dots$$

then
$$\text{vec }\mathbf{T}=\tfrac{1}{2}(\mathbf{A}\wedge\mathbf{X})+\tfrac{1}{2}(\mathbf{B}\wedge\mathbf{Y})+\tfrac{1}{2}(\mathbf{C}\wedge\mathbf{Z})+\dots.$$

If $\mathbf{T}$ is self-conjugate, its vector is zero.

Example. $$\operatorname{vec} \bar{\mathbf{T}} = -\operatorname{vec} \mathbf{T}.$$

Theorem *: If $\mathbf{T}$ is an anti-symmetrical tensor, its three non-zero components in any triad T_{23}, T_{31}, T_{12} are the components of a vector in the same triad.

For since $T_{23} = -T_{32}$, etc., the components of vec $\mathbf{T}$ are just (T_{23}, T_{31}, T_{12}).

The following alternative proof, avoiding the use of the $\mathbf{A}$-tensor is of some interest.

By the definition of an anti-symmetrical tensor, its expression in terms of the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of any orthogonal triad is

$$\mathbf{T} = f(\mathbf{jk}-\mathbf{kj})+g(\mathbf{ki}-\mathbf{ik})+h(\mathbf{ij}+\mathbf{ji}).$$

We shall show that

$$f\mathbf{i}+g\mathbf{j}+h\mathbf{k}$$

is a vector. Let $\mathbf{r}$ be an arbitrary vector. Then

$$\begin{aligned} \mathbf{T.r} &= f[\mathbf{j}(\mathbf{k.r})-\mathbf{k}(\mathbf{j.r})]+\ldots+\ldots \\ &= -f(\mathbf{j}\wedge\mathbf{k})\wedge\mathbf{r}+\ldots+\ldots \\ &= -(f\mathbf{i}+g\mathbf{j}+h\mathbf{k})\wedge\mathbf{r}. \end{aligned}$$

In any other triad $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ let $\mathbf{T}$ be of the form

$$\mathbf{T} = f'(\mathbf{j}'\mathbf{k}'-\mathbf{k}'\mathbf{j}')+g'(\mathbf{k}'\mathbf{i}'-\mathbf{i}'\mathbf{k}')+h'(\mathbf{i}'\mathbf{j}'-\mathbf{j}'\mathbf{i}').$$

Then, as above, $$\mathbf{T.r} = -(f'\mathbf{i}'+g'\mathbf{j}'+h'\mathbf{k}')\wedge\mathbf{r}.$$

Hence for all $\mathbf{r}$, $$(\Sigma f\mathbf{i}-\Sigma f'\mathbf{i}')\wedge\mathbf{r} = 0,$$

whence $$\Sigma f\mathbf{i} = \Sigma f'\mathbf{i}'.$$

Hence $$f' = f(\mathbf{i}'.\mathbf{i})+g(\mathbf{i}'.\mathbf{j})+h(\mathbf{i}'.\mathbf{k}).$$

Thus (f, g, h) obeys the law of vector transformation.

64. *The tensor of a vector.* If P_α are the components of a vector $\mathbf{P}$ in any triad, the set of nine numbers $A_{\mu\nu\alpha}P_\alpha$ are the components of a tensor of rank 2. We call this tensor tens $\mathbf{P}$, and accordingly

$$(\operatorname{tens} \mathbf{P})_{\mu\nu} = A_{\mu\nu\alpha}P_\alpha.$$

The components of tens $\mathbf{P}$ are:

$$\begin{array}{c|ccc} & \nu\rightarrow & & \\ \hline \mu & 0 & P_3 & -P_2 \\ \downarrow & -P_3 & 0 & P_1 \\ & P_2 & -P_1 & 0 \end{array}$$

* This has already been established from first principles in § 54.

Clearly tens $\mathbf{P}$ is an anti-symmetrical tensor.

Example (1). $(\text{tens } \mathbf{P}).\mathbf{Q} = -(\mathbf{P}\wedge\mathbf{Q}) = \mathbf{P}.(\text{tens } \mathbf{Q})$.

Example (2). $\text{tens } (\mathbf{P}\wedge\mathbf{Q}) = \mathbf{PQ} - \mathbf{QP}$.

Example (3). $\text{vec } \mathbf{T} = \text{vec } \overset{\times}{\mathbf{T}}$.

Example (4). $\text{vec tens } \mathbf{P} = \mathbf{P}$.

Example (5). $\text{tens vec } \mathbf{T} = \overset{\times}{\mathbf{T}}$.

65. *Stokes's transformation.* The following transformation has important applications in hydrodynamics and elasticity.

Theorem : If $\mathbf{T}$ is any tensor of rank 2, $\mathbf{P}$ any vector, then

$$\mathbf{T}.\mathbf{P} = \overset{=}{\mathbf{T}}.\mathbf{P} - (\text{vec } \mathbf{T})\wedge\mathbf{P}$$

$$\mathbf{P}.\mathbf{T} = \mathbf{P}.\overset{=}{\mathbf{T}} - \mathbf{P}\wedge(\text{vec } \mathbf{T}).$$

This is most conveniently proved by first establishing the following lemma.

Lemma : If $\mathbf{T}$ is an anti-symmetrical tensor, $\mathbf{P}$ any vector, then

$$\mathbf{T}.\mathbf{P} = -(\text{vec } \mathbf{T})\wedge\mathbf{P}$$
$$\mathbf{P}.\mathbf{T} = -\mathbf{P}\wedge(\text{vec } \mathbf{T}).$$

For
$$\begin{aligned}[(\text{vec } \mathbf{T})\wedge\mathbf{P}]_\alpha &= A_{\alpha\beta\gamma}(\text{vec } \mathbf{T})_\beta P_\gamma \\ &= \tfrac{1}{2}A_{\alpha\beta\gamma}A_{\beta\mu\nu}T_{\mu\nu}P_\gamma \\ &= \tfrac{1}{2}A_{\beta\mu\nu}A_{\beta\gamma\alpha}T_{\mu\nu}P_\gamma \\ &= \tfrac{1}{2}[U_{\mu\gamma}U_{\nu\alpha} - U_{\mu\alpha}U_{\nu\gamma}]T_{\mu\nu}P_\gamma \\ &= \tfrac{1}{2}(T_{\gamma\alpha} - T_{\alpha\gamma})P_\gamma,\end{aligned}$$

or, since $\mathbf{T}$ is anti-symmetrical,

$$[(\text{vec } \mathbf{T})\wedge\mathbf{P}]_\alpha = -T_{\alpha\gamma}P_\gamma = -(\mathbf{T}.\mathbf{P})_\alpha.$$

The second part of the lemma may be proved similarly.

Applying the decomposition process of § 54 and using the above lemma, we now have, if $\mathbf{T}$ is *any* tensor (of rank 2),

$$\begin{aligned}\mathbf{T}.\mathbf{P} &= (\overset{=}{\mathbf{T}} + \overset{\times}{\mathbf{T}}).\mathbf{P} \\ &= \overset{=}{\mathbf{T}}.\mathbf{P} - (\text{vec } \overset{\times}{\mathbf{T}})\wedge\mathbf{P} \\ &= \overset{=}{\mathbf{T}}.\mathbf{P} - (\text{vec } \mathbf{T})\wedge\mathbf{P},\end{aligned}$$

by § 64, Example 3. The second part of the theorem follows similarly.

66. *Cross-products of tensors and vectors.* We have seen that the expression $\mathbf{P}(\mathbf{Q}.\mathbf{X})$, where $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{X}$ are vectors, can be written as $(\mathbf{PQ}).\mathbf{X}$, which is the inner product of the dyad $\mathbf{PQ}$ and the vector $\mathbf{X}$. As it were, we succeed in taking $\mathbf{X}$ out as a factor. The question arises whether we can perform a similar operation for the dyad $\mathbf{P}(\mathbf{Q}\wedge\mathbf{X})$. We want to

be able to give a meaning to the formal expression $(\mathbf{PQ})\wedge\mathbf{X}$, and hence by addition of dyads $\mathbf{PQ}$ to a formal expression $\mathbf{T}\wedge\mathbf{X}$, the 'cross' product of a tensor and a vector, which is itself to be a tensor.

This procedure is most readily carried out as follows. Given a vector $\mathbf{P}$ and a tensor $\mathbf{T}$, we can construct out of their components the tensors whose components are

$$A_{\alpha\beta\gamma}P_\beta T_{\gamma\mu},$$

and

$$T_{\alpha\beta}P_\gamma A_{\beta\gamma\mu}.$$

We call these the *cross-product* of $\mathbf{P}$ and $\mathbf{T}$ and the cross-product of $\mathbf{T}$ and $\mathbf{P}$, respectively, and we denote them by $\mathbf{P}\wedge\mathbf{T}$ and $\mathbf{T}\wedge\mathbf{P}$. Thus our definitions are

$$(\mathbf{P}\wedge\mathbf{T})_{\alpha\mu}=A_{\alpha\beta\gamma}P_\beta T_{\gamma\mu},$$

$$(\mathbf{T}\wedge\mathbf{P})_{\alpha\mu}=T_{\alpha\beta}P_\gamma A_{\beta\gamma\mu}.$$

Here $A_{\alpha\beta\gamma}$, etc., denotes as usual the alternate tensor. These combinations are in many ways analogous to the vector product of two vectors.

In the triad in which the components of $\mathbf{T}$ are $T_{\alpha\beta}$, those of $\mathbf{P}$ P_γ, the components of $\mathbf{P}\wedge\mathbf{T}$ are

		$(\mu=1)$	$(\mu=2)$	$(\mu=3)$
	$(\alpha=1)$	$P_2T_{31}-P_3T_{21}$,	$P_2T_{32}-P_3T_{22}$,	$P_2T_{33}-P_3T_{23}$,
$(\mathbf{P}\wedge\mathbf{T})_{\alpha\mu}$	$(\alpha=2)$	$P_3T_{11}-P_1T_{31}$,	$P_3T_{12}-P_1T_{32}$,	$P_3T_{13}-P_1T_{33}$,
	$(\alpha=3)$	$P_1T_{21}-P_2T_{11}$,	$P_1T_{22}-P_2T_{12}$,	$P_1T_{23}-P_2T_{13}$.

The components of $\mathbf{T}\wedge\mathbf{P}$ are similarly

		$(\mu=1)$	$(\mu=2)$	$(\mu=3)$
	$(\alpha=1)$	$T_{12}P_3-T_{13}P_2$,	$T_{13}P_1-T_{11}P_3$,	$T_{11}P_2-T_{12}P_1$,
$(\mathbf{T}\wedge\mathbf{P})_{\alpha\mu}$	$(\alpha=2)$	$T_{22}P_3-T_{23}P_2$,	$T_{23}P_1-T_{21}P_3$,	$T_{21}P_2-T_{22}P_1$,
	$(\alpha=3)$	$T_{32}P_3-T_{33}P_2$	$T_{33}P_1-T_{31}P_3$	$T_{31}P_2-T_{32}P_1$.

It is quite unnecessary to memorize the expressions of $\mathbf{P}\wedge\mathbf{T}$ and $\mathbf{T}\wedge\mathbf{P}$ in terms of the components of $\mathbf{T}$ and $\mathbf{P}$. The essential properties of cross-products are contained in the following theorem.

Theorem: If $\mathbf{T}$ is a dyad $\mathbf{XY}$, then

$$(\mathbf{XY})\wedge\mathbf{P}=\mathbf{X}(\mathbf{Y}\wedge\mathbf{P})$$

$$\mathbf{P}\wedge(\mathbf{XY})=(\mathbf{P}\wedge\mathbf{X})\mathbf{Y}.$$

These equalities follow directly from the definitions of cross-products, on taking the $\alpha\mu$-components of each side. It is to be noted that $\mathbf{X}(\mathbf{Y}\wedge\mathbf{P})$ is a dyad, the product of the vectors $\mathbf{X}$ and $\mathbf{Y}\wedge\mathbf{P}$. Similarly $(\mathbf{P}\wedge\mathbf{X})\mathbf{Y}$ is a dyad.

67. Any tensor is a sum of nine dyads, namely multiples of the nine fundamental dyads $\mathbf{ii}, \mathbf{ij}, \ldots$ obtained from the triad $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of reference. It will

be shown later, by a procedure independent of the theory of cross-products, that any tensor $\mathbf{T}$ can in fact be expressed as the sum of *three* dyads in the form $\mathbf{PX}+\mathbf{QY}+\mathbf{RZ}$, where either $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ or $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ form a given set of linearly independent vectors. It follows that to establish any theorem concerning cross-products of tensors and vectors, it is sufficient to establish it for the case when the tensor is a dyad, and then arrange the result so that it may be generalized by addition of dyads. Alternatively, any such theorem may be established directly from the definition of a cross-product.

As an example of these alternative procedures, we give proofs of the following theorem:

$$\overline{\mathbf{T}\wedge\mathbf{P}} = -\mathbf{P}\wedge\overline{\mathbf{T}}$$
$$\overline{\mathbf{P}\wedge\mathbf{T}} = -\overline{\mathbf{T}}\wedge\mathbf{P}.$$

In terms of components,

$$\begin{aligned}(\overline{\mathbf{T}\wedge\mathbf{P}})_{\alpha\beta} &= (\mathbf{T}\wedge\mathbf{P})_{\beta\alpha} = T_{\beta\mu}P_{\nu}A_{\mu\nu\alpha}\\ &= -A_{\alpha\nu\mu}P_{\nu}T_{\beta\mu} = -A_{\alpha\nu\mu}P_{\nu}\overline{T}_{\mu\beta}\\ &= -(\mathbf{P}\wedge\overline{\mathbf{T}})_{\alpha\beta}.\end{aligned}$$

Alternatively, in terms of dyads, consider first the particular case when $\mathbf{T}$ is a dyad $\mathbf{AX}$, where $\mathbf{A}$, $\mathbf{X}$ are vectors. Then

$$\begin{aligned}\overline{(\mathbf{AX})\wedge\mathbf{P}} &= \overline{\mathbf{A}(\mathbf{X}\wedge\mathbf{P})} = (\mathbf{X}\wedge\mathbf{P})\mathbf{A} = -(\mathbf{P}\wedge\mathbf{X})\mathbf{A}\\ &= -\mathbf{P}\wedge(\mathbf{XA}) = -\mathbf{P}\wedge(\overline{\mathbf{AX}}).\end{aligned}$$

If now $\mathbf{T}$ is a sum of dyads, $\mathbf{T}=\mathbf{AX}+\mathbf{BY}+\ldots$, then by addition of equalities of the type just established we have

$$(\overline{\mathbf{T}\wedge\mathbf{P}}) = (\overline{\Sigma\mathbf{AX})\wedge\mathbf{P}} = -\mathbf{P}\wedge(\Sigma\overline{\mathbf{AX}}) = -\mathbf{P}\wedge\overline{\mathbf{T}}.$$

We prove the second equality of the theorem similarly.

If we replace $\overline{\mathbf{T}}$ by $\mathbf{T}$ in the enunciation of the theorem we have

$$\mathbf{P}\wedge\mathbf{T} = -\overline{\overline{\mathbf{T}}\wedge\mathbf{P}},$$
$$\mathbf{T}\wedge\mathbf{P} = -\overline{\mathbf{P}\wedge\overline{\mathbf{T}}}.$$

68. The reader will readily establish by these methods the following results, which are here called theorems, though it is not worth while to memorize them. Their analogies with corresponding theorems involving vectors only will be noted. The verification of the theorems affords useful practice in the use of the $(\mathbf{A}, \mathbf{U})$ theorem, or alternatively in the manipulation of dyads. Throughout $\mathbf{T}$ denotes a tensor, $\mathbf{P}$ and $\mathbf{Q}$ vectors.

Theorem :

$$(\mathbf{T}\wedge\mathbf{P}).\mathbf{Q} = \mathbf{T}.(\mathbf{P}\wedge\mathbf{Q}),$$
$$\mathbf{P}.(\mathbf{Q}\wedge\mathbf{T}) = (\mathbf{P}\wedge\mathbf{Q}).\mathbf{T}.$$

(In this theorem we interchange brackets and at the same time interchange the signs . and $\wedge$; $\mathbf{T}$ is a *lateral* factor. The analogy with triple products will be noted.)

Corollary. $(\mathbf{T}\wedge\mathbf{P}).\mathbf{Q}$ and $\mathbf{P}.(\mathbf{Q}\wedge\mathbf{T})$ vanish if $\mathbf{P}$ and $\mathbf{Q}$ are parallel.

Theorem:
$$\mathbf{P}.(\mathbf{T}\wedge\mathbf{Q})=(\mathbf{P}.\mathbf{T})\wedge\mathbf{Q},$$
$$(\mathbf{P}\wedge\mathbf{T}).\mathbf{Q}=\mathbf{P}\wedge(\mathbf{T}.\mathbf{Q}).$$

(In this theorem we interchange brackets but leave the signs . and $\wedge$ unaltered ; $\mathbf{T}$ is a *central* factor.)

It should be noted that
$$\mathbf{T}.(\mathbf{P}\wedge\mathbf{Q})\neq(\mathbf{T}.\mathbf{P})\wedge\mathbf{Q},\quad (\mathbf{P}\wedge\mathbf{Q}).\mathbf{T}\neq\mathbf{P}\wedge(\mathbf{Q}.\mathbf{T}).$$

Theorem:
$$\mathbf{P}\wedge(\mathbf{T}\wedge\mathbf{Q})=(\mathbf{P}\wedge\mathbf{T})\wedge\mathbf{Q}.$$

(This is the continued product theorem with $\mathbf{T}$ central.)

Theorem:
$$(\mathbf{T}\wedge\mathbf{P})\wedge\mathbf{Q}=-\mathbf{T}(\mathbf{P}.\mathbf{Q})+(\mathbf{T}.\mathbf{Q})\mathbf{P},$$
$$\mathbf{P}\wedge(\mathbf{Q}\wedge\mathbf{T})=-(\mathbf{P}.\mathbf{Q})\mathbf{T}+\mathbf{Q}(\mathbf{P}.\mathbf{T}).$$

(This is the continued product theorem with $\mathbf{T}$ lateral.)

Theorem:
$$\mathbf{T}\wedge(\mathbf{P}\wedge\mathbf{Q})=-\mathbf{T}.(\mathbf{PQ}-\mathbf{QP})=-(\mathbf{T}.\mathbf{P})\mathbf{Q}+(\mathbf{T}.\mathbf{Q})\mathbf{P},$$
$$(\mathbf{P}\wedge\mathbf{Q})\wedge\mathbf{T}=-(\mathbf{PQ}-\mathbf{QP}).\mathbf{T}=-\mathbf{P}(\mathbf{Q}.\mathbf{T})+\mathbf{Q}(\mathbf{P}.\mathbf{T}).$$

Examples.
$$(\text{tens }\mathbf{P})\wedge\mathbf{Q}=(\mathbf{P}.\mathbf{Q})\mathbf{U}-\mathbf{QP},$$
$$\mathbf{P}\wedge(\text{tens }\mathbf{Q})=(\mathbf{P}.\mathbf{Q})\mathbf{U}-\mathbf{QP},$$
$$(\text{tens }\mathbf{P}).\mathbf{T}=-\mathbf{P}\wedge\mathbf{T},$$
$$\mathbf{T}.(\text{tens }\mathbf{P})=-\mathbf{T}\wedge\mathbf{P},$$
$$\text{vec }(\mathbf{T}\wedge\mathbf{P})=\tfrac{1}{2}[\mathbf{P}.\mathbf{T}-\mathbf{P}(\text{sca }\mathbf{T})],$$
$$\text{vec }(\mathbf{P}\wedge\mathbf{T})=\tfrac{1}{2}[\mathbf{T}.\mathbf{P}-\mathbf{P}(\text{sca }\mathbf{T})].$$

We have also the following theorems involving two tensors $\mathbf{T}$ and $\mathbf{S}$.

Theorem:
$$(\mathbf{P}\wedge\mathbf{T}).\mathbf{S}=\mathbf{P}\wedge(\mathbf{T}.\mathbf{S}),$$
$$\mathbf{T}.(\mathbf{S}\wedge\mathbf{P})=(\mathbf{T}.\mathbf{S})\wedge\mathbf{P}.$$

Theorem:
$$(\mathbf{T}\wedge\mathbf{P}).\mathbf{S}=\mathbf{T}.(\mathbf{P}\wedge\mathbf{S}).$$

69. *Cross-product properties of the idem tensor* **U.** If we reverse the order of the terms in the cross-product of a tensor and a vector, in general we alter the value of the cross-product, as we have seen. But for the particular case of the cross-products $\mathbf{U}\wedge\mathbf{P}$ and $\mathbf{P}\wedge\mathbf{U}$ we have by definition
$$(\mathbf{U}\wedge\mathbf{P})_{\alpha\beta}=U_{\alpha\mu}P_\nu A_{\mu\nu\beta}=A_{\alpha\nu\beta}P_\nu=-A_{\alpha\beta\nu}P_\nu=-(\text{tens }\mathbf{P})_{\alpha\beta},$$
$$(\mathbf{P}\wedge\mathbf{U})_{\alpha\beta}=A_{\alpha\mu\nu}P_\mu U_{\nu\beta}=A_{\alpha\mu\beta}P_\mu=-A_{\alpha\beta\mu}P_\mu=-(\text{tens }\mathbf{P})_{\alpha\beta}.$$

Thus
$$\mathbf{U}\wedge\mathbf{P}=\mathbf{P}\wedge\mathbf{U}=-\text{tens }\mathbf{P}.$$

If in the theorem
$$\mathbf{P}.(\mathbf{Q}\wedge\mathbf{T})=(\mathbf{P}\wedge\mathbf{Q}).\mathbf{T},$$
we put $\mathbf{T}=\mathbf{U}$, we get
$$\mathbf{P}\wedge\mathbf{Q}=\mathbf{P}.(\mathbf{Q}\wedge\mathbf{U}).$$

If in the theorem

$$\mathbf{P}.(\mathbf{T}\wedge\mathbf{Q})=(\mathbf{P}.\mathbf{T})\wedge\mathbf{Q},$$

we put $\mathbf{T}=\mathbf{U}$, we get

$$\mathbf{P}\wedge\mathbf{Q}=\mathbf{P}.(\mathbf{U}\wedge\mathbf{Q}).$$

The result

$$\mathbf{P}\wedge\mathbf{Q}=\mathbf{P}.(\mathbf{Q}\wedge\mathbf{U})=\mathbf{P}.(\mathbf{U}\wedge\mathbf{Q}),$$

or the similar result

$$\mathbf{P}\wedge\mathbf{Q}=(\mathbf{U}\wedge\mathbf{P}).\mathbf{Q}=(\mathbf{P}\wedge\mathbf{U}).\mathbf{Q}$$

expresses any vector product as the inner product of a tensor and a vector, in either order.

Example (1). Prove that

$$(\mathbf{U}\wedge\mathbf{P}).(\mathbf{U}\wedge\mathbf{Q})=\mathbf{QP}-(\mathbf{P}.\mathbf{Q})\mathbf{U}.$$

Example (2). Prove that

$$(\mathbf{U}\wedge\mathbf{P}).(\mathbf{U}\wedge\mathbf{Q}).(\mathbf{U}\wedge\mathbf{R})=\mathbf{Q}(\mathbf{P}\wedge\mathbf{R})-(\mathbf{P}.\mathbf{Q})\mathbf{U}\wedge\mathbf{R}.$$

Example (3). Prove that

$$\mathbf{U}\wedge(\mathbf{P}\wedge\mathbf{Q})=\mathbf{QP}-\mathbf{PQ}.$$

Example (4). Prove that if $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are a positive orthogonal triad of unit vectors, then

$$\mathbf{jk}-\mathbf{kj}=-\mathbf{U}\wedge\mathbf{i}.$$

For, from first principles,

$$\begin{aligned}\mathbf{jk}-\mathbf{kj}&=+\mathbf{j}(\mathbf{i}\wedge\mathbf{j})-\mathbf{k}(\mathbf{k}\wedge\mathbf{i})=-\mathbf{j}(\mathbf{j}\wedge\mathbf{i})-\mathbf{k}(\mathbf{k}\wedge\mathbf{i})\\&=-(\mathbf{jj}+\mathbf{kk})\wedge\mathbf{i}=-(\mathbf{U}-\mathbf{ii})\wedge\mathbf{i}=-\mathbf{U}\wedge\mathbf{i}.\end{aligned}$$

It follows that any anti-symmetrical tensor, say $\Sigma f(\mathbf{jk}-\mathbf{kj})$, can be put in the form $-\mathbf{U}\wedge(f\mathbf{i}+g\mathbf{j}+h\mathbf{k})=-(f\mathbf{i}+g\mathbf{j}+h\mathbf{k})\wedge\mathbf{U}$.

70.* Let us enquire whether $\mathbf{U}$ is the most general tensor $\mathbf{T}$ which makes

$$\mathbf{T}\wedge\mathbf{P}=\mathbf{P}\wedge\mathbf{T},$$

for any $\mathbf{P}$. If this is satisfied, we have on taking the $\alpha\beta$-components

$$T_{\alpha\mu}P_{\nu}A_{\mu\nu\beta}=A_{\alpha\mu\nu}P_{\mu}T_{\nu\beta},$$

or, changing a dummy suffix,

$$[T_{\alpha\mu}A_{\mu\nu\beta}-T_{\mu\beta}A_{\alpha\nu\mu}]P_{\nu}=0.$$

This is to hold for all vectors $\mathbf{P}$. Hence

$$T_{\alpha\mu}A_{\mu\nu\beta}-T_{\mu\beta}A_{\alpha\nu\mu}=0.$$

Multiply by $A_{\nu\beta\gamma}$ and carry out the implied summations. We get

$$2\,T_{\alpha\gamma}-T_{\beta\beta}U_{\gamma\alpha}+T_{\gamma\alpha}=0.$$

Interchanging γ and α, we have

$$2\,T_{\gamma\alpha}-T_{\beta\beta}U_{\alpha\gamma}+T_{\alpha\gamma}=0.$$

Subtracting, we get

$$T_{\alpha\gamma}=T_{\gamma\alpha}.$$

Hence $$T_{\alpha\gamma}=\tfrac{1}{3}T_{\beta\beta}U_{\alpha\gamma}.$$

* An application of the result of this section occurs in § 237.

Since $T_{\beta\beta}$ is a scalar multiplier, **T** is a scalar multiple of **U**. The last relation is easily seen to be self-consistent on contracting by setting $\gamma=\alpha$.

71. *Expression of an anti-symmetrical tensor as the difference between two conjugate dyads.* Consider the tensor $\mathbf{U}\wedge \operatorname{vec} \mathbf{T}$. The $\alpha\beta$-component of this in any triad is given by

$$\begin{aligned}[\mathbf{U}\wedge \operatorname{vec} \mathbf{T}]_{\alpha\beta} &= U_{\alpha\mu}(\operatorname{vec} T)_\nu A_{\mu\nu\beta}\\ &= A_{\alpha\nu\beta}(\tfrac{1}{2}A_{\nu\rho\sigma}T_{\rho\sigma})\\ &= \tfrac{1}{2}A_{\nu\beta\alpha}A_{\nu\rho\sigma}T_{\rho\sigma}\\ &= \tfrac{1}{2}(T_{\beta\alpha}-T_{\alpha\beta}).\end{aligned}$$

Hence if **T** is anti-symmetrical,

$$\mathbf{T}=-\mathbf{U}\wedge \operatorname{vec} \mathbf{T}.$$

Since vec **T** is a vector, it may be expressed in a triple infinity of ways as a vector product,

$$\operatorname{vec} \mathbf{T}=\mathbf{P}\wedge\mathbf{Q},$$

and
$$-\mathbf{U}\wedge(\mathbf{P}\wedge\mathbf{Q})=+\mathbf{U}.(\mathbf{PQ}-\mathbf{QP})=\mathbf{PQ}-\mathbf{QP},$$

by a theorem of § 68, on putting **U** for **T**. Hence

$$\mathbf{T}=\mathbf{PQ}-\mathbf{QP}.$$

Example. If **T** is a tensor of rank 2 (not necessarily anti-symmetrical), prove that

$$\mathbf{T}+\mathbf{U}\wedge \operatorname{vec} \mathbf{T}$$

is self-conjugate and equal to $\overline{\overline{\mathbf{T}}}$.

72. *Expression of* **U** *in terms of any three linearly independent vectors and their reciprocals.* We have seen that for any vector **P**, $\mathbf{P}=\mathbf{U}.\mathbf{P}$ where **U** is the tensor $\mathbf{ii}+\mathbf{jj}+\mathbf{kk}$. Let us inquire whether **U** is the only tensor with this property. Let **T** be any tensor for which, for an arbitrary vector **P**,

$$\mathbf{P}=\mathbf{T}.\mathbf{P}.$$

Then
$$(\mathbf{T}-\mathbf{U}).\mathbf{P}=0,$$

or
$$(T_{\alpha\mu}-U_{\alpha\mu})P_\mu=0.$$

Since P_μ is an arbitrary vector, the coefficients in this linear relation must be separately zero, and so for fixed α and $\mu=1, 2, 3$ we must have

$$T_{\alpha\mu}=U_{\alpha\mu}.$$

But α is arbitrary. Hence $\mathbf{T}=\mathbf{U}$. Similarly if $\mathbf{P}=\mathbf{P}.\mathbf{T}$, then $\mathbf{T}=\mathbf{U}$.

We have seen (§ 34) that given any three linearly independent vectors **A**, **B**, **C**, ($\mathbf{A}\wedge\mathbf{B}.\mathbf{C}\neq 0$), we can construct the reciprocal vectors **A′**, **B′**, **C′**, given by

$$\mathbf{A}'=\frac{\mathbf{B}\wedge\mathbf{C}}{\mathbf{A}\wedge\mathbf{B}.\mathbf{C}},\qquad \mathbf{B}'=\frac{\mathbf{C}\wedge\mathbf{A}}{\mathbf{A}\wedge\mathbf{B}.\mathbf{C}},\qquad \mathbf{C}'=\frac{\mathbf{A}\wedge\mathbf{B}}{\mathbf{A}\wedge\mathbf{B}.\mathbf{C}},$$

and express any arbitrary vector **P** in the form

$$\mathbf{P}=(\mathbf{P}.\mathbf{A}')\mathbf{A}+(\mathbf{P}.\mathbf{B}')\mathbf{B}+(\mathbf{P}.\mathbf{C}')\mathbf{C}.$$

This may be written, using the theorem of § 52.

$$\mathbf{P}=\mathbf{P}.(\mathbf{A}'\mathbf{A}+\mathbf{B}'\mathbf{B}+\mathbf{C}'\mathbf{C}).$$

It now follows that the tensor in the bracket is the idem tensor **U**, or

$$\mathbf{U}=\mathbf{A}'\mathbf{A}+\mathbf{B}'\mathbf{B}+\mathbf{C}'\mathbf{C}.$$

Similarly, we have had

$$\mathbf{P}=(\mathbf{P}.\mathbf{A})\mathbf{A}'+(\mathbf{P}.\mathbf{B})\mathbf{B}'+(\mathbf{P}.\mathbf{C})\mathbf{C}',$$

whence also $$\mathbf{U}=\mathbf{A}\mathbf{A}'+\mathbf{B}\mathbf{B}'+\mathbf{C}\mathbf{C}'.$$

The expression $$\mathbf{U}=\mathbf{ii}+\mathbf{jj}+\mathbf{kk}$$

is a particular case of this, since **i, j, k** form a self-reciprocal triad.

A sum of products of vectors, i.e. a sum of dyads, is called a *dyadic.* We know that any tensor **T** may be expressed as a dyadic consisting of the nine dyads **ii, jj,** etc. We have just seen that the particular tensor **U** may be expressed as a dyadic of three terms in which three corresponding factors of the dyads can be chosen as arbitrary linearly independent vectors. We proceed to inquire as to the corresponding property for a tensor in general.

Example. If **X, Y** are two non-parallel vectors, show that

$$\frac{\mathbf{XX}(\mathbf{Y}^2)+\mathbf{YY}(\mathbf{X}^2)-(\mathbf{XY}+\mathbf{YX})(\mathbf{X}.\mathbf{Y})+(\mathbf{X}\wedge\mathbf{Y})(\mathbf{X}\wedge\mathbf{Y})}{(\mathbf{X}\wedge\mathbf{Y})^2}=\mathbf{U}.$$

(Express an arbitrary vector **P** in the form $\mathbf{P}=\lambda\mathbf{X}+\mu\mathbf{Y}+\nu(\mathbf{X}\wedge\mathbf{Y})$ and evaluate λ, μ, ν. The result then follows.)

73. *Expression of an arbitrary tensor* **T** *as a dyadic involving three linearly independent vectors.* We shall now show that given three linearly independent vectors **A, B, C** and an arbitrary tensor **T,** then three vectors **X, Y, Z** may be found such that

$$\mathbf{T}=\mathbf{XA}+\mathbf{YB}+\mathbf{ZC}.$$

For, assume that the decomposition is possible. Form the inner product of each side with $\mathbf{B}\wedge\mathbf{C}$. Then

$$\mathbf{T}.(\mathbf{B}\wedge\mathbf{C})=\mathbf{X}(\mathbf{A}.\mathbf{B}\wedge\mathbf{C})+\mathbf{Y}(\mathbf{B}.\mathbf{B}\wedge\mathbf{C})+\mathbf{Z}(\mathbf{C}.\mathbf{B}\wedge\mathbf{C})=\mathbf{X}(\mathbf{A}\wedge\mathbf{B}.\mathbf{C})$$

whence $$\mathbf{X}=\mathbf{T}.\mathbf{A}'.$$

Similarly $$\mathbf{Y}=\mathbf{T}.\mathbf{B}', \qquad \mathbf{Z}=\mathbf{T}.\mathbf{C}'.$$

It now follows that with these values of **X, Y, Z,** the dyadic

$$\mathbf{XA}+\mathbf{YB}+\mathbf{ZC}$$

has the value $$(\mathbf{T}.\mathbf{A}')\mathbf{A}+(\mathbf{T}.\mathbf{B}')\mathbf{B}+(\mathbf{T}.\mathbf{C}')\mathbf{C}$$

which is just $$\mathbf{T}.(\mathbf{A}'\mathbf{A}+\mathbf{B}'\mathbf{B}+\mathbf{C}'\mathbf{C})=\mathbf{T}.\mathbf{U}=\mathbf{T}.$$

Thus the decomposition is possible. Similarly, any tensor **T** may be put in the form

$$\mathbf{T}=\mathbf{AX}+\mathbf{BY}+\mathbf{CZ}$$

where $$\mathbf{X}=\mathbf{A}'.\mathbf{T}, \qquad \mathbf{Y}=\mathbf{B}'.\mathbf{T}, \qquad \mathbf{Z}=\mathbf{C}'.\mathbf{T}.$$

Similarly, if **A, B, C** are linearly independent, any tensor **T** may be expressed in the form

$$\mathbf{T}=\mathbf{X}(\mathbf{B}\wedge\mathbf{C})+\mathbf{Y}(\mathbf{C}\wedge\mathbf{A})+\mathbf{Z}(\mathbf{A}\wedge\mathbf{B}),$$

where $$\mathbf{X}=\frac{\mathbf{T}.\mathbf{A}}{\mathbf{A}\wedge\mathbf{B}.\mathbf{C}}$$

etc. ; and it may be expressed also in the form

$$\mathbf{T}=(\mathbf{B}\wedge\mathbf{C})\mathbf{X}+(\mathbf{C}\wedge\mathbf{A})\mathbf{Y}+(\mathbf{A}\wedge\mathbf{B})\mathbf{Z},$$

with $$\mathbf{X}=\frac{\mathbf{A}.\mathbf{T}}{\mathbf{A}\wedge\mathbf{B}.\mathbf{C}},$$

etc.

Example. If **A, B, C** are three linearly independent vectors, and if

$$\mathbf{T}.\mathbf{A}=\mathbf{A}, \qquad \mathbf{T}.\mathbf{B}=\mathbf{B}, \qquad \mathbf{T}.\mathbf{C}=\mathbf{C},$$

then $\mathbf{T}=\mathbf{U}$.

For, expand **T** in the form

$$\mathbf{T}=\mathbf{X}(\mathbf{B}\wedge\mathbf{C})+\mathbf{Y}(\mathbf{C}\wedge\mathbf{A})+\mathbf{Z}(\mathbf{A}\wedge\mathbf{B}).$$

Then $$\mathbf{X}=\frac{\mathbf{T}.\mathbf{A}}{\mathbf{A}\wedge\mathbf{B}.\mathbf{C}}=\frac{\mathbf{A}}{\mathbf{A}\wedge\mathbf{B}.\mathbf{C}}$$

Hence $$\mathbf{T}=\Sigma\frac{\mathbf{A}(\mathbf{B}\wedge\mathbf{C})}{\mathbf{A}\wedge\mathbf{B}.\mathbf{C}}=\Sigma\mathbf{A}\mathbf{A}'=\mathbf{U}.$$

74. *Alternative expressions for an arbitrary tensor.* If **A** is a given vector, **T** any tensor, then **T** may always be expressed in the form

$$\mathbf{T}=\mathbf{X}\mathbf{A}+\mathbf{S}\wedge\mathbf{A},$$

where **X** is some vector, **S** a tensor satisfying $\mathbf{S}.\mathbf{A}=0$. For, assume the decomposition is possible. Forming the inner product with **A** as a right-hand factor we have

$$\mathbf{T}.\mathbf{A}=\mathbf{X}\mathbf{A}^2+(\mathbf{S}\wedge\mathbf{A}).\mathbf{A}.$$

But by a theorem of § 68,

$$(\mathbf{S}\wedge\mathbf{A}).\mathbf{A}=\mathbf{S}.(\mathbf{A}\wedge\mathbf{A})=0.$$

Hence $$\mathbf{X}=(\mathbf{T}.\mathbf{A})/\mathbf{A}^2.$$

Forming the cross-product with **A**, we have

$$\mathbf{T}\wedge\mathbf{A}=\mathbf{X}(\mathbf{A}\wedge\mathbf{A})+(\mathbf{S}\wedge\mathbf{A})\wedge\mathbf{A}$$

or, again by a theorem of § 68,

$$\mathbf{T}\wedge\mathbf{A}=-\mathbf{S}\mathbf{A}^2+\mathbf{A}(\mathbf{S}.\mathbf{A}),$$

whence, if $\mathbf{S}.\mathbf{A}=0$, $$\mathbf{S}=-(\mathbf{T}\wedge\mathbf{A})/\mathbf{A}^2.$$

We can now verify that the tensor

$$\frac{1}{\mathbf{A}^2}[\mathbf{T}.\mathbf{A}-(\mathbf{T}\wedge\mathbf{A})\wedge\mathbf{A}]$$

is precisely **T**.

Similarly, given $\mathbf{A}$, any tensor $\mathbf{T}$ may be expressed in the form

$$\mathbf{T}=\mathbf{AX}+\mathbf{A}\wedge\mathbf{S}, \qquad (\mathbf{A.S}=0),$$

where $$\mathbf{X}=(\mathbf{A.T})/\mathbf{A}^2, \qquad \mathbf{S}=-(\mathbf{A}\wedge\mathbf{T})/\mathbf{A}^2.$$

Again, given *two* vectors $\mathbf{A}$ and $\mathbf{B}$, any tensor $\mathbf{T}$ may be expressed in the forms

$$\mathbf{T}=\mathbf{XA}+\mathbf{YB}+\mathbf{Z}(\mathbf{A}\wedge\mathbf{B})$$

$$\mathbf{T}=\mathbf{AX}+\mathbf{BY}+(\mathbf{A}\wedge\mathbf{B})\mathbf{Z}.$$

The evaluation of $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ is left to the reader.

If $\mathbf{A}$ and $\mathbf{B}$ are given vectors, an arbitrary tensor $\mathbf{T}$ cannot be in general expressed in the form $\mathbf{T}=\mathbf{XA}+\mathbf{YB}$. For this would require $\mathbf{T}.(\mathbf{A}\wedge\mathbf{B})=0$, which will not in general be true.

Example. Solve for the tensor $\mathbf{T}$ the equation

$$\mathbf{T.P}=\mathbf{A}\wedge\mathbf{P},$$

where $\mathbf{A}$ and $\mathbf{P}$ are given vectors.

By Example (1) § 64, we have

$$\mathbf{A}\wedge\mathbf{P}=-(\text{tens }\mathbf{A}).\mathbf{P}.$$

Hence from the given equation, $\mathbf{T}$ satisfies

$$(\mathbf{T}+\text{tens }\mathbf{A}).\mathbf{P}=0.$$

Put $$\mathbf{T}+\text{tens }\mathbf{A}=\mathbf{S}.$$

Then $\mathbf{S}$, being a tensor, may be put in the form

$$\mathbf{S}=\mathbf{XP}+\mathbf{W}\wedge\mathbf{P},$$

where $\mathbf{X}$ is a vector and $\mathbf{W}$ a tensor satisfying $\mathbf{W.P}=0$. But the tensor $\mathbf{S}$ has been shown to satisfy $\mathbf{S.P}=0$. Hence $\mathbf{X}=0$. Hence the most general solution is

$$\mathbf{T}=-\text{tens }\mathbf{A}+\mathbf{W}\wedge\mathbf{P},$$

where $\mathbf{W}$ is any tensor satisfying $\mathbf{W.P}=0$.

75. *The triple product of the three inner products of a given tensor with three given vectors.*

Theorem: If $\mathbf{T}$ is any tensor, $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ three vectors, then

$$(\mathbf{T.P})\wedge(\mathbf{T.Q}).\mathbf{T.R}=(\det \mathbf{T})\mathbf{P}\wedge\mathbf{Q.R}.$$

For the left-hand side is equal to

$$\begin{aligned}&A_{\alpha\beta\gamma}(\mathbf{T.P})_\alpha(\mathbf{T.Q})_\beta(\mathbf{T.R})_\gamma\\ &=A_{\alpha\beta\gamma}T_{\alpha\mu}T_{\beta\nu}T_{\gamma\sigma}P_\mu Q_\nu R_\sigma\\ &=A_{\mu\nu\sigma}(\det \mathbf{T})P_\mu Q_\nu R_\sigma\\ &=(\det \mathbf{T})\mathbf{P}\wedge\mathbf{Q.R}.\end{aligned}$$

If in this theorem we replace the tensor $\mathbf{T}$ by the tensor $\mathbf{T.S}$, where $\mathbf{S}$ is a second tensor, then since

$$(\mathbf{T.S}).\mathbf{P}=\mathbf{T}.(\mathbf{S.P}),$$

we have on applying the theorem

$$(\det \mathbf{T.S})\mathbf{P}\wedge\mathbf{Q.R} = (\det \mathbf{T})[(\mathbf{S.P})\wedge(\mathbf{S.Q}).(\mathbf{S.R})]$$
$$= (\det \mathbf{T})(\det \mathbf{S})\mathbf{P}\wedge\mathbf{Q.R}.$$

This gives an independent proof of the law of multiplication of determinants

$$(\det \mathbf{T.S}) = (\det \mathbf{T})(\det \mathbf{S}).$$

Similarly, we have

$$(\mathbf{P.S})\wedge(\mathbf{Q.S}).(\mathbf{R.S}) = (\mathbf{P}\wedge\mathbf{Q.R})\det \mathbf{S}.$$

Then, using

$$(\mathbf{T.\bar{S}}).\mathbf{P} = \mathbf{T}.(\mathbf{\bar{S}.P}) = \mathbf{T}.(\mathbf{P.S})$$

we have

$$\det (\mathbf{T.\bar{S}})\mathbf{P}\wedge\mathbf{Q.R} = (\det \mathbf{T})(\det \mathbf{S})\mathbf{P}\wedge\mathbf{Q.R}$$

whence $$\det \mathbf{T.\bar{S}} = (\det \mathbf{T})(\det \mathbf{S}) = \det \mathbf{S.\bar{T}}.$$

Corollary (1). It follows from the theorem that if **P**, **Q**, **R** are linearly independent vectors, then so are **T.P**, **T.Q**, **T.R** unless $\det \mathbf{T} = 0$. Hence if **T** is expressed in the form

$$\mathbf{T} = \mathbf{X}(\mathbf{Q}\wedge\mathbf{R}) + \mathbf{Y}(\mathbf{R}\wedge\mathbf{P}) + \mathbf{Z}(\mathbf{P}\wedge\mathbf{Q}),$$

then **X**, **Y**, **Z** are linearly independent if $\det \mathbf{T} \neq 0$.

Corollary (2). Replacing **P**, **Q**, **R** by the three linearly independent vectors $\mathbf{Q}\wedge\mathbf{R}$, $\mathbf{R}\wedge\mathbf{P}$, $\mathbf{P}\wedge\mathbf{Q}$, we have

$$[\mathbf{T}.(\mathbf{Q}\wedge\mathbf{R})]\wedge[\mathbf{T}.(\mathbf{R}\wedge\mathbf{P})].[\mathbf{T}.(\mathbf{P}\wedge\mathbf{Q})] = (\det \mathbf{T})(\mathbf{P}\wedge\mathbf{Q.R})^2.$$

Hence if **T** is expressed in the form

$$\mathbf{T} = \mathbf{XP} + \mathbf{YQ} + \mathbf{ZR}$$

where **P**, **Q**, **R** are linearly independent, then **X**, **Y**, **Z** are linearly independent provided $\det \mathbf{T} \neq 0$.

Corollary (3). If $$\mathbf{T} = \mathbf{XP} + \mathbf{YQ} + \mathbf{ZR},$$
then $$\det \mathbf{T} = (\mathbf{X}\wedge\mathbf{Y.Z})(\mathbf{P}\wedge\mathbf{Q.R}).$$

This follows from Corollary (2), since $\mathbf{T}.(\mathbf{Q}\wedge\mathbf{R}) = \mathbf{X}(\mathbf{P}\wedge\mathbf{Q.R})$. It also follows from the definition of det **T**. This determines the relative signs of the triads **X**, **Y**, **Z**; **P**, **Q**, **R** in terms of the sign of det **T**.

It follows from the above that if $\det \mathbf{T} \neq 0$, **T** cannot be expressed as the sum of less than three dyads. For if $\mathbf{T} = \mathbf{XA} + \mathbf{YB} + \mathbf{ZC}$ is to be expressed as $\mathbf{LP} + \mathbf{MQ}$, then $\mathbf{T.B}\wedge\mathbf{C} = \mathbf{X}(\mathbf{A}\wedge\mathbf{B.C}) = \mathbf{L}(\mathbf{P.B}\wedge\mathbf{C}) + \mathbf{M}(\mathbf{Q.B}\wedge\mathbf{C})$. Hence **X**, and similarly **Y** and **Z** are coplanar with **L** and **M**, and so are linearly dependent, which contradicts $\det \mathbf{T} \neq 0$. If $\det \mathbf{T} = 0$, the tensor **T** may be expressed as the sum of two dyads in an infinity of ways.

Example. If **T** is any tensor, **P** any vector, then $\det (\mathbf{T}\wedge\mathbf{P}) = 0$. Hence $\mathbf{T}\wedge\mathbf{P}$ can be expressed as the sum of two dyads.

76. *The tensor as a linear vector operator.* Let **T** be a given tensor, **P** a variable vector. Then the vector $\mathbf{Q} = \mathbf{T.P}$ is a function of the vector **P**, and moreover a *linear* function of **P**. Accordingly, **T** may be regarded

as an *operator* converting the vector $\mathbf{P}$ into the vector $\mathbf{Q}$. In some treatments this aspect of a tensor is considered as fundamental: the tensor $\mathbf{T}$ is *defined* as an operator. We have, however, preferred to consider $\mathbf{T}$ as an entity in itself, on the same footing as, though a generalization of the notion of, a vector. The aspect of $\mathbf{T}$ as an operator will, however, be more prominent in what follows.

Similarly, given $\mathbf{P}$ and $\mathbf{T}$, we may construct the vector $\mathbf{P.T}$, which is also a linear function of $\mathbf{P}$, and different from $\mathbf{T.P}$ unless $\mathbf{T}$ is self-conjugate. The 'operation' $\mathbf{T}$ may thus be applied as a *pre-factor* or as a *post-factor*. The operator $\mathbf{T}$ is called linear because if $\mathbf{P}=\Sigma\alpha_\nu\mathbf{P}_\nu$, then $\mathbf{Q}=\Sigma\alpha_\nu\mathbf{Q}_\nu$, where $\mathbf{Q}_\nu=\mathbf{T.P}_\nu$; similarly if $\mathbf{T}$ is a post-factor.

77. *The tensor inverse to a given tensor.* As the vector $\mathbf{P}$ varies in any way, the vector $\mathbf{Q}=\mathbf{T.P}$ varies, and there is a correspondence giving a unique $\mathbf{Q}$ for every $\mathbf{P}$. We may inquire conversely whether $\mathbf{P}$ may be regarded as a linear function of $\mathbf{Q}$, expressible similarly. In symbols, if $\mathbf{Q}=\mathbf{T.P}$, does a tensor $\mathbf{S}$ exist such that $\mathbf{P}=\mathbf{S.Q}$?

If this is so, then

$$\mathbf{P}=\mathbf{S.(T.P)}=\mathbf{(S.T).P},$$

for any $\mathbf{P}$, so that

$$\mathbf{S.T}=\mathbf{U}.$$

If $\mathbf{T}$ is given by its components in a triad $\mathbf{i}, \mathbf{j}, \mathbf{k}$, putting $\mathbf{ii}+\mathbf{jj}+\mathbf{kk}$ for $\mathbf{U}$ we can *in general* solve the nine equations expressed by $\mathbf{S.T}=\mathbf{U}$ for the nine components of $\mathbf{S}$. (We shall carry out this solution later.) Since $\mathbf{S.T}=\mathbf{U}$ is a tensor equation, the nine numbers $S_{\alpha\beta}$ thus found will describe a tensor. This tensor $\mathbf{S}$ will be written $\mathbf{T}^{-1}$, so that

$$\mathbf{T}^{-1}.\mathbf{T}=\mathbf{U}.$$

Thus if $\mathbf{Q}=\mathbf{T.P}$, then $\mathbf{P}=\mathbf{T}^{-1}.\mathbf{Q}$. It follows that

$$\mathbf{Q}=\mathbf{T.T}^{-1}.\mathbf{Q}=(\mathbf{T.T}^{-1}).\mathbf{Q}$$

so that also

$$\mathbf{T.T}^{-1}=\mathbf{U}.$$

Hence $(\mathbf{T}^{-1})^{-1}=\mathbf{T}$. For this reason $\mathbf{T}^{-1}$ is called the *inverse* of $\mathbf{T}$.

78. *Inverse of a product of tensors.* The following theorem is true of any operators possessing inverses; in our context we need it only for tensors of rank 2.

Theorem: If

$$\mathbf{R}=\mathbf{T.S.} \ldots \mathbf{V.W},$$

where $\mathbf{T}, \mathbf{S}, \ldots \mathbf{V}, \mathbf{W}$ are tensors, then

$$\mathbf{R}^{-1}=\mathbf{W}^{-1}.\mathbf{V}^{-1}. \ldots \mathbf{S}^{-1}.\mathbf{T}^{-1}.$$

For, since

$$\mathbf{R}^{-1}.\mathbf{R}=\mathbf{U},$$

we have

$$[\mathbf{R}^{-1}.(\mathbf{T.S.} \ldots \mathbf{V.W})].\mathbf{W}^{-1}=\mathbf{U.W}^{-1}=\mathbf{W}^{-1}.$$

Hence

$$\mathbf{R}^{-1}.(\mathbf{T.S.} \ldots \mathbf{V})=\mathbf{W}^{-1}.$$

Operate on the right similarly with $\mathbf{V}^{-1}$. We find

$$\mathbf{R}^{-1}.(\mathbf{T.S.} \ldots)=\mathbf{W}^{-1}.\mathbf{V}^{-1}.$$

Proceeding similarly, and operating lastly with $\mathbf{S}^{-1}$ and $\mathbf{T}^{-1}$, we get the theorem as stated.

79. *The inverse of a tensor when expressed as the sum of three dyads.* Let **A, B, C** be three linearly independent vectors. Then any tensor **T** may be expressed in the form

$$\mathbf{T}=\mathbf{XA}+\mathbf{YB}+\mathbf{ZC},$$

where, if det $\mathbf{T} \neq 0$, **X, Y, Z** are linearly independent. By the definition of the inverse tensor, we have

$$\mathbf{U}=\mathbf{T}^{-1}.\mathbf{T}=(\mathbf{T}^{-1}.\mathbf{X})\mathbf{A}+(\mathbf{T}^{-1}.\mathbf{Y})\mathbf{B}+(\mathbf{T}^{-1}.\mathbf{Z})\mathbf{C}.$$

Forming the inner product on the right with the vector $\mathbf{B}\wedge\mathbf{C}$, we get

$$\mathbf{B}\wedge\mathbf{C}=(\mathbf{T}^{-1}.\mathbf{X})(\mathbf{A}\wedge\mathbf{B}.\mathbf{C})$$

whence
$$\mathbf{T}^{-1}.\mathbf{X}=\frac{\mathbf{B}\wedge\mathbf{C}}{\mathbf{A}\wedge\mathbf{B}.\mathbf{C}}=\mathbf{A}',$$

and similarly
$$\mathbf{T}^{-1}.\mathbf{Y}=\mathbf{B}', \qquad \mathbf{T}^{-1}.\mathbf{Z}=\mathbf{C}',$$

where $\mathbf{A}'$, $\mathbf{B}'$, $\mathbf{C}'$ are the vectors reciprocal to **A, B, C.**

Since we thus know the vectors $\mathbf{T}^{-1}.\mathbf{X}$, etc., it suggests itself that we expand $\mathbf{T}^{-1}$ as a dyadic with post-factors $(\mathbf{Y}\wedge\mathbf{Z})$, $(\mathbf{Z}\wedge\mathbf{X})$, $(\mathbf{X}\wedge\mathbf{Y})$, in the form

$$\mathbf{T}^{-1}=\mathbf{P}(\mathbf{Y}\wedge\mathbf{Z})+\mathbf{Q}(\mathbf{Z}\wedge\mathbf{X})+\mathbf{R}(\mathbf{X}\wedge\mathbf{Y}).$$

Operating on the vector **X** on each side, we get

$$\mathbf{T}^{-1}.\mathbf{X}=\mathbf{P}(\mathbf{Y}\wedge\mathbf{Z}.\mathbf{X})$$

or
$$\mathbf{P}=\frac{\mathbf{A}'}{\mathbf{X}\wedge\mathbf{Y}.\mathbf{Z}}$$

We have similar expressions for **Q** and **R,** whence

$$\mathbf{T}^{-1}=\frac{\mathbf{A}'(\mathbf{Y}\wedge\mathbf{Z})+\mathbf{B}'(\mathbf{Z}\wedge\mathbf{X})+\mathbf{C}'(\mathbf{X}\wedge\mathbf{Y})}{\mathbf{X}\wedge\mathbf{Y}.\mathbf{Z}}.$$

But the second vectors in the three dyads are just

$$\mathbf{X}', \mathbf{Y}', \mathbf{Z}',$$

where $\mathbf{X}'$, $\mathbf{Y}'$, $\mathbf{Z}'$ are the vectors reciprocal to **X, Y, Z,** namely

$$\mathbf{X}'=\frac{\mathbf{Y}\wedge\mathbf{Z}}{\mathbf{X}\wedge\mathbf{Y}.\mathbf{Z}},$$

etc. Accordingly, if the inverse $\mathbf{T}^{-1}$ of the tensor T given by

$$\mathbf{T}=\mathbf{XA}+\mathbf{YB}+\mathbf{ZC}$$

exists, its form is
$$\mathbf{T}^{-1}=\mathbf{A}'\mathbf{X}'+\mathbf{B}'\mathbf{Y}'+\mathbf{C}'\mathbf{Z}'.$$

But we know that
$$\mathbf{X}'\mathbf{X}+\mathbf{Y}'\mathbf{Y}+\mathbf{Z}'\mathbf{Z}=\mathbf{U}$$
$$\mathbf{A}'\mathbf{A}+\mathbf{B}'\mathbf{B}+\mathbf{C}'\mathbf{C}=\mathbf{U}.$$

We immediately verify that $\mathbf{T}^{-1}.\mathbf{T}$ is in fact **U,** and so is $\mathbf{T}.\mathbf{T}^{-1}$, since $\mathbf{X}'.\mathbf{X}=1$, $\mathbf{A}'.\mathbf{A}=1$, etc.

We thus have the following rule for forming the inverse of a tensor **T** expressed as the sum of three dyads: *the inverse tensor is equal to a dyadic formed of the reciprocals of the sets of vectors forming the dyads taken in the opposite order.*

Corollary (1). The inverse of the conjugate of a tensor is the conjugate of the inverse.

For $$\bar{\mathbf{T}}^{-1}=(\mathbf{XA}+\mathbf{YB}+\mathbf{ZC})^{-1}=\mathbf{A'X'}+\mathbf{B'Y'}+\mathbf{C'Z'}=\overline{\mathbf{T}^{-1}}.$$

Corollary (2). The inverse of a self-conjugate tensor is self-conjugate.

Corollary (3). The determinant of the inverse tensor is the inverse of the determinant of the original tensor.

For, if $$\mathbf{T}=\mathbf{XA}+\mathbf{YB}+\mathbf{ZC},$$
then $$\det \mathbf{T}=(\mathbf{X}\wedge\mathbf{Y}.\mathbf{Z})(\mathbf{A}\wedge\mathbf{B}.\mathbf{C})$$
and $$\det \mathbf{T}^{-1}=(\mathbf{A}'\wedge\mathbf{B}'.\mathbf{C}')(\mathbf{X}'\wedge\mathbf{Y}'.\mathbf{Z}')=[(\mathbf{A}\wedge\mathbf{B}.\mathbf{C})(\mathbf{X}\wedge\mathbf{Y}.\mathbf{Z})]^{-1}=(\det \mathbf{T})^{-1}.$$

80. *The inverse of a tensor in terms of its components with respect to a given triad of reference.* Let the tensor **T** have components $t_{\alpha\beta}$ with respect to the triad **i, j, k**. Then **T** may be written in dyadic form as

$$\mathbf{i}(t_{11}\mathbf{i}+t_{12}\mathbf{j}+t_{13}\mathbf{k})+\mathbf{j}(t_{21}\mathbf{i}+t_{22}\mathbf{j}+t_{23}\mathbf{k})+\mathbf{k}(t_{31}\mathbf{i}+t_{32}\mathbf{j}+t_{33}\mathbf{k})$$

or, say $$\mathbf{iX}+\mathbf{jY}+\mathbf{kZ}.$$

Its inverse is accordingly $\mathbf{X'i'}+\mathbf{Y'j'}+\mathbf{Z'k'}$.

But $$\mathbf{i}'=\mathbf{i},\qquad \mathbf{j}'=\mathbf{j},\qquad \mathbf{k}'=\mathbf{k},$$

and $$\mathbf{X}'=\frac{(t_{21}\mathbf{i}+t_{22}\mathbf{j}+t_{23}\mathbf{k})\wedge(t_{31}\mathbf{i}+t_{32}\mathbf{j}+t_{33}\mathbf{k})}{\begin{vmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33}\end{vmatrix}}$$
$$=\frac{\mathbf{i}(t_{22}t_{33}-t_{23}t_{32})+\mathbf{j}(t_{23}t_{31}-t_{21}t_{33})+\mathbf{k}(t_{21}t_{32}-t_{22}t_{31})}{\det \mathbf{T}}$$
$$=(T_{11}\mathbf{i}+T_{12}\mathbf{j}+T_{13}\mathbf{k})/\det \mathbf{T},$$

where T_{11}, T_{12}, ... are the co-factors of t_{11}, t_{12}, ... etc., in det **T**.

Thus $$\begin{aligned}\mathbf{T}^{-1}=[&(T_{11}\mathbf{i}+T_{12}\mathbf{j}+T_{13}\mathbf{k})\mathbf{i}\\ &+(T_{21}\mathbf{i}+T_{22}\mathbf{j}+T_{23}\mathbf{k})\mathbf{j}\\ &+(T_{31}\mathbf{i}+T_{32}\mathbf{j}+T_{33}\mathbf{k})\mathbf{k}]/\det \mathbf{T},\\ =(&T_{11}\mathbf{ii}+T_{21}\mathbf{ij}+T_{31}\mathbf{ik}\\ &+T_{12}\mathbf{ji}+T_{22}\mathbf{jj}+T_{32}\mathbf{jk}\\ &+T_{13}\mathbf{ki}+T_{23}\mathbf{kj}+T_{33}\mathbf{kk})/\det \mathbf{T}.\end{aligned}$$

Thus, to obtain $\mathbf{T}^{-1}$, given **T** in the form

$$\begin{aligned}\mathbf{T}=&t_{11}\mathbf{ii}+t_{12}\mathbf{ij}+t_{13}\mathbf{ik}\\ &+t_{21}\mathbf{ji}+t_{22}\mathbf{jj}+t_{23}\mathbf{jk}\\ &+t_{31}\mathbf{ki}+t_{32}\mathbf{kj}+t_{33}\mathbf{kk},\end{aligned}$$

the rule is: change the coefficients into their co-factors in det $\mathbf{T}$, interchange the order of the factors in each dyad and divide by det $\mathbf{T}$.

It may readily be verified that this expression for $\mathbf{T}^{-1}$ satisfies

$$\mathbf{T}^{-1}.\mathbf{T}=\mathbf{T}.\mathbf{T}^{-1}=\mathbf{ii}+\mathbf{jj}+\mathbf{kk}.$$

Example. If
$$\mathbf{T}=a\mathbf{ii}+b\mathbf{jj}+c\mathbf{kk},$$
then
$$\mathbf{T}^{-1}=a^{-1}\mathbf{ii}+b^{-1}\mathbf{jj}+c^{-1}\mathbf{kk}.$$

We see that $\mathbf{T}^{-1}$ exists whenever det $\mathbf{T}\neq 0$. If det $\mathbf{T}=0$, $\mathbf{T}$ has no inverse. A tensor which is the sum of two dyads, or reduces to a single dyad, has accordingly no inverse. An anti-symmetrical tensor, being the difference of two dyads, namely of two conjugate dyads, has determinant zero, and hence has no inverse.

81. The foregoing theory of the inverse of a dyadic of three terms depends essentially on the number of spatial dimensions being equal to three. A dyadic $\mathbf{T}=\mathbf{AX}+\mathbf{BY}$ has no inverse in three dimensions, where $\mathbf{U}=\mathbf{ii}+\mathbf{jj}+\mathbf{kk}$, but it has an inverse in two dimensions, where $\mathbf{U}=\mathbf{ii}+\mathbf{jj}$. To find the inverse of $\mathbf{T}=\mathbf{AX}+\mathbf{BY}$ in two dimensions we proceed thus. Suppose the tensor $\mathbf{T}^{-1}$ expanded as a dyadic in two dimensions of the form

$$\mathbf{T}^{-1}=\mathbf{P}(\mathbf{k}\wedge\mathbf{A})+\mathbf{Q}(\mathbf{k}\wedge\mathbf{B}),$$

where $\mathbf{k}$ is a unit vector perpendicular to the 2-space concerned. Then since

$$\mathbf{T}^{-1}.\mathbf{T}=\mathbf{U},$$

we have
$$\mathbf{PY}(\mathbf{k}\wedge\mathbf{A}.\mathbf{B})+\mathbf{QX}(\mathbf{k}\wedge\mathbf{B}.\mathbf{A})=\mathbf{U}.$$

Operating on $\mathbf{k}\wedge\mathbf{X}$, we get

$$\mathbf{P}(\mathbf{Y}.\mathbf{k}\wedge\mathbf{X})(\mathbf{k}\wedge\mathbf{A}.\mathbf{B})=\mathbf{k}\wedge\mathbf{X},$$

and similarly
$$\mathbf{Q}(\mathbf{X}.\mathbf{k}\wedge\mathbf{Y})(\mathbf{k}\wedge\mathbf{B}.\mathbf{A})=\mathbf{k}\wedge\mathbf{Y}.$$

These determine $\mathbf{P}$ and $\mathbf{Q}$, and we have then

$$\mathbf{T}^{-1}=\frac{(\mathbf{k}\wedge\mathbf{X})(\mathbf{k}\wedge\mathbf{A})+(\mathbf{k}\wedge\mathbf{Y})(\mathbf{k}\wedge\mathbf{B})}{(\mathbf{k}\wedge\mathbf{A}.\mathbf{B})(\mathbf{k}\wedge\mathbf{X}.\mathbf{Y})}.$$

It may be verified that in two dimensions,

$$\frac{(\mathbf{k}\wedge\mathbf{X})\mathbf{Y}-(\mathbf{k}\wedge\mathbf{Y})\mathbf{X}}{\mathbf{k}\wedge\mathbf{X}.\mathbf{Y}}=\mathbf{U}=\mathbf{ii}+\mathbf{jj}$$

for any pair of non-parallel vectors $\mathbf{X}$, $\mathbf{Y}$. For on forming the inner product with an arbitrary vector $\mathbf{P}$ in the plane of $\mathbf{X}$ and $\mathbf{Y}$, it will be found that $\mathbf{P}$ is reproduced identically. It may then be verified that $\mathbf{T}.\mathbf{T}^{-1}=\mathbf{T}^{-1}.\mathbf{T}=\mathbf{U}$.

The relationship of these results to the corresponding three-dimensional results is obtained by noting that if we have a triad $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ and take $\mathbf{Z}$ to be a unit vector $\mathbf{k}$ normal to the plane of $\mathbf{X}$ and $\mathbf{Y}$, then

$$\mathbf{Z}'=\frac{\mathbf{X}\wedge\mathbf{Y}}{\mathbf{X}\wedge\mathbf{Y}.\mathbf{k}}=\mathbf{k}.$$

Hence in three dimensions, since

$$\mathbf{XX}'+\mathbf{YY}'+\mathbf{ZZ}'=\mathbf{U}=\mathbf{ii}+\mathbf{jj}+\mathbf{kk},$$

we have on subtracting the dyad $\mathbf{kk}$ from each side

$$\mathbf{XX}'+\mathbf{YY}'=\mathbf{ii}+\mathbf{jj},$$

which reduces to the two-dimensional result found above.

Example (1). We have seen (§ 16, Example (3)) that the solution $\mathbf{X}$ of the vector equation

$$\alpha\mathbf{X}+\mathbf{A}(\mathbf{X}.\mathbf{B})=\mathbf{C}$$

is

$$\mathbf{X}=\frac{\mathbf{C}}{\alpha}-\frac{\mathbf{C}.\mathbf{B}}{\alpha(\alpha+\mathbf{A}.\mathbf{B})}\mathbf{A}.$$

But the given equation may be written

$$(\alpha\mathbf{U}+\mathbf{AB}).\mathbf{X}=\mathbf{C},$$

whence

$$\mathbf{X}=(\alpha\mathbf{U}+\mathbf{AB})^{-1}.\mathbf{C}.$$

But the solution $\mathbf{X}$ may be written

$$\mathbf{X}=\left[\frac{\mathbf{U}}{\alpha}-\frac{\mathbf{AB}}{\alpha(\alpha+\mathbf{A}.\mathbf{B})}\right].\mathbf{C}.$$

Since $\mathbf{C}$ is arbitrary, this yields

$$(\alpha\mathbf{U}+\mathbf{AB})^{-1}=\frac{\mathbf{U}}{\alpha}-\frac{\mathbf{AB}}{\alpha(\alpha+\mathbf{A}.\mathbf{B})}.$$

We can now use this result to solve *tensor* equations. Suppose that $\mathbf{S}$ is a given tensor, $\mathbf{A}$, $\mathbf{B}$ given vectors, and that it is required to solve for $\mathbf{T}$ the tensor equation

$$\alpha\mathbf{T}+\mathbf{A}(\mathbf{B}.\mathbf{T})=\mathbf{S}.$$

This may be written

$$(\alpha\mathbf{U}+\mathbf{AB}).\mathbf{T}=\mathbf{S},$$

whence

$$\mathbf{T}=(\alpha\mathbf{U}+\mathbf{AB})^{-1}.\mathbf{S}=\left[\frac{\mathbf{U}}{\alpha}-\frac{\mathbf{AB}}{\alpha(\alpha+\mathbf{A}.\mathbf{B})}\right].\mathbf{S}$$

$$=\frac{\mathbf{S}}{\alpha}-\frac{\mathbf{A}(\mathbf{B}.\mathbf{S})}{\alpha(\alpha+\mathbf{A}.\mathbf{B})}.$$

This solution may also be found from first principles.

Example (2). We have seen (§ 31, Example (14)) that the solution $\mathbf{X}$ of the vector equation

$$\alpha\mathbf{X}+\mathbf{X}\wedge\mathbf{A}=\mathbf{B}$$

is

$$\mathbf{X}=\frac{\alpha^2\mathbf{B}-\alpha(\mathbf{B}\wedge\mathbf{A})+\mathbf{A}(\mathbf{B}.\mathbf{A})}{\alpha(\alpha^2+\mathbf{A}^2)}.$$

But the given equation may be written

$$(\alpha\mathbf{U}+\text{tens }\mathbf{A}).\mathbf{X}=\mathbf{B},$$

whence

$$\mathbf{X}=(\alpha\mathbf{U}+\text{tens }\mathbf{A})^{-1}.\mathbf{B}.$$

But the solution may be written

$$\mathbf{X}=\frac{\alpha^2\mathbf{U}-\alpha\text{ tens }\mathbf{A}+\mathbf{AA}}{\alpha(\alpha^2+\mathbf{A}^2)}.\mathbf{B}.$$

Since $\mathbf{B}$ is arbitrary, we have

$$(\alpha\mathbf{U}+\text{tens }\mathbf{A})^{-1}=\frac{\alpha^2\mathbf{U}-\alpha\text{ tens }\mathbf{A}+\mathbf{AA}}{\alpha(\alpha^2+\mathbf{A}^2)}.$$

Example (3). Show that the solution $\mathbf{T}$ of the equation

$$\alpha\mathbf{T}+\mathbf{A}\wedge\mathbf{T}=\mathbf{S},$$

where $\mathbf{S}$ is a given tensor, $\mathbf{A}$ a given vector, is

$$\mathbf{T}=\frac{\alpha^2\mathbf{S}-\alpha(\mathbf{A}\wedge\mathbf{S})+\mathbf{A}(\mathbf{A.S})}{\alpha(\alpha^2+\mathbf{A}^2)}.$$

[Use $\mathbf{A}\wedge\mathbf{T}=-\text{tens }\mathbf{A.T}$].

82. *Geometrical interpretation of a self-conjugate tensor.* We have seen that if $\mathbf{T}$ is a fixed tensor, $\mathbf{P}$ a variable vector, then the equation $\mathbf{Q}=\mathbf{T.P}$ defines a vector function of $\mathbf{P}$. Now, by Stokes's transformation (§ 65),

$$\mathbf{Q}=\bar{\bar{\mathbf{T}}}\mathbf{.P}-(\text{vec }\mathbf{T})\wedge\mathbf{P}.$$

A geometrical meaning will later * be attached to the term $-(\text{vec }\mathbf{T})\wedge\mathbf{P}$. We now investigate the geometrical meaning of $\bar{\bar{\mathbf{T}}}\mathbf{.P}$. For simplicity of notation, take $\mathbf{T}$ to be self-conjugate, so that $\mathbf{T}=\bar{\mathbf{T}}=\bar{\bar{\mathbf{T}}}$. Then, if we take a representation OP of $\mathbf{P}$ with O a fixed point, as P describes any locus Q describes another locus. Now impose on $\mathbf{P}$ the restriction

$$(\mathbf{T.P}).\mathbf{P}=\text{const.},$$

or

$$\mathbf{T}:\mathbf{PP}=\text{const.}$$

Since $\mathbf{T}$ is self-conjugate, if $\mathbf{P}$ has components (x, y, z) in any triad of reference, $\mathbf{T}$ components $T_{\alpha\beta}$ in the same triad, then the above restriction gives the relation

$$T_{11}x^2+T_{22}y^2+T_{33}z^2+2T_{23}yz+2T_{31}zx+2T_{12}xy=\text{const.}$$

The locus of (x, y, z) is accordingly a quadric, centre O. Since $\mathbf{T}:\mathbf{PP}=$ const. is a tensor relation, the locus of P is the same surface whatever the triad of reference; in other words, referred to a triad of reference $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ in which $\mathbf{T}$ has components $T'_{\alpha\beta}$, the equation of the same quadric is

$$T'_{11}x'^2+\ \dots\ +2T_{23}y'z'+\ \dots\ =\text{const.}$$

The quadric thus affords a representation of $\mathbf{T}$ independent of the triad of reference.

Now consider a neighbouring vector $\mathbf{P}+d\mathbf{P}$, satisfying the same relation. The 'small' vector $d\mathbf{P}$ then satisfies the relation

$$\mathbf{T}:(\mathbf{P}d\mathbf{P}+d\mathbf{PP})=0,$$

or, since $\mathbf{T}$ is self-conjugate,

$$(\mathbf{T.P}).d\mathbf{P}=0,$$

i.e.

$$\mathbf{Q}.d\mathbf{P}=0.$$

Hence all small vectors $d\mathbf{P}$ satisfying this relation are perpendicular to $\mathbf{Q}$. But all such small vectors lie in the tangent plane at P to the quadric.

* See Chapter VIII.

Hence $\mathbf{Q}$ is perpendicular to the tangent plane at P to the quadric $\mathbf{T{:}PP}=$ const. Thus the result of operating with $\mathbf{T}$ on $\mathbf{P}$ is to yield a vector $\mathbf{Q}$ normal to the tangent plane at $\mathbf{P}$ to the quadric defined by $\mathbf{T}$; the relation between the *directions* of $\mathbf{P}$ and $\mathbf{Q}$ is that of radius vector and corresponding normal.

It is a property of a quadric that it possesses three mutually perpendicular *principal axes*, with the property that they are respectively parallel to the normals at their extremities. Referred to the triad $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ defined by these axes, the quadric considered above has for its equation

$$T_{11}x^2+T_{22}y^2+T_{33}z^2=\text{const.},$$

and this suggests that any self-conjugate tensor $\mathbf{T}$ has associated with it a triad $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ such that referred to $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the tensor is of the form

$$\mathbf{T}=T_{11}\mathbf{ii}+T_{22}\mathbf{jj}+T_{33}\mathbf{kk},$$

with $T_{23}=T_{31}=T_{12}=0$. We proceed to establish this result without appealing to known geometrical properties of quadrics, simply from properties of vectors and tensors already established.

83. *Principal axes of a tensor.*

Theorem: Any self-conjugate tensor $\mathbf{T}$ has associated with it an orthogonal triad of unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ such that referred to this triad the tensor takes the form

$$\mathbf{T}=a\mathbf{ii}+b\mathbf{jj}+c\mathbf{kk}.$$

If such a representation of $\mathbf{T}$ is possible, then

$$\mathbf{T.i}=a\mathbf{i}, \qquad \mathbf{T.j}=b\mathbf{j}, \qquad \mathbf{T.k}=c\mathbf{k},$$

and hence the vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ must be unit vectors satisfying

$$\mathbf{T.r}=\lambda\mathbf{r}.$$

The solution of this equation constitutes what is sometimes called an *eigenwert* problem. We proceed to investigate the solutions of this equation considered as an equation in $\mathbf{r}$.

Consider the vectors $\mathbf{r}$ for which $\mathbf{T{:}rr}$ is a maximum or a minimum subject to $\mathbf{r}^2=1$. (The present scalar under discussion, $\mathbf{T{:}rr}$, is, of course, a variable, as contrasted with the *constant* value occurring in the equation of the quadric, $\mathbf{T{:}rr}=$const.) Since $\mathbf{T{:}rr}$ is bounded, there is at least one value of $\mathbf{r}$ for which $\mathbf{T{:}rr}$ is a maximum, and at least one value of $\mathbf{r}$ for which it is a minimum. At such a maximum or minimum

$$\mathbf{T}{:}(\mathbf{r}d\mathbf{r}+d\mathbf{r}\mathbf{r})=0,$$

or, since $\mathbf{T}$ is self-conjugate, $\quad \mathbf{T.r}.d\mathbf{r}=0,$

for all vectors $\mathbf{r}$ satisfying $\quad \mathbf{r}.d\mathbf{r}=0.$

Any vector $d\mathbf{r}$ satisfying the latter equation, being perpendicular to $\mathbf{r}$, must be of the form

$$d\mathbf{r}=\mathbf{s}\wedge\mathbf{r},$$

where $\mathbf{s}$ is arbitrary. Hence

$$(\mathbf{T.r}).(\mathbf{s}\wedge\mathbf{r})=0,$$

so that $\mathbf{T.r}$, $\mathbf{r}$ and $\mathbf{s}$ are coplanar for all vectors $\mathbf{s}$. Hence $\mathbf{T.r}$ and $\mathbf{r}$ must be parallel, or

$$\mathbf{T.r}=\lambda\mathbf{r}.$$

This holds good at any value of $\mathbf{r}$ for which $\mathbf{T{:}rr}$ is a maximum or a minimum.* Moreover at such a value of $\mathbf{r}$, $\mathbf{T{:}rr}=\lambda\mathbf{r}^2=\lambda$, so that λ is the numerical value of the corresponding maximum or minimum.

We have now different cases to consider.

If the maximum and minimum of $\mathbf{T{:}rr}$ coincide, $\mathbf{T{:}rr}$ must be constant for all $\mathbf{r}$ satisfying $\mathbf{r}^2=1$, and hence $\mathbf{T.r}$ is parallel to $\mathbf{r}$ for all $\mathbf{r}$. Hence $\mathbf{T.r}=\lambda\mathbf{r}$ for all $\mathbf{r}$, where λ is some constant and so

$$(\mathbf{T}-\lambda\mathbf{U}).\mathbf{r}=0$$

for all $\mathbf{r}$. Hence $$\mathbf{T}=\lambda\mathbf{U}=\lambda(\mathbf{ii}+\mathbf{jj}+\mathbf{kk}),$$

and the theorem is established for any triad of reference.

If the maximum and minimum of $\mathbf{T{:}rr}$ do not coincide, there must be at least two distinct vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ such that

$$\mathbf{T.r}_1=\lambda_1\mathbf{r}_1, \qquad \mathbf{T.r}_2=\lambda_2\mathbf{r}_2,$$

where, since λ_1 (say) corresponds to a maximum and λ_2 to a minimum we must have $\lambda_1>\lambda_2$. It follows that

$$(\mathbf{T.r}_1).\mathbf{r}_2=\lambda_1(\mathbf{r}_1.\mathbf{r}_2), \qquad (\mathbf{T.r}_2).\mathbf{r}_1=\lambda_2(\mathbf{r}_2.\mathbf{r}_1).$$

But since $\mathbf{T}$ is self-conjugate,

$$(\mathbf{T.r}_1).\mathbf{r}_2=(\mathbf{T.r}_2).\mathbf{r}_1,$$

whence $$(\lambda_1-\lambda_2)\mathbf{r}_1.\mathbf{r}_2=0.$$

But $\lambda_1\neq\lambda_2$. Hence $\mathbf{r}_1.\mathbf{r}_2=0$.

Write $\mathbf{r}_1=\mathbf{i}$, $\mathbf{r}_2=\mathbf{j}$, and take $\mathbf{k}=\mathbf{i}\wedge\mathbf{j}$. Then $\mathbf{T.i}=\lambda_1\mathbf{i}$, $\mathbf{T.j}=\lambda_2\mathbf{j}$. Now since T is self-conjugate, $(\mathbf{T.k}).\mathbf{i}=(\mathbf{T.i}).\mathbf{k}$. The right-hand side here is $(\lambda_1\mathbf{i}).\mathbf{k}$ which is zero. Hence $\mathbf{T.k}$ is perpendicular to $\mathbf{i}$. Similarly $\mathbf{T.k}$ is perpendicular to $\mathbf{j}$. Hence $\mathbf{T.k}$ must be parallel to $\mathbf{k}$, say

$$\mathbf{T.k}=\lambda_3\mathbf{k}.$$

But since $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are linearly independent, $\mathbf{T}$ can be expressed in dyadic form as

$$\mathbf{T}=\mathbf{Ai}+\mathbf{Bj}+\mathbf{Ck}$$

where $$\mathbf{T.i}=\mathbf{A}, \qquad \mathbf{T.j}=\mathbf{B}, \qquad \mathbf{T.k}=\mathbf{C}.$$

Hence $$\mathbf{T}=\lambda_1\mathbf{ii}+\lambda_2\mathbf{jj}+\lambda_3\mathbf{kk}.$$

This is the required result.

Let the unit vector $\mathbf{r}$ be of the form

$$\mathbf{r}=\alpha\mathbf{i}+\beta\mathbf{j}+\gamma\mathbf{k},$$

* The converse of this, as we shall see shortly, is not true.

where $$\alpha^2+\beta^2+\gamma^2=1.$$
Then $$\mathbf{T}:\mathbf{rr}=\lambda_1\alpha^2+\lambda_2\beta^2+\lambda_3\gamma^2.$$
But $\mathbf{T}:\mathbf{rr}$ reaches its greatest value λ_1 for $\mathbf{r}=\mathbf{i}$. Hence
$$\lambda_1\alpha^2+\lambda_2\beta^2+\lambda_3\gamma^2\leqslant\lambda_1=\lambda_1(\alpha^2+\beta^2+\gamma^2).$$
Hence $$\beta^2(\lambda_2-\lambda_1)+\gamma^2(\lambda_3-\lambda_1)\leqslant 0$$
for all β, γ satisfying $\beta^2+\gamma^2\leqslant 1$. Taking $\beta=0$ we have
$$\lambda_3\leqslant\lambda_1.$$
Similarly, $\mathbf{T}:\mathbf{rr}$ reaches its least value λ_2 for $\mathbf{r}=\mathbf{j}$; hence we can show similarly that
$$\lambda_3\geqslant\lambda_2.$$
Thus $$\lambda_1\geqslant\lambda_3\geqslant\lambda_2.$$

In the case where, say, $\lambda_3=\lambda_1>\lambda_2$, the vector $\mathbf{i}$ is clearly not unique. In fact, if $\mathbf{r}$ is any vector in the plane of $\mathbf{i}$ and $\mathbf{k}$, say $\mathbf{r}=\alpha\mathbf{i}+\gamma\mathbf{k}$, then
$$\mathbf{T}.\mathbf{r}=\alpha(\mathbf{T}.\mathbf{i})+\gamma(\mathbf{T}.\mathbf{k})=\alpha\lambda_1\mathbf{i}+\gamma\lambda_3\mathbf{k}=\lambda_1\mathbf{r},$$
since $\lambda_3=\lambda_1$, and so we may take as solution of $\mathbf{T}.\mathbf{r}=\lambda_1\mathbf{r}$ *any* vector perpendicular to $\mathbf{j}$. For all such vectors,
$$\mathbf{T}:\mathbf{rr}=\lambda_1\mathbf{r}^2=\lambda_1,$$
and thus $\mathbf{T}:\mathbf{rr}$ attains its greatest value for all vectors perpendicular to $\mathbf{j}$. The corresponding quadric is then an oblate spheroid with $\mathbf{j}$ for axis. Similarly, if $\lambda_3=\lambda_2<\lambda_1$, $\mathbf{T}:\mathbf{rr}$ takes its least value for all vectors perpendicular to $\mathbf{i}$, and the corresponding quadric is a prolate spheroid with $\mathbf{i}$ for axis.

To calculate the values λ_1, λ_2, λ_3 we proceed as follows. For any one of these, $\mathbf{T}.\mathbf{r}=\lambda\mathbf{r}$, where $\mathbf{r}$ is one of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, that is,
$$(\mathbf{T}-\lambda\mathbf{U}).\mathbf{r}=0.$$
Now suppose that in any other triad $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$, $\mathbf{T}$ has components T'_{11}, T'_{12}, ... and let the $\mathbf{r}$ corresponding to λ have the form
$$\mathbf{r}=\alpha'\mathbf{i}'+\beta'\mathbf{j}'+\gamma'\mathbf{k}'$$
in this triad. Then, since in this triad $\mathbf{U}=\mathbf{i}'\mathbf{i}'+\mathbf{j}'\mathbf{j}'+\mathbf{k}'\mathbf{k}'$, we have
$$[(T'_{11}-\lambda)\mathbf{i}'\mathbf{i}'+T'_{12}\mathbf{i}'\mathbf{j}'+\ \dots\ ].(\alpha'\mathbf{i}'+\beta'\mathbf{j}'+\gamma'\mathbf{k}')=0.$$
Forming the various inner products we obtain a linear function of $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ equal to zero, in which therefore the coefficients must vanish. This gives
$$\begin{aligned}(T'_{11}-\lambda)\alpha'+T'_{12}\beta'+T'_{13}\gamma'&=0\\ T'_{21}\alpha'+(T'_{22}-\lambda)\beta'+T'_{23}\gamma'&=0\\ T'_{31}\alpha'+T'_{32}\beta'+(T'_{33}-\lambda)\gamma'&=0.\end{aligned}$$
Eliminating α', β', γ' we see that λ is a root of the equation
$$\begin{vmatrix} T'_{11}-\lambda & T'_{12} & T'_{13} \\ T'_{21} & T'_{22}-\lambda & T'_{23} \\ T'_{31} & T'_{32} & T'_{33}-\lambda \end{vmatrix}=0,$$

where, it must be remembered, $T'_{23}=T'_{32}$, etc. This is a cubic in λ, whose three roots must be λ_1, λ_2, λ_3. Since we have shown that the numbers λ_1, λ_2, λ_3 exist, the roots of this cubic must be all real. The form of the cubic is

$$-\lambda^3+\lambda^2(\text{sca }\mathbf{T})-\tfrac{1}{2}\lambda[(\text{sca }\mathbf{T})^2-\mathbf{T}:\mathbf{T}]+\det \mathbf{T}=0.$$

The coefficients are, of course, scalar invariants.

The values of λ are called the characteristic values or eigen-values of the tensor. The vectors **i, j, k** are called the principal directions of the tensor.

84. *Principal axes of the inverse tensor.*

Theorem: The characteristic values of $\mathbf{T}^{-1}$ are the reciprocals of the characteristic values of **T**, and their principal directions coincide.

For, if λ is a characteristic value of **T**, **r** a unit vector along the associated principal direction, then

$$\mathbf{T}.\mathbf{r}=\lambda\mathbf{r}.$$

If λ' is a characteristic value of $\mathbf{T}^{-1}$, $\mathbf{r}'$ a unit vector along the associated principal direction, then

$$\mathbf{T}^{-1}.\mathbf{r}'=\lambda'\mathbf{r}'.$$

Operate on both sides of this equality with **T**. Then

$$\mathbf{r}'=\lambda'(\mathbf{T}.\mathbf{r}')$$

or

$$\mathbf{T}.\mathbf{r}'=(\lambda')^{-1}\mathbf{r}'.$$

But the roots of this equation in $\mathbf{r}'$ are known to be **i, j, k,** the principal directions of **T**, and the associated characteristic values are given by $(\lambda')^{-1}=\lambda_1$, λ_2, λ_3. Hence the values of λ' are λ_1^{-1}, λ_2^{-1}, λ_3^{-1} and the principal directions of $\mathbf{T}^{-1}$ are **i, j, k,** the principal directions of **T**.

*Example.** If **T** is a constant tensor, the necessary and sufficient condition that $\int \mathbf{r}.\mathbf{T}.d\mathbf{r}$ taken round any closed curve shall vanish is that **T** shall be self-conjugate.

For,

$$d(\mathbf{r}.\mathbf{T}.\mathbf{r})=d\mathbf{r}.\mathbf{T}.\mathbf{r}+\mathbf{r}.\mathbf{T}.d\mathbf{r},$$

whence integrating round any circuit

$$\int d\mathbf{r}.\mathbf{T}.\mathbf{r}=-\int \mathbf{r}.\mathbf{T}.d\mathbf{r}.$$

Hence

$$\begin{aligned}\int \mathbf{r}.\mathbf{T}.d\mathbf{r}&=\tfrac{1}{2}\int [\mathbf{r}.\mathbf{T}.d\mathbf{r}-d\mathbf{r}.\mathbf{T}.\mathbf{r}]\\&=\tfrac{1}{2}\int \mathbf{r}.(\mathbf{T}-\overline{\mathbf{T}}).d\mathbf{r}\\&=-\int (\mathbf{r}\wedge\text{vec }\mathbf{T}).d\mathbf{r}\\&=(\text{vec }\mathbf{T}).\int \mathbf{r}\wedge d\mathbf{r}.\end{aligned}$$

But $\int \mathbf{r}\wedge d\mathbf{r}$ is equal to twice the vector area of the open surface bounded by the closed curve. This is non-zero. Hence vec $\mathbf{T}=0$. Hence **T** must be self-conjugate.

*Due to D. R. Hartree.

85. *Complex numbers as tensors in two dimensions.* The well-known elementary theory of the Argand diagram for the representation of complex numbers shows that a complex number may be regarded as a two-dimensional vector, i.e. a vector restricted to lie in a given plane. The *sum* of two complex numbers is then represented by the vector sum of the vectors representing the two complex numbers. The *product* of two complex numbers is, however, less easily connected with a purely vectorial representation of complex numbers. The following considerations distinguish between the notion of a complex number as an argument or *operand*, and the notion of a complex number as an *operator* ; and they serve as an interesting application of the theory of tensors and dyadics.*

86. We shall denote by I the symbol $\sqrt{-1}$ as ordinarily used in the notation of the complex variable. (We avoid using i, to avoid confusion with the unit vector **i**.)

Consider two complex numbers z and α, given by

$$z = x + Iy,$$

$$\alpha = a + Ib.$$

Then z may be represented by the vector

$$\mathbf{z} = x\mathbf{i} + y\mathbf{j},$$

where **i**, **j** are unit vectors in the direction of the real and imaginary axes in the Argand diagram. Similarly α may be considered as the vector

$$\boldsymbol{\alpha} = a\mathbf{i} + b\mathbf{j}.$$

Now consider the product of αz, given by

$$\alpha z = (a + Ib)(x + Iy) = (ax - by) + I(bx + ay).$$

This, being a complex number, may similarly be represented by the vector

$$(ax - by)\mathbf{i} + (bx + ay)\mathbf{j}.$$

The latter vector is derived from the vector $x\mathbf{i} + y\mathbf{j}$ by operating with the complex number α on the complex number z. It suggests itself, therefore, that we should be able to represent α, alternatively, by a linear vector operator or tensor **T**, whose inner product with the vector **z** generates the vector representing αz. And the question arises as to the determination of this tensor **T** in terms of the vector $\boldsymbol{\alpha}$.

To investigate this question, let w denote the product of complex numbers αz, **w** the corresponding vector, and write

$$\mathbf{T}.\mathbf{z} = \mathbf{w}.$$

We now attempt to represent **w** as the result of a two-dimensional tensor acting on **z**, by rearrangement of the terms of **w**, thus :

$$\mathbf{w} = (ax - by)\mathbf{i} + (bx + ay)\mathbf{j}$$

$$= a(x\mathbf{i} + y\mathbf{j}) + b(x\mathbf{j} - y\mathbf{i}).$$

* Cf. E. T. Copson, *Theory of Functions of a Complex Variable* (Oxford, 1935).

Now $$\mathbf{x i}+\mathbf{y j}=(\mathbf{ii}+\mathbf{jj}).(\mathbf{x i}+\mathbf{y j}),$$
and $$\mathbf{x j}-\mathbf{y i}=(-\mathbf{ij}+\mathbf{ji}).(\mathbf{x i}+\mathbf{y j}).$$
Hence $$\mathbf{w}=[\mathrm{a}(\mathbf{ii}+\mathbf{jj})+\mathrm{b}(-\mathbf{ij}+\mathbf{ji})].\mathbf{z}.$$

It follows that the operation of multiplying the complex number z by the complex number α can also be represented by the inner product of the tensor $\mathbf{T}$ with the vector $\mathbf{z}$, where

$$\mathbf{T}=\mathrm{a}(\mathbf{ii}+\mathbf{jj})+\mathrm{b}(-\mathbf{ij}+\mathbf{ji}).$$

We now wish to express $\mathbf{T}$ in terms of $\boldsymbol{\alpha}$. We have

$$\begin{aligned}\mathbf{T}&=(\mathrm{a}\mathbf{i}+\mathrm{b}\mathbf{j})\mathbf{i}+(\mathrm{a}\mathbf{j}-\mathrm{b}\mathbf{i})\mathbf{j}\\ &=\boldsymbol{\alpha}\mathbf{i}+(\mathbf{k}\wedge\boldsymbol{\alpha})\mathbf{j},\end{aligned}$$

where $\mathbf{k}$ is the vector normal to the plane of the complex variable z and such that $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form a positive triad, i.e. such that $\mathbf{k}=\mathbf{i}\wedge\mathbf{j}$. This gives $\mathbf{T}$ as a dyadic.

Since we have now expressed $\mathbf{T}$ uniquely in terms of $\boldsymbol{\alpha}$, it follows that $\mathbf{T}$ affords a representation of α. The components of $\mathbf{T}$ with respect to the fundamental vectors $\mathbf{i}, \mathbf{j}$ are

$$T_{\mu\nu}=\begin{array}{c|cc} & \nu=1 & \nu=2\\ \hline \mu=1 & \mathrm{a} & -\mathrm{b}\\ \mu=2 & \mathrm{b} & \mathrm{a}.\end{array}$$

We now see that a complex number is capable of two distinct representations, one a vector, the other a tensor. The one is appropriate to the complex number as *argument*, the other to the complex number as *operator*.

87. The product $w=\alpha z$ may also be regarded as the product $z\alpha$. We can therefore alternatively represent z as a tensor $\mathbf{S}$, of components

$$S_{\mu\nu}=\begin{array}{c|cc} & \nu=1 & \nu=2\\ \hline \mu=1 & \mathrm{x} & -\mathrm{y}\\ \mu=2 & \mathrm{y} & \mathrm{x},\end{array}$$

or, in dyadic form, $$\mathbf{S}=\mathbf{z}\mathbf{i}+(\mathbf{k}\wedge\mathbf{z})\mathbf{j}.$$

Since the product w is independent of the order of the factors, we must have

$$\mathbf{w}=\mathbf{T}.\mathbf{z}=\mathbf{S}.\boldsymbol{\alpha}.$$

This gives us the identity

$$[\boldsymbol{\alpha}\mathbf{i}+(\mathbf{k}\wedge\boldsymbol{\alpha})\mathbf{j}].\mathbf{z}=[\mathbf{z}\mathbf{i}+(\mathbf{k}\wedge\mathbf{z})\mathbf{j}].\boldsymbol{\alpha},$$

where $\mathbf{z}, \boldsymbol{\alpha}$ are any two vectors in the plane of $\mathbf{i}$ and $\mathbf{j}$. The verification of this identity is left as an exercise for the reader.

88. The product $\alpha_1\alpha_2 z$, where α_1, α_2 are two complex numbers, if $\mathbf{T}_1, \mathbf{T}_2$ are their corresponding tensor representations, may be represented as

$$\mathbf{T}_1.(\mathbf{T}_2.\mathbf{z}),$$

or as $$\mathbf{T}.\mathbf{z},$$

where $\mathbf{T}$ represents $\alpha_1\alpha_2$. It follows that

$$\mathbf{T}=\mathbf{T}_1.\mathbf{T}_2, \qquad \mathbf{T}=\mathbf{T}_2.\mathbf{T}_1,$$

and similarly for any number of factors. Thus the product of two complex numbers regarded as operators is represented by the inner product of the corresponding tensors.

89. *Conjugate complex numbers.* The complex number $\bar{\alpha}$ conjugate to α is $\mathrm{a}-\mathrm{Ib}$. This is accordingly represented by the tensor

$$\begin{aligned}&\mathrm{a}(\mathbf{ii}+\mathbf{jj})+\mathrm{b}(\mathbf{ij}-\mathbf{ji})\\ &=\mathbf{i}(\mathrm{a}\mathbf{i}+\mathrm{b}\mathbf{j})+\mathbf{j}(\mathrm{a}\mathbf{j}-\mathrm{b}\mathbf{i})\\ &=\mathbf{i}\boldsymbol{\alpha}+\mathbf{j}(\mathbf{k}\wedge\boldsymbol{\alpha}).\end{aligned}$$

This is just $\overline{\mathbf{T}}$, the tensor conjugate to $\mathbf{T}$. Thus, if $\mathbf{T}$ represents a complex number α, the conjugate tensor $\overline{\mathbf{T}}$ represents the conjugate complex number $\bar{\alpha}$. (The word conjugate has, of course, a different significance in these two phrases.)

90. *The inverse of a complex number.* Whilst we can represent a complex number by a vector, and its inverse by another vector, we cannot say that the two vectors are inverse of one another, for we have given no meaning to the inverse of a vector. A tensor, however, has in general an inverse, and we therefore inquire whether the inverse of a complex number is represented by the inverse of the corresponding tensor.

The inverse α^{-1} of the complex number $\alpha=\mathrm{a}+\mathrm{Ib}$ is

$$\frac{1}{\alpha}=\frac{\mathrm{a}-\mathrm{Ib}}{\mathrm{a}^2+\mathrm{b}^2},$$

which is

$$\frac{\bar{\alpha}}{|\alpha|^2}.$$

The corresponding tensor, if $\mathbf{T}$ represents α, is

$$\frac{\overline{\mathbf{T}}}{|\boldsymbol{\alpha}|^2}, \quad\text{or}\quad \frac{\mathbf{i}\boldsymbol{\alpha}+\mathbf{j}(\mathbf{k}\wedge\boldsymbol{\alpha})}{|\boldsymbol{\alpha}|^2},$$

The question is whether this is equal to $\mathbf{T}^{-1}$.

We have considered in § 81 the inverse of a two-dimensional dyadic, and shown that in two dimensions if

$$\mathbf{T}=\mathbf{AX}+\mathbf{BY},$$

then

$$\mathbf{T}^{-1}=\frac{(\mathbf{k}\wedge\mathbf{X})(\mathbf{k}\wedge\mathbf{A})+(\mathbf{k}\wedge\mathbf{Y})(\mathbf{k}\wedge\mathbf{B})}{(\mathbf{k}\wedge\mathbf{A}.\mathbf{B})(\mathbf{k}\wedge\mathbf{X}.\mathbf{Y})}.$$

Taking the form

$$\mathbf{T}=\boldsymbol{\alpha}\mathbf{i}+(\mathbf{k}\wedge\boldsymbol{\alpha})\mathbf{j}$$

and forming its inverse we have

$$\mathbf{T}^{-1}=\frac{\mathbf{j}(\mathbf{k}\wedge\boldsymbol{\alpha})+(-\mathbf{i})(-\boldsymbol{\alpha})}{\boldsymbol{\alpha}^2}=\frac{\overline{\mathbf{T}}}{\boldsymbol{\alpha}^2}.$$

Thus the inverse tensor, taken in the sense of two dimensions, represents the inverse of the complex number. This actually follows from the relations

$$\mathbf{T}.\mathbf{z}=\mathbf{w}, \qquad \mathbf{z}=\mathbf{T}^{-1}.\mathbf{w}, \qquad z=\frac{1}{\alpha}(\alpha z),$$

but we have thought it worth while to link up with our standard tensor analysis.

91. The above two-dimensional considerations may be regarded as illustrating the circumstance that the complex number is in effect a particular case of the three-dimensional tensor, simplified to two dimensions. Just as complex numbers may be added, multiplied and divided, so tensors in three dimensions may be added, multiplied and divided. The tensor is therefore a genuine generalization to three dimensions of the notion of a complex number. But its rôle is that of an *operator*, and the idea of a tensor needs to be completed by the idea of a vector as *operand*. The complete parallel with complex numbers is obscured by the simplifying circumstance that a complex number plays the parts of both operator and operand.

CHAPTER IV

INTEGRAL THEOREMS

92. *Definition of a function of a vector.* Suppose that φ denotes a number which is known whenever a vector $\mathbf{r}$ is specified. We have already seen (§ 19) that φ is said to be a scalar function of the vector $\mathbf{r}$, $\varphi(\mathbf{r})$. If we wish, we may regard $\mathbf{r}$ as the position vector OP of a variable point P with respect to some fixed point O, and we may then regard φ as a function of the position of P. The analysis which follows is, however, independent of this interpretation. If, with respect to a triad of reference $\mathbf{i}, \mathbf{j}, \mathbf{k}$, $\mathbf{r}$ has components x, y, z, then φ may be regarded as a function of the three variables x, y, z. More concisely, we can say that in the triad $\mathbf{i}_\alpha$, $\mathbf{r}$ has components r_α, and that φ or $\varphi(x, y, z)$ may be written $\varphi(r_\alpha)$. The function φ is not, however, a *general* function of three variables x, y, z, for its value at $\mathbf{r}$ is independent of the triad of reference ; φ is thus a scalar invariant.

We may define similarly a vector function of a vector. If $\mathbf{F}$ is a vector whose value is given when $\mathbf{r}$ is given, $\mathbf{F}$ is said to be a vector function of $\mathbf{r}$. If r_α are the components of $\mathbf{r}$ in a triad $\mathbf{i}_\alpha$, the components of $\mathbf{F}$ in the same triad will be three functions $F_1(r_\alpha)$, $F_2(r_\alpha)$, $F_3(r_\alpha)$, or, say $\mathbf{F}(\mathbf{r})$.

93. *The gradient of a scalar function.*

Theorem : The three partial derivatives

$$\frac{\partial\varphi}{\partial r_\alpha} \qquad (\alpha=1, 2, 3)$$

of a scalar function φ with respect to the components r_α of $\mathbf{r}$ in any triad are the components of a vector in the same triad.

For, take a neighbouring vector $\mathbf{r}+d\mathbf{r}$, of components $r_\alpha+dr_\alpha$ in the same triad. Then by the standard formula of partial differentiation, the differential $d\varphi$ of the function φ is given by

$$d\varphi=\frac{\partial\varphi}{\partial r_\alpha}dr_\alpha.$$

But dr_α, $(\alpha=1, 2, 3)$, are the components of an arbitrary vector $d\mathbf{r}$, and $d\varphi$ is a scalar. Hence, by the quotient theorem (§ 56) the numbers $\dfrac{\partial\varphi}{\partial r_\alpha}$, $(\alpha=1, 2, 3)$, are the components of a vector in the given triad.

This vector is a vector function of $\mathbf{r}$. It is called the gradient of φ and is written grad φ, or $\nabla\varphi$, or $\dfrac{\partial\varphi}{\partial\mathbf{r}}$. In the triad $\mathbf{i}, \mathbf{j}, \mathbf{k}$ we have

$$\frac{\partial\varphi}{\partial r}=\frac{\partial\varphi}{\partial x}\mathbf{i}+\frac{\partial\varphi}{\partial y}\mathbf{j}+\frac{\partial\varphi}{\partial z}\mathbf{k}.$$

It is instructive to verify by direct transformation that the three numbers $\dfrac{\partial\varphi}{\partial r_\alpha}$ are the components of a vector. In any other triad $(\mathbf{i}', \mathbf{j}', \mathbf{k}')$ the numbers $\dfrac{\partial\varphi}{\partial r'_\alpha}$ are given by

$$\frac{\partial\varphi}{\partial r'_\alpha}=\frac{\partial\varphi}{\partial r_\mu}\,\frac{\partial r_\mu}{\partial r'_\alpha}.$$

But since r_α are the components of a vector, we have

$$r_\mu=l_{\nu\mu}r'_\nu, \qquad (l_{\nu\mu}=\mathbf{i}'_\nu.\mathbf{i}_\mu)$$

and

$$\frac{\partial r'}{\partial r'_\alpha}=\delta_{\nu\alpha}.$$

Hence

$$\frac{\partial r_\mu}{\partial r'_\alpha}=l_{\nu\mu}\delta_{\nu\alpha}=l_{\alpha\mu}.$$

Hence

$$\frac{\partial\varphi}{\partial r'_\alpha}=l_{\alpha\mu}\frac{\partial\varphi}{\partial r_\mu},$$

so that the numbers $\partial\varphi/\partial r'_\alpha$ obey the vector transformation.

We now have, for any scalar function φ, a function of a vector $\mathbf{r}$, the general relation

$$d\varphi=(\text{grad }\varphi).d\mathbf{r}.$$

Example (1). If $\mathbf{A}$ is a constant vector, and $\varphi=\mathbf{r}.\mathbf{A}$, then grad $\varphi=\mathbf{A}$.

Example (2). If $\varphi=\mathbf{r}^2$, grad $\varphi=2\mathbf{r}$.

Example (3). If $\psi=|\mathbf{r}|$, $\varphi=\mathbf{r}^2$, then $\psi=\varphi^{\frac{1}{2}}$ and grad $\psi=\frac{1}{2}\varphi^{-\frac{1}{2}}$ grad φ, or grad $\psi=\mathbf{r}/|\mathbf{r}|$, by Example (2). Cf. § 19.

94. *The gradient of a vector function.*

Theorem: If F_β, $(\beta=1, 2, 3)$, are the components, in any triad, of a vector function $\mathbf{F}$ of $\mathbf{r}$ (whose components in a triad $\mathbf{i}_\alpha$ are r_α), then the nine partial differential coefficients

$$\frac{\partial F_\beta}{\partial r_\alpha}$$

are the components $T_{\alpha\beta}$ of a tensor $\mathbf{T}$ in the same triad.

For, as before, we have

$$dF_\beta=dr_\alpha\frac{\partial F_\beta}{\partial r_\alpha},$$

and since the numbers dF_β form a vector for arbitrary vectors $d\mathbf{r}$, partial differential coefficients $\partial F_{\beta\alpha}/\partial r$ must be the components of

The components may be written $(\text{grad } \mathbf{F})_{\alpha\beta}$, and the tensor may be described as grad $\mathbf{F}$, $\nabla\mathbf{F}$ or $\partial\mathbf{F}/\partial\mathbf{r}$. Thus

$$d\mathbf{F} = d\mathbf{r}.\text{ grad }\mathbf{F}.$$

The components $(\text{grad } \mathbf{F})_{\alpha\beta}$ in any triad are given by the scheme :

	$\beta=1$	$\beta=2$	$\beta=3$
$\alpha=1$	$\frac{\partial F_1}{\partial x}$,	$\frac{\partial F_2}{\partial x}$,	$\frac{\partial F_3}{\partial x}$,
$\alpha=2$	$\frac{\partial F_1}{\partial y}$,	$\frac{\partial F_2}{\partial y}$,	$\frac{\partial F_3}{\partial y}$,
$\alpha=3$	$\frac{\partial F_1}{\partial z}$,	$\frac{\partial F_2}{\partial z}$,	$\frac{\partial F_3}{\partial z}$.

Example (1). If $\mathbf{F} = \mathbf{r}.\mathbf{T}$, where $\mathbf{T}$ is a constant tensor, then

$$\text{grad } \mathbf{F} = \mathbf{T}.$$

Example (2). If $\mathbf{F} = \mathbf{T}.\mathbf{r}$, where $\mathbf{T}$ is a constant tensor, then

$$\text{grad } \mathbf{F} = \overline{\mathbf{T}}.$$

Example (3). If $\mathbf{F} = \mathbf{r}\wedge\mathbf{X}$, where $\mathbf{X}$ is a constant vector, then

$$\text{grad } \mathbf{F} = \mathrm{U}\wedge\mathbf{X} = -\text{tens } \mathbf{X}.$$

For $$d\mathbf{F} = d\mathbf{r}\wedge\mathbf{X} = d\mathbf{r}.(\mathrm{U}\wedge\mathbf{X}) = -d\mathbf{r}.\text{ tens } \mathbf{X}.$$

This is readily verified directly. For

$$[\text{grad } (\mathbf{r}\wedge\mathbf{X})]_{\alpha\beta} = \frac{\partial}{\partial r_\alpha}[A_{\beta\mu\nu}r_\mu X_\nu] = A_{\beta\mu\nu}\delta_{\mu\alpha}X_\nu = -A_{\alpha\beta\nu}X_\nu = -(\text{tens } \mathbf{X})_{\alpha\beta}.$$

95. *The divergence of a vector function of a vector.* If $\mathbf{F}$ is a vector function of $\mathbf{r}$, its gradient (grad $\mathbf{F}$), being a tensor, possesses a 'scalar'—the sum of the diagonal terms in any triad of reference—obtained by equating indices and summing. This invariant is called div $\mathbf{F}$. Thus

$$\text{div } \mathbf{F} = \frac{\partial F_\alpha}{\partial r_\alpha} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

It is sometimes convenient to write this in the form

$$\text{div } \mathbf{F} = \nabla.\mathbf{F},$$

by analogy with the scalar product ; the operator ∇ has the formal properties of a vector, as has already appeared in the notation $\nabla\varphi$ for the vector grad φ and $\nabla\mathbf{F}$ for the tensor grad $\mathbf{F}$. The operator ∇ is just the operator $\partial/\partial\mathbf{r}$, and its components in any triad are $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$.

Example (1). If $\mathbf{F} = \mathbf{r}$, div $\mathbf{F} = 3$.

Example (2). If $\mathbf{F} = \mathbf{r}.\mathbf{T}$, where $\mathbf{T}$ is a constant tensor, then

$$\text{div } \mathbf{F} = \text{sca } \mathbf{T}.$$

Example (3). If $\mathbf{F} = \mathbf{r}\varphi$, where φ is a scalar function,

$$\text{div } \mathbf{F} = 3\varphi + \mathbf{r}.\text{ grad } \varphi.$$

96. *The divergence of a tensor function.* Similarly we may form the gradient of a tensor function **T** of rank 2 (which will be a tensor of rank 3), and by contraction obtain a vector. This vector is called the divergence of the original tensor, and written div **T** or $\nabla.\mathbf{T}$. Thus

$$(\text{div }\mathbf{T})_\alpha=(\nabla.\mathbf{T})_\alpha=\frac{\partial}{\partial r_\mu}T_{\mu\alpha}.$$

Thus in the triad in which the components of $\mathbf{r}$ are x, y, z, the x-component of div **T** is

$$\frac{\partial T_{11}}{\partial x}+\frac{\partial T_{21}}{\partial y}+\frac{\partial T_{31}}{\partial z},$$

and similarly for the y- and z-components.

97. *The Laplacian of a scalar function.* If φ is a scalar function of a vector, the divergence of its gradient is another scalar, which is called the *Laplacian* of φ and is written $\nabla^2\varphi$. Thus, in the triad in which the components of $\mathbf{r}$ are x, y, z, the value of $\nabla^2\varphi$ is given by

$$\nabla^2\varphi=\frac{\partial^2\varphi}{\partial r_\alpha\partial r_\alpha}=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=\text{div (grad }\varphi).$$

It should be noted that it is convenient to write $\frac{\partial^2\varphi}{\partial r_\alpha\partial r_\alpha}$ and *not* $\frac{\partial^2\varphi}{\partial {r_\alpha}^2}$, so as to put in evidence the repetition of the suffix α which is necessary to secure summation.

Example. If $\varphi=\frac{1}{|\mathbf{r}|}$, $\nabla^2\varphi=0$.

For
$$\text{grad }\frac{1}{|\mathbf{r}|}=-\frac{1}{|\mathbf{r}|^2}\text{ grad }|\mathbf{r}|=-\frac{\mathbf{r}}{|\mathbf{r}|^3},$$

and so
$$\text{div grad }\frac{1}{|\mathbf{r}|}=-\frac{1}{|\mathbf{r}|^3}\text{ div }\mathbf{r}-\mathbf{r}.\text{ grad }\left(\frac{1}{|\mathbf{r}|^3}\right)$$
$$=-\frac{3}{|\mathbf{r}|^3}+3\frac{\mathbf{r}}{|\mathbf{r}|^4}.\frac{\mathbf{r}}{|\mathbf{r}|}\equiv 0.$$

This is a fundamental result in potential theory.

98. Similarly we can define the Laplacian of a vector function as the divergence of its gradient. The Laplacian of a vector function **F** is itself a vector whose components are given by

$$(\nabla^2\mathbf{F})_\beta=\frac{\partial^2F_\beta}{\partial x_\alpha\partial x_\alpha}=\frac{\partial^2F_\beta}{\partial x^2}+\frac{\partial^2F_\beta}{\partial y^2}+\frac{\partial^2F_\beta}{\partial z^2}.$$

99. *The curl* (*or rotation*) *of a vector function.* If **F** is a vector function of $\mathbf{r}$, its gradient grad **F**, being a tensor, possesses an associated vector, vec grad **F**. It is customary to call *twice* this vector the curl or rotation

of **F**, and to write it as curl **F** (sometimes as rot **F**). This definition (cf. § 63) gives us

$$(\text{curl } \mathbf{F})_\alpha = A_{\alpha\beta\gamma}\frac{\partial}{\partial r_\beta}F_\gamma.$$

In the triad **i, j, k** in which the components of **r** are x, y, z and those of **F** are F_1, F_2, F_3, curl **F** has for its expansion

$$\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

Symbolically, we may write

$$\text{curl } \mathbf{F} = \nabla\wedge\mathbf{F} = 2 \text{ vec } (\text{grad } \mathbf{F}).$$

Once again, the operator ∇, or $\partial/\partial\mathbf{r}$, plays the part of a vector.

Example. If **X** is a constant vector, curl $(\mathbf{r}\wedge\mathbf{X}) = -2\mathbf{X}$.

For $$[\text{curl } (\mathbf{r}\wedge\mathbf{X})]_\alpha = A_{\alpha\beta\gamma}\frac{\partial}{\partial r_\beta}(\mathbf{r}\wedge\mathbf{X})_\gamma = A_{\alpha\beta\gamma}\frac{\partial}{\partial r_\beta}(A_{\gamma\mu\nu}r_\mu X_\nu)$$
$$= A_{\gamma\alpha\beta}A_{\gamma\mu\nu}\delta_{\mu\beta}X_\nu = A_{\gamma\alpha\beta}A_{\gamma\beta\nu}X_\nu = -2U_{\nu\alpha}X_\nu = -2X_\alpha.$$

In a simple example such as the foregoing, it happens to be easier to write down the three components of $\mathbf{r}\wedge\mathbf{X}$,

$$yX_3-zX_2, \qquad zX_1-xX_3, \qquad xX_2-yX_1$$

and form the components of curl $(\mathbf{r}\wedge\mathbf{X})$ from the definition. But in more complicated cases the manipulation by use of general suffixes is by far the simplest and safest, and the student is recommended to familiarize himself with this procedure. The (**A, U**) theorem will be found to be of constant application.

We can in this example attempt to proceed by formal manipulation of the symbol ∇. Thus, since

$$\mathbf{P}\wedge(\mathbf{Q}\wedge\mathbf{R}) = \mathbf{P}.(\mathbf{RQ}-\mathbf{QR}),$$

we may anticipate that

$$\begin{aligned}\nabla\wedge(\mathbf{r}\wedge\mathbf{X}) &= \nabla.(\mathbf{Xr}-\mathbf{rX})\\ &= \nabla.(\mathbf{Xr})-(\nabla.\mathbf{r})\mathbf{X}\\ &= \mathbf{X}-3\mathbf{X} = -2\mathbf{X}.\end{aligned}$$

But to verify the validity of this symbolic procedure we have in effect to go back to the definition in terms of the alternate tensor. Further, great care is required in interpreting the symbols. For example, although $\mathbf{P}.(\mathbf{QR}) = (\mathbf{P}.\mathbf{Q})\mathbf{R}$, it is *not* true that $\nabla.(\mathbf{Xr}) = (\nabla.\mathbf{X})\mathbf{r}$; for since **X** is a constant, $\nabla.\mathbf{X} = 0$, whilst actually $\nabla.(\mathbf{Xr}) = \mathbf{X}$.

100. It is customary at this stage in treatises on vector analysis to state and prove a large number of theorems involving the differential operators grad, div, curl. For example, if **X** and **Y** are vector functions

of $\mathbf{r}$, φ a scalar function of $\mathbf{r}$, $\mathbf{T}$ a tensor function of $\mathbf{r}$, we may obtain expressions for

$$\begin{array}{llll} \text{grad}\,(\mathbf{X.Y}), & \text{grad}\,(\mathbf{X}\wedge\mathbf{Y}), & \text{grad}\,(\mathbf{T{:}XY}), & \\ \text{div}\,(\mathbf{X}\wedge\mathbf{Y}), & \text{div}\,(\varphi\mathbf{X}), & \text{div}\,(\mathbf{X.T}), & \text{div}\,(\mathbf{T.X}), \\ \text{curl}\,(\mathbf{X}\wedge\mathbf{Y}), & \text{curl}\,(\varphi\mathbf{X}), & \text{curl}\,(\mathbf{X.T}), & \text{curl}\,(\mathbf{T.X})\,; \end{array}$$

and also expressions for the repeated operations

$$\text{grad div}\,\mathbf{X},\quad \text{div grad}\,\varphi,\quad \text{div curl}\,\mathbf{X},\quad \text{curl grad}\,\varphi,$$
$$\text{curl curl}\,\mathbf{X},$$

etc. Some of these are given below as examples. But the student is strongly advised not to burden his memory with them ; the more striking ones will survive in his memory without conscious effort. Instead, he should establish each expansion as he requires it, using the suffix notation and the (**A**, **U**) theorem. The suffix notation always automatically suggests the desired transformation.

Example (1). $\text{grad}\,(\mathbf{X.Y}) = (\text{grad}\,\mathbf{X}).\mathbf{Y} + (\text{grad}\,\mathbf{Y}).\mathbf{X}$.

Example (2). $\text{grad}\,(\mathbf{X}\wedge\mathbf{Y}) = (\text{grad}\,\mathbf{X})\wedge\mathbf{Y} - (\text{grad}\,\mathbf{Y})\wedge\mathbf{X}$.

Example (3). $\text{div}\,(\mathbf{X}\wedge\mathbf{Y}) = (\text{curl}\,\mathbf{X}).\mathbf{Y} - \mathbf{X}.\,\text{curl}\,\mathbf{Y}$.

Example (4). $\text{div}\,\varphi\mathbf{X} = (\text{grad}\,\varphi).\mathbf{X} + \varphi\,\text{div}\,\mathbf{X}$.

Example (5).

$$\text{curl}\,(\mathbf{X}\wedge\mathbf{Y}) = (\mathbf{Y}.\,\text{grad}\,\mathbf{X} - \mathbf{X}.\,\text{grad}\,\mathbf{Y}) + (\mathbf{X}\,\text{div}\,\mathbf{Y} - \mathbf{Y}\,\text{div}\,\mathbf{X}).$$

Example (6). $\text{curl}\,\varphi\mathbf{X} = (\text{grad}\,\varphi)\wedge\mathbf{X} + \varphi\,\text{curl}\,\mathbf{X}$.

Example (7). $\mathbf{X}\wedge\text{curl}\,\mathbf{Y} = (\text{grad}\,\mathbf{Y}).\mathbf{X} - \mathbf{X}.\,\text{grad}\,\mathbf{Y}$.

Example (8). $\text{div curl}\,\mathbf{X} = 0, \qquad \text{curl grad}\,\varphi = 0.$

Example (9). $\text{curl curl}\,\mathbf{X} = \text{grad div}\,\mathbf{X} - \nabla^2\mathbf{X}$.

The above examples, whilst exhibiting the relations between the symbols in vector or tensor form, conceal the nature of the identities. A little gain in insight is obtained occasionally if the symbol $\boldsymbol{\nabla}$ is employed. E.g. Example (9) may be written

$$\boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\mathbf{X}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}.\mathbf{X}) - \boldsymbol{\nabla}^2\mathbf{X},$$

which bears an obvious analogy to

$$\mathbf{Q}\wedge(\mathbf{Q}\wedge\mathbf{X}) = \mathbf{Q}(\mathbf{Q.X}) - \mathbf{Q}^2\mathbf{X}.$$

On the other hand Example (5) may be written

$$\boldsymbol{\nabla}\wedge(\mathbf{X}\wedge\mathbf{Y}) = (\mathbf{Y}.\boldsymbol{\nabla}\mathbf{X} - \mathbf{X}.\boldsymbol{\nabla}\mathbf{Y}) + [\mathbf{X}(\boldsymbol{\nabla}.\mathbf{Y}) - \mathbf{Y}(\boldsymbol{\nabla}.\mathbf{X})]$$

which bears no obvious analogy to

$$\mathbf{Q}\wedge(\mathbf{X}\wedge\mathbf{Y}) = \mathbf{X}(\mathbf{Q.Y}) - \mathbf{Y}(\mathbf{Q.X}).$$

To obtain a better analogy one would have to write

$$\mathbf{Q}\wedge(\mathbf{X}\wedge\mathbf{Y}) = \mathbf{Q}.(\mathbf{YX} - \mathbf{XY})$$

and replace $\mathbf{Q}$ by $\boldsymbol{\nabla}$.

101. *Green's theorem.* We shall prove this famous theorem using the suffix notation, and then deduce the well-known particular cases.

Theorem: Let S denote the surface of any closed domain V in the space defined by a position vector **r**. Let **n** denote a unit vector along the *outward* normal to any element dS of S, $d\tau$ any volume element of V. Let φ be any function (scalar, vector, tensor, ...) of **r**, of components $\varphi_{\alpha\beta\gamma\ldots}$ in any given triad. Then in the same triad

$$\int_S \varphi_{\alpha\beta\gamma\ldots} n_\mu dS = \int_V \frac{\partial \varphi_{\alpha\beta\gamma\ldots}}{\partial r_\mu} d\tau.$$

Here the suffix μ may or may not coincide with one of the suffixes $\alpha, \beta, \gamma, \ldots$ of φ; if it does so coincide, summation is implied.

*Proof.** Take an arbitrary constant unit vector **a**. Take a line parallel to **a**, meeting the surface S in two points A and B (Fig. 14), and with this line AB as axis construct an elementary cylinder C, of any small cross-section, with its generators parallel to **a**.

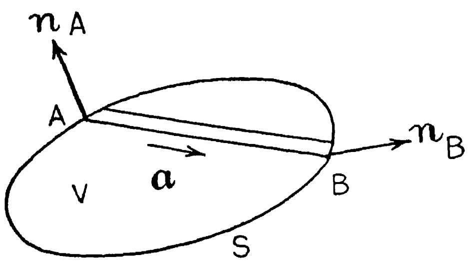

Fig. 14

For any two neighbouring points **r**, **r**+d**r**, we have

$$d\varphi_{\alpha\beta\gamma\ldots} = \frac{\partial \varphi_{\alpha\beta\gamma\ldots}}{\partial r_\mu} dr_\mu,$$

where μ is a suffix different from $\alpha, \beta, \gamma, \ldots$ and where accordingly summation is implied with respect to μ. Apply this relation to two points close together on the axis of the elementary cylinder C, so that

$$dr_\mu = a_\mu ds,$$

where ds is an element of length along the axis of C in the sense of **a**. Now integrate along this axis from A to B. The result is

$$(\varphi_{\alpha\beta\gamma\ldots})_B - (\varphi_{\alpha\beta\gamma\ldots})_A = a_\mu \int \frac{\partial \varphi_{\alpha\beta\gamma\ldots}}{\partial r_\mu} ds.$$

Now let $\mathbf{n}_A$, $\mathbf{n}_B$ be unit vectors along the outward normals at A, B, respectively, dS_A, dS_B the corresponding areas of the two ends of the cylinder, σ the normal cross-sectional area of the cylinder. Then

$$\sigma = (-\mathbf{n}_A.\mathbf{a})dS_A = -(n_\mu a_\mu dS)_A$$
$$= (+\mathbf{n}_B.\mathbf{a})dS_B = +(n_\mu a_\mu dS)_B.$$

Multiply the result of the integration by σ and use the foregoing relations.

* The idea of this method of proof was communicated to E. A. M. by Professor D. R Hartree.

Then

$$(\varphi_{\alpha\beta\gamma\ldots}n_\mu a_\mu dS)_B+(\varphi_{\alpha\beta\gamma\ldots}n_\mu a_\mu dS)_A=\sigma a_\mu\int\frac{\partial\varphi_{\alpha\beta\gamma\ldots}}{\partial r_\mu}ds.$$

But

$$\sigma ds=d\tau,$$

where $d\tau$ is an elementary volume of the cylinder. Summing the foregoing result for a set of elementary parallel cylinders completely occupying the volume V, we get

$$a_\mu\int_S\varphi_{\alpha\beta\gamma\ldots}n_\mu dS=a_\mu\int_V\frac{\partial\varphi_{\alpha\beta\gamma\ldots}}{\partial r_\mu}d\tau.$$

But a_μ is an arbitrary unit vector. Hence

$$\int_S\varphi_{\alpha\beta\gamma\ldots}n_\mu dS=\int_V\frac{\partial\varphi_{\alpha\beta\gamma\ldots}}{\partial r_\mu}d\tau.$$

This being a tensor relation, we can now contract it by replacing μ by any one of the suffixes α, β, γ, ..., and carrying out the implied summation. The theorem thus follows.

The above proof assumes that the lines parallel to **a** meet S in only two points. But since S is closed, any line parallel to **a** must meet S in an *even* number of points, such as the pairs A, B ; C, D (Fig. 15) ; and consequently the interior V can be distributed amongst cylindrical segments of the types AB, CD, ... The same result then follows.

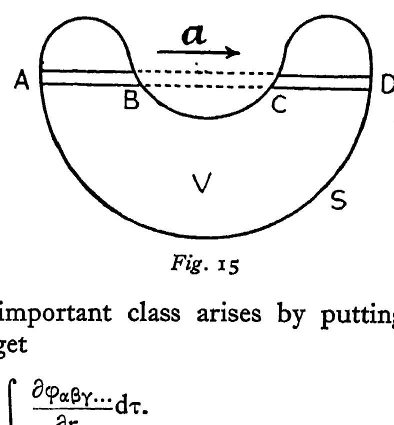

Fig. 15

The above theorem contains a large number of particular cases, which, though often separately enumerated, do not need to be separately remembered. The most important class arises by putting μ equal to α and summing, when we get

$$\int_S\varphi_{\alpha\beta\gamma\ldots}n_\alpha dS=\int_V\frac{\partial\varphi_{\alpha\beta\gamma\ldots}}{\partial r_\alpha}d\tau.$$

Some of these particular cases we now enumerate, φ (without suffix) denoting a scalar function, **Q** a vector function, **T** a tensor function of rank 2.

Example (1). For $\varphi_{\alpha\beta\gamma\ldots}$ take the scalar φ. Then

$$\int_S\varphi n_\alpha dS=\int_V\frac{\partial\varphi}{\partial r_\alpha}d\tau,$$

or, in vector notation

$$\int_S\varphi\mathbf{n}dS=\int_V\text{grad }\varphi\, d\tau.$$

Example (2). For $\varphi_{\alpha\beta\gamma\ldots}$ take the components Q_α of a vector **Q**. Then using the contracted form of the theorem we have

$$\int_S Q_\alpha n_\alpha\, dS = \int_V \frac{\partial Q_\alpha}{\partial r_\alpha} d\tau$$

or in vector notation

$$\int_S (\mathbf{Q}.\mathbf{n})dS = \int_V \text{div } \mathbf{Q}\ d\tau.$$

This is often written in the form

$$\int_S Q_n dS = \int_V \text{div } \mathbf{Q}\ d\tau,$$

where Q_n is the normal component of **Q** at the element dS.

Example (3). For $\varphi_{\alpha\beta\gamma\ldots}$ take the tensor $A_{\alpha\beta\gamma}Q_\beta$. Using the contracted form of the theorem we have

$$\int_S A_{\alpha\beta\gamma} Q_\beta n_\gamma dS = \int_V A_{\alpha\beta\gamma}\frac{\partial Q_\beta}{\partial r_\gamma} d\tau,$$

or, interchanging the suffixes β and γ in A,

$$\int_S (\mathbf{n}\wedge\mathbf{Q})dS = \int_V \text{curl } \mathbf{Q}\ d\tau.$$

Example (4). For $\varphi_{\alpha\beta\gamma\ldots}$ take the tensor $T_{\alpha\beta}$. The contracted form gives

$$\int_S T_{\alpha\beta} n_\alpha dS = \int_V \frac{\partial T_{\alpha\beta}}{\partial r_\alpha} d\tau,$$

or

$$\int_S \mathbf{n}.\mathbf{T}dS = \int_V \text{div } \mathbf{T}\ d\tau,$$

where it must be remembered that both sides are vectors.

In order to arrive rapidly at any of the foregoing special cases, the most expeditious procedure is to replace **n** by n_α and introduce suitable suffixes in the remainder of the left-hand side. The desired theorem then follows by substituting for n_α the differential operator $\partial/\partial r_\alpha$ and replacing dS by $d\tau$. The right-hand side then needs to be interpreted. It is clear that the suffix notation is the more convenient analytical expression of the facts of differential and integral calculus contained in the theorem, whilst the vector notation gives the greater physical meaning in each particular case.

102. *Consequences of Green's theorem.* Let φ, f be two scalar functions of the vector **r**. Then grad φ is a vector function of **r**, and so f grad φ is a vector function of **r**. Applying Example (2) above to this vector function we have

$$\int_S f(\mathbf{n}.\text{grad } \varphi)dS = \int_V \text{div } (f \text{ grad } \varphi)d\tau.$$

But by Example (4), § 100, or from first principles,

$$\operatorname{div}(f \operatorname{grad} \varphi) = (\operatorname{grad} f).(\operatorname{grad} \varphi) + f\nabla^2\varphi.$$

Thus
$$\int_S f(\mathbf{n}.\operatorname{grad} \varphi)dS = \int_V [(\operatorname{grad} f).(\operatorname{grad} \varphi) + f\nabla^2\varphi]d\tau.$$

Similarly,

$$\int_S \varphi(\mathbf{n}.\operatorname{grad} f)dS = \int_V [(\operatorname{grad} \varphi).(\operatorname{grad} f) + \varphi\nabla^2 f]d\tau.$$

Subtracting,

$$\int_S (f \operatorname{grad} \varphi - \varphi \operatorname{grad} f).\mathbf{n}dS = \int_V (f\nabla^2\varphi - \varphi\nabla^2 f)d\tau.$$

This result itself is sometimes known as Green's theorem.

103. *Further consequences.* In the result of § 102, put $f = 1/|\mathbf{r}|$. We have seen (§ 97, Example), that $\nabla^2(1/|\mathbf{r}|) = 0$. Take for V any volume-domain not containing the origin O ($\mathbf{r} = 0$). Then

$$\int_S \left[\frac{1}{|\mathbf{r}|} \operatorname{grad} \varphi - \varphi \operatorname{grad} \frac{1}{|\mathbf{r}|}\right].\mathbf{n}dS = \int_V \frac{\nabla^2\varphi}{|\mathbf{r}|}d\tau.$$

Now apply this result by taking for S a pair of surfaces S_1 and S_2, of which S_1 is a small sphere surrounding O, of centre O, and S_2 is any surface surrounding S_1. The domain V is then the volume-domain lying between S_1 and S_2. The element dS_1 is given by $dS_1 = r^2 d\omega$, where $d\omega$ is the element of solid angle subtended by dS_1 at O, and hence, as $r \to 0$,

$$\int_{S_1} \left[\frac{1}{|\mathbf{r}|} \operatorname{grad} \varphi\right].\mathbf{n}dS \to 0.$$

But on S_1,
$$n_\alpha = -\frac{x_\alpha}{|\mathbf{r}|}.$$

Hence
$$\int_{S_1} \varphi\left(\operatorname{grad} \frac{1}{|\mathbf{r}|}\right).\mathbf{n}dS = \int_{S_1} \varphi\left(-\frac{x_\alpha}{|\mathbf{r}|} \frac{\partial}{\partial x_\alpha} \frac{1}{|\mathbf{r}|}\right) r^2 d\omega$$
$$= \int_{S_1} \varphi\left(-\frac{x_\alpha}{|\mathbf{r}|}\right)\left(-\frac{x_\alpha}{|\mathbf{r}|^3}\right)|\mathbf{r}|^2 d\omega \to 4\pi\varphi_0,$$

where φ_0 denotes the value of φ at O. Hence, proceeding to the limit and writing now S for S_2,

$$\varphi_0 = -\frac{1}{4\pi}\int_V \frac{\nabla^2\varphi}{|\mathbf{r}|}d\tau + \frac{1}{4\pi}\int_S \left[\frac{1}{|\mathbf{r}|} \operatorname{grad} \varphi - \varphi \operatorname{grad} \frac{1}{|\mathbf{r}|}\right].\mathbf{n}dS,$$

where now V denotes the whole domain interior to S_2. The volume integral is easily seen to converge near $r = 0$ if $\nabla^2\varphi$ is bounded.

104. In particular, if $\nabla^2\varphi = 0$ throughout V, then

$$\varphi_0 = \frac{1}{4\pi}\int_S \frac{(\operatorname{grad} \varphi).\mathbf{n}}{|\mathbf{r}|}dS - \frac{1}{4\pi}\int_S \varphi\left(\operatorname{grad} \frac{1}{|\mathbf{r}|}\right).\mathbf{n}dS.$$

When $\nabla^2\varphi=0$ throughout a region, φ is said to be harmonic throughout this region. Accordingly the above formula gives the value of a harmonic function φ at any point (here the origin O) in terms of the values of φ and grad φ over any surface S surrounding O.

105. Particularizing further, if $\nabla^2\varphi=0$ throughout V and if φ is constant over S with the constant value φ_S, then

$$\varphi_0=\frac{1}{4\pi}\int_S \frac{(\text{grad }\varphi).\mathbf{n}}{|\mathbf{r}|}\,dS-\frac{\varphi_S}{4\pi}\int_S\left((\text{grad }\frac{1}{|\mathbf{r}|}\right).\mathbf{n}dS.$$

To evaluate the coefficient of φ_S in this relation, we may apply Green's theorem to the region between S and some included sphere Σ with centre O. Then

$$\int_S\left(\text{grad }\frac{1}{|\mathbf{r}|}\right).\mathbf{n}dS+\int_\Sigma\left(\text{grad}\frac{1}{|\mathbf{r}|}\right).\mathbf{n}d\Sigma=\int_{(S)-(\Sigma)}\nabla^2\frac{1}{|\mathbf{r}|}d\tau=0.$$

Remembering that, over Σ, $\mathbf{n}$ is directed inwards, we have

$$\int_\Sigma\left(\text{grad }\frac{1}{|\mathbf{r}|}\right).\mathbf{n}d\Sigma=\int_\Sigma\left(-\frac{\mathbf{r}}{|\mathbf{r}|^3}\right).\left(-\frac{\mathbf{r}}{|\mathbf{r}|}\right)d\Sigma=\int_\Sigma\frac{d\Sigma}{|\mathbf{r}|^2}=\int d\omega=4\pi.$$

Hence
$$\int_S\left(-\text{grad }\frac{1}{|\mathbf{r}|}\right).\mathbf{n}dS=4\pi.$$

Hence
$$\varphi_0=\frac{1}{4\pi}\int\frac{(\text{grad }\varphi).\mathbf{n}}{|\mathbf{r}|}dS+\varphi_S.$$

106. *Gauss's theorem.* The relation expressed by the formula

$$\int_S\left(-\text{grad }\frac{1}{|\mathbf{r}|}\right).\mathbf{n}dS=4\pi$$

where S is a closed surface surrounding the origin O, is known as *Gauss's theorem.* It is readily proved from first principles, without appeal to Green's theorem. For, the vector $-\text{grad }(1/|\mathbf{r}|)$ being given by

$$-\text{grad }\frac{1}{|\mathbf{r}|}=\frac{\mathbf{r}}{|\mathbf{r}|^3},$$

and since
$$\frac{\mathbf{r}.}{|\mathbf{r}|}\mathbf{n}dS=d\sigma,$$

where $d\sigma$ is the normal cross-sectional area of the elementary cone subtended by dS at O, we have

$$\int_S\left(-\text{grad }\frac{1}{|\mathbf{r}|}\right).\mathbf{n}dS=\int\frac{d\sigma}{|\mathbf{r}|^2}=\int d\omega=4\pi.$$

The proof can obviously be extended to the case where the radius vector from O meets the surface in more than one point. Since the surface S surrounds O, the number of such points is necessarily odd.

If S does not surround O, the corresponding surface integral is zero. For any elementary cone with vertex O contributes equal and oppositely signed amounts $\pm d\omega$ to the integral from each pair of the even number of points in which a radius vector from O meets the surface.

107. Returning to the last formula of § 103 if $\varphi \sim (\text{const})/|\mathbf{r}|$ for $|\mathbf{r}|$ large, then as $|\mathbf{r}| \to \infty$,

$$\int_S \left[\frac{1}{|\mathbf{r}|} \operatorname{grad} \varphi - \varphi \operatorname{grad}\left(\frac{1}{|\mathbf{r}|}\right)\right] . \mathbf{n} dS \to 0,$$

where S is a large sphere whose radius $\to \infty$. Hence

$$\varphi_0 = -\frac{1}{4\pi}\int \frac{\nabla^2 \varphi}{|\mathbf{r}|} d\tau,$$

where the volume integral now extends to the whole of space.

The results of §§ 103–107 have immediate applications in the theory of the potential in the contexts of gravitation and electrostatics. They are collected here so that the student may see them derived as consequences of Green's theorem, independent of their physical setting.

108. *Stokes's theorem.* The following theorem is of fundamental importance in the theory of magnetism.

Theorem: Let S be any unclosed surface in three dimensions, C its boundary curve. Let φ be any vector or tensor function of $\mathbf{r}$, the position vector of any point in the three-dimensional domain considered. Then if $\varphi_{\mu\alpha\ldots}$ are the components of φ in any triad, in the same triad

$$\int_C \varphi_{\mu\alpha\ldots} dr_\mu = \int_S A_{\kappa\sigma\tau} \frac{\partial \varphi_{\tau\alpha\ldots}}{\partial r_\sigma} n_\kappa dS,$$

where $\mathbf{n}$ is a unit vector normal to the element dS, and the sense of $\mathbf{n}$ is to be such that if $\mathbf{r}$ is a small vector drawn from an interior point of S near C to the boundary curve C, $d\mathbf{r}$ the element of C in the chosen sense of the integration, then $\mathbf{r}$, $d\mathbf{r}$, $\mathbf{n}$ form a positive triad.

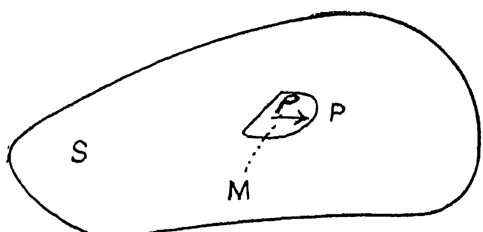

Fig 16.

Proof. Consider a small curvilinear element of S, and take a point M (Fig. 16) in the neighbourhood of this element but not necessarily in the element of surface itself. Let P be any point in the contour of the element, $\boldsymbol{\rho}$ its position vector with respect to M. Then

$$(\varphi_{\mu\alpha\ldots})_P = (\varphi_{\mu\alpha\ldots})_M + \rho_\nu \left(\frac{\partial \varphi_{\mu\alpha\ldots}}{\partial r_\nu}\right)_M + O(\rho^2),$$

which we can arrange as

$$(\varphi_{\mu\alpha\ldots})_P = (\varphi_{\mu\alpha\ldots})_M + \tfrac{1}{2}\rho_\nu \left[\frac{\partial \varphi_{\mu\alpha\ldots}}{\partial r_\nu} + \frac{\partial \varphi_{\nu\alpha\ldots}}{\partial r_\mu}\right]_M + \tfrac{1}{2}\rho_\nu \left[\frac{\partial \varphi_{\mu\alpha\ldots}}{\partial r_\nu} - \frac{\partial \varphi_{\nu\alpha\ldots}}{\partial r_\mu}\right]_M + O(\rho^2)$$

Now multiply both sides by dr_μ, the vector element of arc of the boundary of the mesh, carrying out the implied summation with respect to μ, and integrate round the boundary of the mesh. The right-hand side contributes four terms to the result. We shall prove that the first and second terms vanish and that the fourth (the remainder term) not only vanishes in the limit but vanishes for a sum of elements amounting to the whole surface S. The third term alone gives a non-zero contribution.

Consider the terms *seriatim*. The first term contributes

$$\int(\varphi_{\mu\alpha\ldots})_M dr_\mu.$$

This vanishes because the integrand is a constant for the element of S considered, and $[r_\mu]$ vanishes on going round its contour.

The second term contributes

$$\tfrac{1}{2}\left[\frac{\partial\varphi_{\mu\alpha\ldots}}{\partial r_\nu}+\frac{\partial\varphi_{\nu\alpha\ldots}}{\partial r_\mu}\right]_M\int\rho_\nu dr_\mu.$$

But $dr_\mu = d\rho_\mu$, and in the second term of the square bracket, μ and ν are dummy suffixes. Interchanging them the whole expression comes to

$$\tfrac{1}{2}\left[\frac{\partial\varphi_{\mu\alpha\ldots}}{\partial r_\nu}\right]_M\int(\rho_\nu d\rho_\mu+\rho_\mu d\rho_\nu),$$

i.e. to

$$\tfrac{1}{2}\left[\frac{\partial\varphi_{\mu\alpha\ldots}}{\partial r_\nu}\right]_M\int d(\rho_\nu\rho_\mu).$$

This vanishes since $[\rho_\nu\rho_\mu]$ vanishes on going round the contour.

The third term may be written, by the (**A**, **U**) theorem,

$$\tfrac{1}{2}A_{\kappa\nu\mu}A_{\kappa\sigma\tau}\left[\frac{\partial\varphi_{\tau\alpha\ldots}}{\partial r_\sigma}\right]_M\int\rho_\nu d\rho_\mu$$

$$=A_{\kappa\sigma\tau}\left[\frac{\partial\varphi_{\tau\alpha\ldots}}{\partial r_\sigma}\right]_M\int\tfrac{1}{2}(\boldsymbol{\rho}\wedge d\boldsymbol{\rho})_\kappa$$

$$=A_{\kappa\sigma\tau}\left[\frac{\delta\varphi_{\tau\alpha\ldots}}{\delta r_\sigma}\right]_M n_\kappa dS,$$

where dS is the scalar area of the element, **n** a unit vector normal to the element in the sense which makes $\boldsymbol{\rho}$, $d\boldsymbol{\rho}$, **n** a positive triad. (We have assumed that the small curvilinear element may be treated as if plane.)

The fourth term we leave alone for the moment.

Dividing S into a set of elementary meshes and summing for all such elements, we see that if the contours of the meshes are all described in the same sense, the left-hand side reduces to an integral over the contour C of S only. Proceeding to the limit, we get

$$\int_C\varphi_{\mu\alpha\ldots}dr_\mu=\int_S A_{\kappa\sigma\tau}\frac{\partial\varphi_{\tau\alpha\ldots}}{\partial r_\sigma}n_\kappa dS+\lim\Sigma\int o(\rho^2)|d\rho|.$$

If the linear dimensions of an element of area into which S is divided are of the order l, the number of such elements is of the order A/l^2, whilst the contribution from each element to the sum Σ is of the order $l^2 \times l$ or l^3. Hence as $l \to 0$, the sum Σ also tends to zero.*

We have therefore

$$\int_C \varphi_{\mu\alpha\ldots} dr_\mu = \int_S A_{\varkappa\sigma\tau} \frac{\partial \varphi_{\tau\alpha\ldots}}{\partial r_\sigma} n_\varkappa \, dS.$$

This result can be written more explicitly as follows. Let the integer $\varkappa$ (one of the numbers 1, 2, 3) have for its successors in the cycle 1, 2, 3, 1, ... the numbers $\varkappa'$ and $\varkappa''$. Let ds be a scalar element of arc of C, l_μ the components of a unit vector along ds in the sense of the integration. Then

$$\sum_{\mu=1,2,3} \int_C \varphi_{\mu\alpha\ldots} l_\mu ds = \sum_{\varkappa=1,2,3} \int_S \left[\frac{\partial \varphi_{\varkappa''\alpha\ldots}}{\partial r_{\varkappa'}} - \frac{\partial \varphi_{\varkappa'\alpha\ldots}}{\partial r_{\varkappa''}} \right] n_\varkappa dS.$$

Here the vector element of length drawn from a neighbouring interior point of S to the boundary C, the tangent to the boundary C and the normal to the element of surface, at a point of C, form a positive triad.

109. *Particular forms of Stoke's theorem.* (1) Take $\varphi_{\mu\alpha\ldots}$ to be a vector Q_μ. Then

$$\int_C Q_\mu dr_\mu = \int_S A_{\varkappa\sigma\tau} \frac{\partial Q_\tau}{\partial r_\sigma} n_\varkappa dS,$$

or, in vector notation,

$$\int_C \mathbf{Q}.d\mathbf{r} = \int_S (\text{curl } \mathbf{Q}).\mathbf{n} dS,$$

where, it may be remembered, an elementary vector drawn in the surface, to the boundary, an elementary arc $d\mathbf{r}$ of the boundary, and the normal $\mathbf{n}$ to S at the boundary form a positive triad. This is the classical form of Stokes's theorem. It is worth while restating it explicitly in Cartesian components. Let the components of Q be X, Y, Z. Then

$$\int_C (X dx + Y dy + Z dz) = \sum_{x,y,z} \iint_S \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} \right) dy\, dz.$$

If the surface S is closed, its bounding curve C reduces to zero, and we deduce

$$\iint_S (\text{curl } \mathbf{Q}).\mathbf{n} dS = 0.$$

This is readily verified by Green's theorem. For, since S is closed, we may apply Green's theorem to the domain enclosed, and obtain

$$\iint_S (\text{curl } \mathbf{Q}).\mathbf{n} dS = \iiint_V \left(\frac{\partial}{\partial \mathbf{r}}.\text{curl } \mathbf{Q} \right) d\tau.$$

* This is not offered as a formally rigorous proof. In applied mathematics the spirit of a demonstration is more important than minutiae of rigour, and the above proof is arranged to show how the form of Stokes's famous theorem arises from the situation for which it is enunciated.

But
$$\frac{\partial}{\partial \mathbf{r}}.\text{curl } \mathbf{Q} = \frac{\partial}{\partial r_\alpha} A_{\alpha\beta\gamma} \frac{\partial}{\partial r_\beta} Q_\gamma,$$
and since $\partial^2 Q_\gamma / \partial r_\alpha \partial r_\beta$ is unaltered by interchange of α and β, whilst $A_{\alpha\beta\gamma}$ changes sign, for each γ the result of the summation with respect to α and β is zero.

(2) Take φ. to be $\varphi U_{\mu\alpha}$, where φ is a scalar. Then
$$\int_C \varphi dr_\alpha = A_{\varkappa\sigma\tau} \int_S \frac{\partial \varphi}{\partial r_\sigma} U_{\tau\alpha} n_\varkappa dS = A_{\alpha\varkappa\sigma} \int_S n_\varkappa \frac{\partial \varphi}{\partial r_\sigma} dS,$$
or, in vector form,
$$\int_C \varphi d\mathbf{r} = \int_S (\mathbf{n} \wedge \text{grad } \varphi) dS.$$
Explicitly, in Cartesian components, it gives for the x-component
$$\int_C \varphi dx = \iint_S \left[\frac{\partial \varphi}{\partial z} dz dx - \frac{\partial \varphi}{\partial y} dy dx \right].$$
In vector form, this is an important alternative statement of Stokes's theorem.

(3) Take $\varphi_{\mu\alpha\ldots}$ to be $A_{\mu\alpha\beta} Q_\alpha$. Then
$$\int_C A_{\mu\alpha\beta} Q_\alpha \, dr_\mu = \int_S A_{\varkappa\sigma\tau} \frac{\partial}{\partial r_\sigma} (A_{\tau\alpha\beta} Q_\alpha) n_\varkappa \, dS,$$
or
$$\int_C A_{\beta\mu\alpha} dr_\mu Q_\alpha = \int_S \left[\frac{\partial Q_\alpha}{\partial r_\beta} n_\alpha - \frac{\partial Q_\alpha}{\partial r_\alpha} n_\beta \right] dS$$
or
$$\int_C d\mathbf{r} \wedge \mathbf{Q} = \int_S [(\text{grad } \mathbf{Q}).\mathbf{n} - \mathbf{n} \text{ div } Q] dS.$$
Explicitly,
$$\int_C (Z dy - Y dz) = \iint_S \left[\left(\frac{\partial X}{\partial x} dy dz + \frac{\partial Y}{\partial x} dz dx + \frac{\partial Z}{\partial x} dx dy \right) - \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \right) dy dz \right]$$
$$= \iint_S \left[\left(\frac{\partial Y}{\partial x} dx dz - \frac{\partial Y}{\partial y} dy dz \right) + \left(\frac{\partial Z}{\partial x} dx dy - \frac{\partial Z}{\partial z} dz dy \right) \right].$$

110. *Connexion of Stokes's theorem with Stokes's transformation.* It will have been noticed that the essential steps in the proof of Stokes's theorem are the decomposition of the tensor $\frac{\partial \varphi_{\mu\alpha\ldots}}{\partial r_\nu}$ into the sum of a self-conjugate and an anti-symmetrical tensor, and then the expression of the inner product of the latter with the differential vector ρ_ν as a vector product; in other words, we have used the same procedure as in Stokes's transformation, § 65. We have preferred not to quote § 65 in order to secure a

ıpplicable generally to tensors of any rank. The essence of the ›wever, may be seen by applying § 65 to the tensor grad **Q**. Then

$$\text{vec}\,(\text{grad}\,\mathbf{Q})=\tfrac{1}{2}\,\text{curl}\,\mathbf{Q},$$

ɿes

$$\boldsymbol{\rho}.\text{grad}\,\mathbf{Q}=\boldsymbol{\rho}.\overline{\overline{\text{grad}\,\mathbf{Q}}}-\tfrac{1}{2}\boldsymbol{\rho}\wedge\text{curl}\,\mathbf{Q},$$

$\overline{\overline{\text{ȝrad}\,\mathbf{Q}}}$ is the self-conjugate tensor $\tfrac{1}{2}(\text{grad}\,\mathbf{Q}+\overline{\text{grad}\,\mathbf{Q}})$. ꞓes's transformation, as it is used in hydrodynamics and elasticity, :t precisely the last-given relation, rewritten in the form

$$\boldsymbol{\rho}.\text{grad}\,\mathbf{Q}=\boldsymbol{\rho}.\overline{\overline{\text{grad}\,\mathbf{Q}}}+\tfrac{1}{2}\,\text{curl}\,\mathbf{Q}\wedge\boldsymbol{\rho}.$$

ll see later that $\tfrac{1}{2}$ curl **Q** or $\tfrac{1}{2}$ rot **Q** plays the part of a small rigid ›tation or of an angular velocity.

. *Applications of Green's and Stokes's theorems.* The following :ion of Green's and Stokes's theorems are required in physics.

:orem : If **Q** is a vector function of **r**, and if

$$\int_S \mathbf{Q}.\mathbf{n}dS=0$$

ver any closed surface S in the domain of **r**, then

$$\text{div}\,\mathbf{Q}=0$$

here in the domain.

, by § 101, Example (2),

$$\int_V \text{div}\,\mathbf{Q}d\tau=0$$

hrough any volume V in the domain of **r**. By applying this to a omain $\Delta\tau$ round a point $\mathbf{r}_0$ it is readily shown that $(\text{div}\,\mathbf{Q})_{\mathbf{r}=\mathbf{r}_0}=0$. the theorem.

:orem : If **Q** is a vector function of **r**, and if

$$\int_C \mathbf{Q}.d\mathbf{r}=0$$

ɔund any closed curve C in the domain of **r**, then

$$\text{curl}\,\mathbf{Q}=0$$

here in the domain.

, by § 109 (1),

$$\int_S (\text{curl}\,\mathbf{Q}).\mathbf{n}dS=0$$

y surface S in the domain of **r**. By taking S to be a small element ce with normal parallel to $(\text{curl}\,\mathbf{Q})_{\mathbf{r}=\mathbf{r}_0}$ we see that $(\text{curl}\,\mathbf{Q})_{\mathbf{r}=\mathbf{r}_0}=0$. the theorem.

RATE OF CHANGE THEOREMS

112. *The rate of change of a volume integral taken through the interior of a moving closed surface.* Let φ be a given scalar, vector or tensor function of a vector $\mathbf{r}$ and a scalar t. It is convenient to regard $\mathbf{r}$ as the position vector of any point in the domain of $\mathbf{r}$, and to regard t as the time. Associate with each point in the domain of $\mathbf{r}$ a vector $\mathbf{u}$, a function of $\mathbf{r}$ and possibly t. Let S be a closed surface in the domain of $\mathbf{r}$ depending on the variable t in such a way that when t varies from t to t+dt, any point P of the surface moves from $\mathbf{P}$ to $\mathbf{P}+d\mathbf{P}$, where $d\mathbf{P}=\mathbf{u}dt$. Then, if S is given for one value t_0 of t, it is determinate, in general, for all values of t. If t is taken to be the time, the domain of $\mathbf{r}$ may be considered as the seat of a medium moving with the velocity $\mathbf{u}$ at P, and S may be considered to be moving with the medium.

We shall denote by the operator D/Dt acting on any symbol the rate of change of the number, vector, tensor, represented by that symbol, considered as a function of t. Let V denote the interior of S. We now prove the following theorem.

Theorem : If in any triad a tensor $I_{\alpha\beta\ldots}$ is defined by the relation

$$I_{\alpha\beta\ldots}=\int_V \varphi_{\alpha\beta\ldots}d\tau,$$

then
$$\frac{D}{Dt}I_{\alpha\beta\ldots}=\int_V\left[\frac{\partial\varphi_{\alpha\beta\ldots}}{\partial t}+\frac{\partial}{\partial r_\mu}(\varphi_{\alpha\beta\ldots}u_\mu)\right]d\tau.$$

For, consider two neighbouring positions S and S′ of the surface, corresponding to the values t and t+dt. Let V′ denote the interior of S′. Then the increment $DI_{\alpha\beta\ldots}$ is given by

$$DI_{\alpha\beta\ldots}=\int_{V'}\varphi_{\alpha\beta\ldots}(t+dt)d\tau-\int_V\varphi_{\alpha\beta\ldots}(t)d\tau,$$

where for simplicity we have suppressed mention of the argument $\mathbf{r}$ of φ at each point of the domains V and V′. To a sufficient order, this can be rewritten as

$$DI_{\alpha\beta\ldots}=\int_{V'}\left[\varphi_{\alpha\beta\ldots}(t)+\frac{\partial\varphi_{\alpha\beta\ldots}}{\partial t}dt\right]d\tau-\int_V\varphi_{\alpha\beta\ldots}(t)d\tau$$
$$=dt\int_{V'}\frac{\partial\varphi_{\alpha\beta\ldots}}{\partial t}d\tau+\left(\int_{V'}-\int_V\right)\varphi_{\alpha\beta\ldots}d\tau.$$

But $\left(\int_{V'}-\int_V\right)\varphi_{\alpha\beta\ldots}d\tau$ is the integral of φ through the shell-shaped region defined by the displacement of the given surface from S to S′, with the appropriate positive or negative sign given to the volume elements of

this shell (Fig. 17). Now let $\mathbf{n}$ denote a unit vector along the outward normal to S at any point. Then the volume element $d\tau$ of the shell, of slant side $\mathbf{u}dt$, with its appropriate sign, is given by

$$d\tau=(\mathbf{n}.\mathbf{u}dt)dS,$$

whence

$$DI_{\alpha\beta\ldots}=dt\int_{V'}\frac{\partial\varphi_{\alpha\beta\ldots}}{\partial t}d\tau+dt\int_S\varphi_{\alpha\beta\ldots}n_\mu u_\mu dS.$$

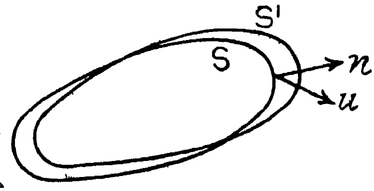

Fig. 17

Apply Green's theorem to the surface integral. We get

$$\int_S\varphi_{\alpha\beta\ldots}n_\mu u_\mu dS=\int_V\frac{\partial}{\partial r_\mu}(\varphi_{\alpha\beta\ldots}u_\mu)d\tau.$$

Letting $dt\to 0$, we get

$$\frac{DI_{\alpha\beta\ldots}}{Dt}=\int_V\left[\frac{\partial\varphi_{\alpha\beta\ldots}}{\partial t}+\frac{\partial}{\partial r_\mu}(\varphi_{\alpha\beta\ldots}u_\mu)\right]d\tau.$$

113. *The rate of change of a function of a vector.* We now investigate the rate of change of the function $\varphi_{\alpha\beta\ldots}$ itself, considered as a function of t only, when P is varying according to $d\mathbf{P}=\mathbf{u}dt$, starting from any given position P_0 at $t=t_0$. This is called 'the rate of change of φ following the motion.' We have

$$\begin{aligned}D\varphi_{\alpha\beta\ldots}&=\varphi_{\alpha\beta\ldots}(\mathbf{r}+d\mathbf{r},\ t+dt)-\varphi_{\alpha\beta\ldots}(\mathbf{r},\ t)\\&=dt\left[\frac{\partial\varphi_{\alpha\beta\ldots}}{\partial t}+\frac{\partial\varphi_{\alpha\beta\ldots}}{\partial r_\mu}u_\mu\right].\end{aligned}$$

Hence

$$\frac{D\varphi_{\alpha\beta\ldots}}{Dt}=\frac{\partial\varphi_{\alpha\beta\ldots}}{\partial t}+u_\mu\frac{\partial}{\partial r_\mu}\varphi_{\alpha\beta\ldots}.$$

This may be written in vector form as

$$\frac{D\varphi}{Dt}=\frac{\partial\varphi}{\partial t}+\mathbf{u}.\mathrm{grad}\ \varphi.$$

114. In terms of $D\varphi/Dt$, the theorem of § 110 may now be written in the form

$$\frac{D}{Dt}\int_V\varphi d\tau=\int_V\left[\frac{D\varphi}{Dt}+\varphi\ \mathrm{div}\ \mathbf{u}\right]d\tau,$$

on using Example (4) § 100. The distinction between D/Dt acting on a function of a point $\mathbf{r}$ and on an integral through a volume V should be noted.

The origin of the second term on the right-hand side can be seen as follows. By taking the function φ to be identically equal to unity, we have, if V denotes the volume enclosed by S,

$$\frac{DV}{Dt}=\int_V\mathrm{div}\ \mathbf{u}\ d\tau$$

or, taking V to be the elementary volume $d\tau$ itself,

$$\frac{D(d\tau)}{Dt}=(\text{div } \mathbf{u})d\tau.$$

Thus
$$\frac{D}{Dt}\int_V \varphi d\tau=\int_V \frac{D}{Dt}(\varphi d\tau)=\int\left[\frac{D\varphi}{Dt}d\tau+\varphi \text{ div } \mathbf{u}\, d\tau\right].$$

Accordingly the term φ div $\mathbf{u}$ arises from the change of volume of the volume element.

115. *The flux of a vector through a surface.* Consider a vector which is a function of a position vector $\mathbf{r}$. Associated with any surface, closed or unclosed, in the domain of $\mathbf{r}$, is the integral of the normal component of the vector function, taken over that surface. This is called the flux of the vector through the surface.

Let $\mathbf{Q}$ denote a vector function of $\mathbf{r}$, which may also be a function of a parameter t, which may be taken to be the time. Let $\mathbf{n}$ be a unit vector normal to an element dS of a surface S in the domain of $\mathbf{r}$. The sense in which $\mathbf{n}$ is to be taken is supposed specified. Then the flux of $\mathbf{Q}$ through S in the specified sense is the scalar f defined by

$$f=\int_S(\mathbf{Q}.\mathbf{n})dS=\int_S Q_\mu n_\mu\, dS.$$

If S is closed, we may transform the surface integral into a volume integral by Green's theorem. Gauss's theorem (§ 106) relates to the flux of a certain vector, grad $(1/|\mathbf{r}|)$ over a closed surface.

116. *Rate of change of flux through a surface.* Whether S is closed or unclosed, it is of interest to investigate the rate of change of f 'following the motion.' We suppose, as in § 112, that the domain of $\mathbf{r}$ is the seat of a vector field of velocity $\mathbf{u}$, a function of $\mathbf{r}$ and possibly t.

Theorem : The rate of change of the flux f of the vector $\mathbf{Q}$ through the moving surface S is given by

$$\frac{Df}{Dt}=\int_S\left[\frac{\partial \mathbf{Q}}{\partial t}+\mathbf{u} \text{ div } \mathbf{Q}+\text{curl } (\mathbf{Q}\wedge\mathbf{u})\right].\mathbf{n}dS.$$

Actually we shall establish the theorem in a somewhat more general form. Let $\varphi_{\alpha\beta\ldots}$ denote any vector or tensor function of the vector $\mathbf{r}$. Then the flux of φ through a surface S is defined to be the tensor of rank one lower than that of φ, given by

$$f_\beta=\int_S \varphi_{\mu\beta\ldots}n_\mu dS.$$

We shall prove that

$$\frac{D}{D}f_\beta=\int_S\left[\frac{\partial\varphi_{\mu\beta\ldots}}{\partial t}+u_\mu\frac{\partial}{\partial r_\nu}\varphi_{\nu\beta\ldots}+A_{\mu\sigma\tau}\frac{\partial}{\partial r_\sigma}\{A_{\tau\kappa\xi}\varphi_{\kappa\beta\ldots}u_\xi\}\right]n_\mu dS.$$

For, let S, S′ (Fig. 18) denote two neighbouring positions of S corresponding to the values t, t+dt of the variable t. Let C denote the bounding curve of S, and let W denote the collar-shaped surface uniting S and S′ formed by the elementary displacements **u**dt of points on the boundary C of S. Let Σ denote the complete closed surface S+W+S′, and let V denote its interior. Then

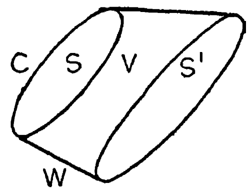

Fig. 18

$$\mathrm{Df}_{\beta\ldots}=\int_{S'}\varphi_{\mu\beta\ldots}(\mathbf{r}',\,t+dt)n'_{\mu}dS'-\int_{S}\varphi_{\mu\beta\ldots}(\mathbf{r},\,t)n_{\mu}dS$$

$$=\int_{S'}\varphi_{\mu\beta\ldots}(\mathbf{r}',\,t)n'_{\mu}dS'-\int_{S}\varphi_{\mu\beta\ldots}(\mathbf{r},\,t)n_{\mu}dS+dt\int_{S'}\frac{\partial\varphi_{\mu\beta\ldots}}{\partial t}(\mathbf{r}',\,t)n'_{\mu}dS'.$$

On the right-hand side, add and subtract the integral

$$\int_{W}\varphi_{\mu\beta\ldots}(\mathbf{r},\,t)n_{\mu}(W)dW,$$

where $n_{\mu}(W)$ are the components of the outward normal to W at dW. If now **n** is taken to denote the outward normal at dΣ, any element of the complete boundary of V, we have

$$\mathrm{Df}_{\beta\ldots}=dt\int_{S'}\frac{\partial\varphi_{\mu\beta\ldots}}{\partial t}n'_{\mu}dS'+\int_{\Sigma}\varphi_{\mu\beta\ldots}(\mathbf{r},\,t)n_{\mu}d\Sigma-\int_{W}\varphi_{\mu\beta\ldots}(\mathbf{r},\,t)n_{\mu}(W)dW.$$

The second term in $\mathrm{Df}_{\beta\ldots}$ being a surface integral over a closed surface Σ, we can apply Green's theorem, when we get

$$\int_{\Sigma}\varphi_{\mu\beta\ldots}(\mathbf{r},\,t)n_{\mu}d\Sigma=\int_{V}\frac{\partial\varphi_{\mu\beta\ldots}}{\partial r_{\mu}}d\tau.$$

On the right-hand side of this relation, the volume element dτ is given to a sufficient order of accuracy by

$$d\tau=n_{\nu}u_{\nu}\,dSdt,$$

where **n** is a unit vector in the sense of the original normal to dS used in defining the flux. Hence

$$\int_{\Sigma}\varphi_{\mu\beta\ldots}(\mathbf{r},\,t)n_{\mu}d\Sigma=dt\int_{S}\frac{\partial\varphi_{\mu\beta\ldots}}{\partial r_{\mu}}n_{\nu}u\ \ dS,$$

in which we may interchange the dummy suffixes μ and ν, obtaining

$$\int_{\Sigma}\varphi_{\mu\beta\ldots}(\mathbf{r},\,t)n_{\mu}d\Sigma=dt\int_{S}\frac{\partial\varphi_{\nu\beta\ldots}}{\partial r_{\nu}}n_{\mu}u_{\mu}dS.$$

In the third term in $\mathrm{Df}_{\beta\ldots}$, $n_{\mu}(W)dW$ is the μ-component of the vector element of area subtended by the vectors **u**dt and d**r**, where d**r** is the

vector element of arc of the boundary curve C in the sense which makes

$$\mathbf{u}dt, \qquad d\mathbf{r}, \qquad \mathbf{n}(W)$$

a positive triad (Fig. 19). Hence

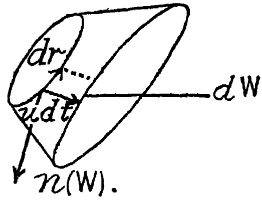

Fig. 19

$$\mathbf{n}(W)dW=(\mathbf{u}\wedge d\mathbf{r})dt,$$

or $$n_\mu(W)dW=A_{\mu\xi\eta}u_\xi(dr)_\eta dt,$$

whence the third term in $Df_{\beta\ldots}$ is given by

$$-\int_W \varphi_{\mu\beta\ldots}(\mathbf{r},\ t)n_\mu(W)dW=-dt\int_C A_{\mu\xi\eta}\varphi_{\mu\beta\ldots}u_\xi(dr)_\eta=-dt\int_C \Phi_{\eta\beta\ldots}dr_\eta$$

say. The last integral being a line integral round a closed contour, we may apply Stokes's theorem, obtaining

$$-\int_W \varphi_{\mu\beta\ldots}(\mathbf{r},\ t)n_\mu(W)dW=-dt\int_S A_{\varkappa\sigma\tau}\left(\frac{\partial}{\partial r_\sigma}\Phi_{\tau\beta\ldots}\right)n'_\varkappa(S)dS,$$

where $\mathbf{n}'(S)$ is the normal to dS (the element of the surface bounded by C) in the sense which makes

$$\boldsymbol{\rho}, \qquad d\mathbf{r}, \qquad \mathbf{n}'(S)$$

a positive triad, $\boldsymbol{\rho}$ being a small vector drawn to C from an interior point of S. But the vector element of area $\boldsymbol{\rho}\wedge d\mathbf{r}$ so defined has for its normal the *outward* normal at this element to V, the volume included by Σ, and this is opposite in direction to $\mathbf{n}(S)$, the originally chosen normal to S used for defining the flux (Fig. 20). Hence $\mathbf{n}'(S)=-\mathbf{n}(S)$. (See figure.) Hence

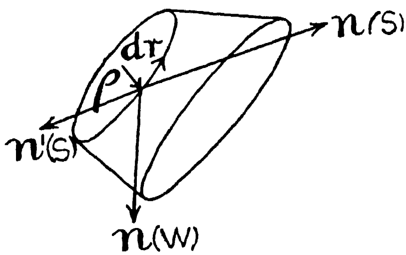

Fig. 20

$$-\int_W \varphi_{\mu\beta}(\mathbf{r},\ t)n_\mu(W)dW=+dt\int_S A_{\varkappa\sigma\tau}\left(\frac{\partial}{\partial r_\sigma}\Phi_{\tau\beta\ldots}\right)n_\varkappa dS$$

where now on the right-hand side $\mathbf{n}$ denotes the original normal to S. But we had written

$$\Phi_{\eta\beta\ldots}=A_{\mu\xi\eta}\varphi_{\mu\beta\ldots}u_\xi$$

whence $$A_{\varkappa\sigma\tau}\left(\frac{\partial}{\partial r_\sigma}\Phi_{\tau\beta\ldots}\right)n_\varkappa=A_{\mu\sigma\tau}\frac{\partial}{\partial r_\sigma}(A_{\varkappa\xi\tau}\varphi_{\varkappa\beta\ldots}u_\xi)n_\mu.$$

Hence we get on returning to the expression for $Df_{\beta\ldots}$

$$\frac{Df_\beta}{Dt}=\int_S\left[\frac{\partial\varphi_{\mu\beta\ldots}}{\partial t}+\frac{\partial\varphi_{\nu\beta}}{\partial r_\nu}u_\mu+A_{\mu\sigma\tau}\frac{\partial}{\partial r_\sigma}(A_{\tau\varkappa\xi}\varphi_{\varkappa\beta\ldots}u_\xi)\right]n_\mu dS.$$

This is the general theorem as stated.

For tensors of any rank we may write, in accordance with our previous notation,

$$(\mathbf{u}\wedge\boldsymbol{\varphi})_{\tau\beta\ldots}=A_{\tau\xi\varkappa}u_\xi\varphi_{\varkappa\beta\ldots}$$
$$=-A_{\tau\varkappa\xi}\varphi_{\varkappa\beta\ldots}u_\xi,$$

and we may define the operation *curl* generally for tensors of any rank by

$$(\text{curl }\boldsymbol{\psi})_{\mu\beta\ldots}=A_{\mu\sigma\tau}\frac{\partial}{\partial r_\sigma}\psi_{\tau\beta\ldots}$$

Then the general theorem may be written in the form

$$\frac{D}{Dt}\left[\int_S \mathbf{n}.\boldsymbol{\varphi}dS\right]=\int_S\left[\frac{\partial\boldsymbol{\varphi}}{\partial t}+(\text{div }\boldsymbol{\varphi})\mathbf{u}-\text{curl }(\mathbf{u}\wedge\boldsymbol{\varphi})\right].\mathbf{n}dS,$$

where $\boldsymbol{\varphi}$ denotes a tensor of any rank. When $\boldsymbol{\varphi}$ reduces to a vector $\mathbf{Q}$, $-\mathbf{u}\wedge\boldsymbol{\varphi}$ reduces to $\mathbf{Q}\wedge\mathbf{u}$, and we get the form of the theorem as first stated,

117. *Case of a closed surface.* As a check on the correctness of this theorem, we consider its application to the case where S is closed. In this case Df_β/Dt, being expressible as a surface integral over a closed surface, can be transformed by Green's theorem into a volume integral through the interior V of S. But f_β itself, being an integral over a closed surface, can be transformed by Green's theorem into an integral through V, and hence its rate of change Df_β/Dt can be expressed in another way, using the theorem of § 110, as a volume integral through V. The two volume integrals so obtained should be identically equal.

When S is closed, we have on applying Green's theorem to the result of § 116,

$$\frac{Df_\beta}{Dt}=\int_V\frac{\partial}{\partial r_\mu}\left[\frac{\partial\varphi_{\mu\beta}}{\partial t}+u_\mu\frac{\partial}{\partial r_\sigma}\varphi_{\sigma\beta}+A_{\mu\sigma\tau}\frac{\partial}{\partial r_\sigma}(A_{\tau\varkappa\xi}\varphi_{\varkappa\beta}u_\xi)\right]d\tau.$$

But since again S is closed, we have by Green's theorem

$$f_\beta=\int_S\varphi_{\sigma\beta}n_\sigma dS=\int_V\frac{\partial}{\partial r_\sigma}\varphi_{\sigma\beta}d\tau.$$

Applying the theorem of § 112 to obtain the rate of change of the last-written volume integral, we have

$$\frac{Df_\beta}{Dt}=\int_V\left[\frac{\partial}{\partial t}\frac{\partial}{\partial r_\sigma}\varphi_{\sigma\beta}+\frac{\partial}{\partial r_\mu}\left\{\left(\frac{\partial\varphi_{\sigma\beta}}{\partial r_\sigma}\right)u_\mu\right\}\right]d\tau.$$

The two expressions so obtained for Df_β/Dt will be equal provided that

$$\int_V\frac{\partial}{\partial r_\mu}\left[A_{\mu\sigma\tau}\frac{\partial}{\partial r_\sigma}(A_{\tau\varkappa\xi}\varphi_{\varkappa\beta}u_\xi)\right]d\tau=0.$$

The integrand here is clearly identically zero, since $A_{\mu\sigma\tau}$ changes sign on interchanging μ and σ; the integrand is, in fact, the divergence of the curl of a vector, which vanishes identically.

118. *The circulation of a vector round a closed curve.* Let C be a closed curve in the domain of a position vector **r**, **Q** a vector function of **r** in this domain. Then the circulation c of **Q** round C is defined as

$$c=\int_C \mathbf{Q}.d\mathbf{r}=\int_C Q_\alpha dr_\alpha.$$

119. *The rate of change of the circulation round a given circuit.* Let **Q** be a function of a variable t, as well as of **r**, and let any point **P** of the domain of **r** be displaced to **P**+d**P**, where d**P**=**u**(**r**)dt, when t increases from t to t+dt. As before, the domain of **r** may be considered to be the seat of a medium with velocity-distribution **u**(**r**), and C may be taken to move with the medium. We may then consider the rate of change of the circulation c round C, following the motion.

Theorem: The rate of change Dc/Dt of the circulation round a closed curve C is given by

$$\frac{Dc}{Dt}=\int_C\left[\frac{\partial \mathbf{Q}}{\partial t}+(\text{curl } \mathbf{Q})\wedge\mathbf{u}\right].d\mathbf{r}.$$

Actually we shall establish the theorem in a slightly more general form. Let $\varphi_{\alpha\rho\ldots}$ be any vector or tensor function of **r** and t. Let the circulation $c_{\rho\ldots}$ be defined by

$$c_{\rho\ldots}=\int_C \varphi_{\alpha\rho\ldots}dr_\alpha.$$

We shall prove that

$$\frac{Dc_{\rho\ldots}}{Dt}=\int_C\left[\frac{\partial\varphi_{\alpha\rho\ldots}}{\partial t}-A_{\alpha\mu\nu}u_\mu A_{\nu\beta\gamma}\frac{\partial}{\partial r_\beta}\varphi_{\gamma\rho\ldots}\right]dr_\alpha.$$

For, let C′ and C denote the positions of the curve at times t and t+dt. Then

$$Dc_{\rho\ldots}=\int_{C'}\varphi_{\alpha\rho\ldots}(\mathbf{r},\, t+dt)dr_\alpha-\int_C\varphi_{\alpha\rho\ldots}(\mathbf{r},\, t)dr_\alpha,$$

$$=\int_{C'}\left[\varphi_{\alpha\rho\ldots}(\mathbf{r},\, t)+\frac{\partial\varphi_{\alpha\rho\ldots}}{\partial t}dt\right]dr_\alpha-\int_C\varphi_{\alpha\rho\ldots}(\mathbf{r},\, t)dr_\alpha.$$

Join each point **P** of C to the associated point **P**′=**P**+d**P**=**r**+**u**(**r**)dt of C′, thus forming a collar surface S (Fig. 21). Divide S into meshes dS bounded by the elements of arc d**r** of C and C′ and the lines of flow PP′, and extend the line integral $\int\varphi_{\alpha\rho\ldots}(\mathbf{r},\, t)dr_\alpha$ to the boundary of each of the meshes so obtained. The sum of these line integrals amounts to just

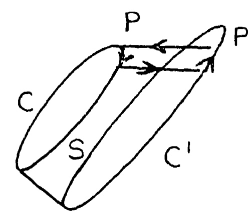

Fig. 21

$$\int_{C'}-\int_C\varphi_{\alpha\rho\ldots}(\mathbf{r},\, t)dr_\alpha,$$

for the integrals along the lines of flow cancel one another when adjacent meshes are considered. We thus have

$$Dc_{\rho\ldots} = dt\int_{C'} \frac{\partial\varphi_{\alpha\rho\ldots}}{\partial t} dr_\alpha + \Sigma\int_{d\Gamma} \varphi_{\alpha\rho\ldots}(\mathbf{r}, t)dr_\alpha,$$

where $d\Gamma$ denotes the boundary of a mesh.

Now transform each integral round a mesh by Stokes's theorem. We get

$$Dc_{\rho\ldots} = dt\int_{C'} \frac{\partial\varphi_{\alpha\rho\ldots}}{\partial t} dr_\alpha + \int_S \left[A_{\alpha\beta\gamma}\frac{\partial}{\partial r_\beta}(\varphi_{\gamma\rho\ldots})\right] n_\alpha\, dS,$$

where the surface integral is now extended to the whole of the collar. But the vector element of area $n_\alpha dS$ is given by

$$\mathbf{n}dS = \mathbf{u}(\mathbf{r})dt\wedge d\mathbf{r},$$

where $d\mathbf{r}$ has the sense in which the original circulation round C was defined. The contribution of the surface integral is accordingly

$$dt\int_C A_{\alpha\beta\gamma}\frac{\partial}{\partial r_\beta}(\varphi_{\gamma\rho\ldots})A_{\alpha\mu\nu}u_\mu\, dr_\nu.$$

Interchanging the dummy suffixes α and ν, this becomes

$$dt\int_C A_{\alpha\nu\mu}u_\mu A_{\nu\beta\gamma}\frac{\partial}{\partial r_\beta}(\varphi_{\gamma\rho\ldots})dr_\alpha.$$

Accordingly, proceeding to the limit $dt\to o$,

$$\frac{Dc_{\rho\ldots}}{Dt} = \int_C \left[\frac{\partial\varphi_{\alpha\rho\ldots}}{\partial t} - A_{\alpha\mu\nu}u_\mu A_{\nu\beta\gamma}\frac{\partial}{\partial r_\beta}(\varphi_{\gamma\rho\ldots})\right] dr_\alpha,$$

which is the general form of the theorem.

When $\varphi_{\alpha\rho\ldots}$ reduces to a vector Q_α, the theorem takes the form

$$\frac{Dc}{Dt} = \int_C \left[\frac{\partial\mathbf{Q}}{\partial t} - \mathbf{u}\wedge \text{curl }\mathbf{Q}\right].d\mathbf{r},$$

which is equivalent to the form enunciated.

120. *Further transformations.* We can transform the second term in the above integrand thus. We have

$$[(\text{curl }\mathbf{Q})\wedge\mathbf{u}]_\alpha = A_{\alpha\beta\gamma}(\text{curl }\mathbf{Q})_\beta u_\gamma$$
$$= A_{\alpha\beta\gamma}A_{\beta\mu\nu}\left(\frac{\partial}{\partial r_\mu}Q_\nu\right)u_\gamma.$$

Applying the $(\mathbf{A}, \mathbf{U})$ theorem, we get now

$$[(\text{curl }\mathbf{Q})\wedge\mathbf{u}]_\alpha = (U_{\mu\gamma}U_{\nu\alpha} - U_{\mu\alpha}U_{\nu\gamma})\left(\frac{\partial}{\partial r_\mu}Q_\nu\right)u_\gamma$$
$$= \left(\frac{\partial Q_\alpha}{\partial r_\gamma} - \frac{\partial Q_\gamma}{\partial r^\alpha}\right)u_\gamma.$$

Thus

$$\frac{Dc}{Dt}=\int_C\left[\frac{\partial Q_\alpha}{\partial t}+\left(u_\gamma\frac{\partial Q_\alpha}{\partial r_\gamma}-u_\gamma\frac{\partial Q_\gamma}{\partial r_\alpha}\right)\right]dr_\alpha.$$

But

$$\frac{\partial Q_\alpha}{\partial t}+u_\gamma\frac{\partial Q_\alpha}{\partial r_\gamma}=\frac{D}{Dt}Q_\alpha,$$

and

$$\frac{\partial Q_\gamma}{\partial r_\alpha}dr_\alpha=dQ_\gamma.$$

Hence

$$\frac{Dc}{Dt}=\int\frac{DQ_\alpha}{Dt}dr_\alpha-\int u_\gamma dQ_\gamma.$$

121. *Particular case.* In the particular case where the vector function **Q** is the velocity at **r** itself, so that $Q_\gamma=u_\gamma$, we have for the second term on the right-hand side

$$\int u_\gamma dQ_\gamma=\int u_\gamma du_\gamma=\int\tfrac{1}{2}d(u_\gamma u_\gamma)=0,$$

if **u** is one-valued. Hence in this case

$$\frac{Dc}{Dt}=\int_C\frac{Du_\alpha}{Dt}dr_\alpha.$$

Now suppose that Du_α/Dt is the derivative with respect to r_α of a one-valued scalar function φ of **r**, so that

$$\frac{Du_\alpha}{Dt}=\frac{\partial\varphi}{\partial r_\alpha}.$$

Then

$$\frac{Dc}{Dt}=[\varphi]_C=0.$$

The circulation round C is thus under these circumstances constant.

The application of this result to hydrodynamics is called Kelvin's theorem.*

* The hydrodynamical equation of motion of a non-viscous fluid is

$$\rho\frac{Du_\alpha}{Dt}=-\frac{\partial p}{\partial r_\alpha}-\rho\frac{\partial\Omega}{\partial r_\alpha},$$

where p is the pressure, Ω the potential of the external field of force, and ρ is the density. When $\int dp/\rho$ is a function of position in the fluid, the circulation round any closed circuit is accordingly constant.

Part II. Systems of Line Vectors

CHAPTER V

LINE CO-ORDINATES

122. *The moment of a line vector about a point.* Consider a line vector **F**, localized in a line l. Let O be any point, taken as origin of position vectors, and let A be any point in l. Let **r** denote the position vector OA of A with respect to O.

Consider the vector product

$$\mathbf{r}\wedge\mathbf{F}.$$

It is constructed out of a line vector and a position vector with respect to a point O. This vector product is independent of the point A chosen. For let A′, of position vector **r′**, be any other point on l (Fig. 22). Then

$$\begin{aligned}\mathbf{r}'\wedge\mathbf{F} &= \mathbf{OA}'\wedge\mathbf{F} = (\mathbf{OA}+\mathbf{AA}')\wedge\mathbf{F}\\ &= \mathbf{OA}\wedge\mathbf{F}+\mathbf{AA}'\wedge\mathbf{F}\\ &= \mathbf{r}\wedge\mathbf{F},\end{aligned}$$

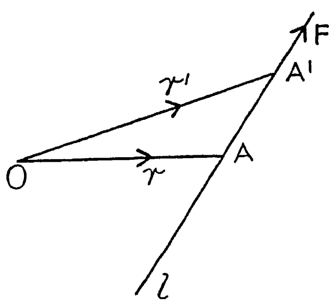

Fig. 22

since $\mathbf{AA}'\wedge\mathbf{F}=0$, **AA′** and **F** being parallel. Hence the vector product in question depends on the line vector **F** and the origin O only. It is called *the moment of* **F** *about* O, and we write

$$\mathbf{M}(\mathrm{O})=\mathbf{r}\wedge\mathbf{F}.$$

Older treatises on statics or line vectors were accustomed to introduce the notion of a *moment about a point* for the two-dimensional case where all the line vectors were coplanar, and then introduce the moment of a line vector about a line for the case where the line vector and the second line were skew to one another in space. But the moment of a line vector about a given *line* is a less fundamental notion than the moment of a line vector about a given *point*, the former being merely the component of the latter (taken with respect to any point in the line) along the line.

123. Before beginning the study of systems of line vectors, it is convenient to discuss some geometrical properties of lines, considered by means of the *moments of the lines* about an origin O. By *the moment of a line about* O we mean the moment of a unit line vector (in the line) about O, a determinate sense having first been assigned to the line.

124. *Line co-ordinates.* Let l (Fig. 23) be a line, O a fixed point, P any point in the line, N the foot of the perpendicular from O on the line. Assign to the line a determinate sense, and let **i** be a unit vector parallel to l and in the sense of l. Then the association of **i** with l constitutes a line vector.

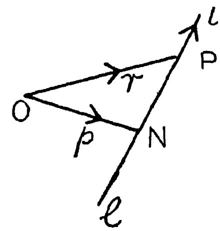

Fig. 23

Let **p** denote the vector **ON**. Then

$$\mathbf{p}.\mathbf{i}=0.$$

Let **r** be the position vector OP of P with respect to O. Then

$$\mathbf{r}=\mathbf{p}+\lambda\mathbf{i},$$

where λ is a parameter, of the dimensions of a length, that can take any real value, positive or negative. It follows that the moment of l about O, namely

$$\mathbf{r}\wedge\mathbf{i},$$

is equal to

$$\mathbf{p}\wedge\mathbf{i}.$$

Call this vector **a**. Then the vectors (**i**, **a**) are called the *line co-ordinates* of l with respect to O. Strictly speaking, the phrase ' line co-ordinates ' refers to the six components of the two vectors **i**, **a**, but it is convenient to retain the term in speaking of the two vectors **i**, **a**. Since $\mathbf{a}=\mathbf{p}\wedge\mathbf{i}$, it follows that

$$\mathbf{a}.\mathbf{i}=0,$$

and so the two line co-ordinates are perpendicular. We have also the relation

$$\mathbf{i}^2=1.$$

Since there are thus two relations between the six components of the two line co-ordinates **i** and **a**, four independent numbers are involved in a pair of line co-ordinates, corresponding to the four degrees of freedom of a line in space.

Since $\mathbf{a}=\mathbf{p}\wedge\mathbf{i}$, $\mathbf{i}^2=1$, $\mathbf{p}.\mathbf{i}=0$, it follows by the continued vector product theorem that

$$\mathbf{i}\wedge\mathbf{a}=\mathbf{i}\wedge(\mathbf{p}\wedge\mathbf{i})=\mathbf{p}$$

consequently

$$\mathbf{r}=\mathbf{i}\wedge\mathbf{a}+\lambda\mathbf{i}. \qquad (\mathbf{i}.\mathbf{a}=0,\ \mathbf{i}^2=1.)$$

This may be considered as the standard form of the equation of a line of line co-ordinates **i**, **a**. Further

$$\mathbf{p}^2=(\mathbf{i}\wedge\mathbf{a})^2=\mathbf{a}^2,$$

so that

$$|\mathbf{p}|=|\mathbf{a}|.$$

The moment of a line vector f**i** in l is by definition

$$\mathbf{r}\wedge f\mathbf{i}$$

$$=f(\mathbf{i}\wedge\mathbf{a}+\lambda\mathbf{i})\wedge\mathbf{i}=f\mathbf{a}.$$

The moment of the line vector $f\mathbf{i}$ in l about any other point of position vector $\mathbf{r}_0$ with respect to O is

$$\begin{aligned}&(\mathbf{r}-\mathbf{r}_0)\wedge f\mathbf{i}\\ &=f[\mathbf{a}-\mathbf{r}_0\wedge\mathbf{i}].\end{aligned}$$

125. *The perpendicular distance between the lines* $(\mathbf{i}, \mathbf{a})$ $(\mathbf{i}', \mathbf{a}')$. The directions of the lines being those of the unit vectors $\mathbf{i}$ and $\mathbf{i}'$, the common perpendicular to the two lines in a certain sense is parallel to $\mathbf{i}\wedge\mathbf{i}'$. If θ is the angle between the lines, as given by the senses of $\mathbf{i}$ and $\mathbf{i}'$, then $\cos\theta=\mathbf{i}.\mathbf{i}'$ and $(\mathbf{i}\wedge\mathbf{i}')/\sin\theta$ is a unit vector along the common perpendicular. If P is any point on $(\mathbf{i}, \mathbf{a})$, P$'$ any point on $(\mathbf{i}', \mathbf{a}')$, then the length of the perpendicular distance between the lines is equal to the projection of the vector $\mathbf{PP}'$ on the unit vector $(\mathbf{i}\wedge\mathbf{i}')/\sin\theta$. Calling this perpendicular distance $\tilde{\omega}$, we have

$$\begin{aligned}\tilde{\omega}&=\left|(\mathbf{r}'-\mathbf{r}).\frac{(\mathbf{i}\wedge\mathbf{i}')}{\sin\theta}\right|\\ &=\frac{|\{(\mathbf{i}'\wedge\mathbf{a}'+\lambda'\mathbf{i})-(\mathbf{i}\wedge\mathbf{a}+\lambda\mathbf{i})\}.(\mathbf{i}\wedge\mathbf{i}')|}{\sin\theta}.\end{aligned}$$

The triple products involving λ and λ' vanish identically, and we get

$$\tilde{\omega}=\frac{1}{\sin\theta}\left|(\mathbf{i}'\wedge\mathbf{a}').(\mathbf{i}\wedge\mathbf{i}')-(\mathbf{i}\wedge\mathbf{a}).(\mathbf{i}\wedge\mathbf{i}')\right|.$$

But $$\begin{aligned}(\mathbf{i}'\wedge\mathbf{a}').(\mathbf{i}\wedge\mathbf{i}')&=[(\mathbf{i}\wedge\mathbf{i}')\wedge\mathbf{i}'].\mathbf{a}'=-\mathbf{a}'.\mathbf{i},\\ (\mathbf{i}\wedge\mathbf{a}).(\mathbf{i}\wedge\mathbf{i}')&=[(\mathbf{i}\wedge\mathbf{i}')\wedge\mathbf{i}].\mathbf{a}=+\mathbf{a}.\mathbf{i}'.\end{aligned}$$

Hence $$\tilde{\omega}=\frac{|\mathbf{a}'.\mathbf{i}+\mathbf{a}.\mathbf{i}'|}{\sin\theta}.$$

It follows that the condition that the two lines $(\mathbf{i}, \mathbf{a})$, $(\mathbf{i}', \mathbf{a}')$ shall intersect is

$$\mathbf{a}'.\mathbf{i}+\mathbf{a}.\mathbf{i}'=0.$$

This can be established directly. For, if the lines intersect, values of λ and λ' can be found so that

$$\mathbf{i}\wedge\mathbf{a}+\lambda\mathbf{i}=\mathbf{i}'\wedge\mathbf{a}'+\lambda'\mathbf{i}'.$$

Eliminate λ and λ' by multiplying scalarly by $\mathbf{i}\wedge\mathbf{i}'$. Then

$$(\mathbf{i}\wedge\mathbf{a}).(\mathbf{i}\wedge\mathbf{i}')=(\mathbf{i}'\wedge\mathbf{a}').(\mathbf{i}\wedge\mathbf{i}'),$$

which reduces to $$\mathbf{a}'.\mathbf{i}+\mathbf{a}.\mathbf{i}'=0.$$

126. *Parallel lines.* If $\mathbf{i}'=\mathbf{i}$, $\sin\theta=0$, and the formula for $\tilde{\omega}$ becomes indeterminate. The lines are now, say, $(\mathbf{i}, \mathbf{a})$ and $(\mathbf{i}, \mathbf{a}')$. The feet N, N$'$ of the perpendiculars from O on to the two lines have for position vectors

$$\mathbf{i}\wedge\mathbf{a} \text{ and } \mathbf{i}\wedge\mathbf{a}'.$$

Hence $$\begin{aligned}\mathrm{NN}'^2&=(\mathbf{i}\wedge\mathbf{a}-\mathbf{i}\wedge\mathbf{a}')^2=[\mathbf{i}\wedge(\mathbf{a}-\mathbf{a}')]^2\\ &=(\mathbf{a}-\mathbf{a}')^2.\end{aligned}$$

To find the direction of the perpendicular to the plane defined by the pair of parallel lines $(\mathbf{i}, \mathbf{a})$, $(\mathbf{i}, \mathbf{a}')$, we proceed thus. Since it is perpendicular to $\mathbf{i}$, a vector parallel to it must be of the form $\mathbf{X}\wedge\mathbf{i}$, where without loss of generality we can take $\mathbf{X}.\mathbf{i}=0$. This vector is perpendicular to the line joining the points

$$\mathbf{i}\wedge\mathbf{a}+\lambda\mathbf{i}, \qquad \mathbf{i}\wedge\mathbf{a}'+\lambda'\mathbf{i},$$

for any λ, λ'. Hence

$$(\mathbf{X}\wedge\mathbf{i}).[(\mathbf{i}\wedge\mathbf{a}+\lambda\mathbf{i})-(\mathbf{i}\wedge\mathbf{a}'+\lambda'\mathbf{i})]=0$$

or

$$(\mathbf{X}\wedge\mathbf{i}).[\mathbf{i}\wedge(\mathbf{a}-\mathbf{a}')]=0,$$

which reduces to

$$\mathbf{X}.(\mathbf{a}-\mathbf{a}')=0.$$

Hence $\mathbf{X}$, being perpendicular to $(\mathbf{a}-\mathbf{a}')$ as well as to $\mathbf{i}$, is parallel to $\mathbf{i}\wedge(\mathbf{a}-\mathbf{a}')$. Hence the perpendicular to the plane of the lines is parallel to

$$[\mathbf{i}\wedge(\mathbf{a}-\mathbf{a}')]\wedge\mathbf{i}$$

i.e. to

$$\mathbf{a}-\mathbf{a}'.$$

127. *The mutual moment of two lines.* Let $(\mathbf{i}, \mathbf{a})$, $(\mathbf{i}', \mathbf{a}')$ be the line co-ordinates of two lines l, l′. Let P be any point on $(\mathbf{i}, \mathbf{a})$, P′ any point on $(\mathbf{i}', \mathbf{a}')$. The moment of the unit vector $\mathbf{i}'$ in l′ about the point P in l is, by definition,

$$\mathrm{PP}'\wedge\mathbf{i}'.$$

The component of this along $\mathbf{i}$ is

$$\mathrm{PP}'\wedge\mathbf{i}'.\mathbf{i},$$

which is equal to

$$\mathrm{P}'\mathrm{P}\wedge\mathbf{i}.\mathbf{i}'.$$

Either of these is called the *mutual moment of the two lines.* It is independent of the order in which the lines are considered, but depends on the sense associated with each line. Its value is independent of the particular points P, P′ chosen on the given lines.

Its value in terms of the line co-ordinates of the lines is

$$[(\mathbf{i}'\wedge\mathbf{a}'+\lambda\mathbf{i}')-(\mathbf{i}\wedge\mathbf{a}+\lambda\mathbf{i})]\wedge\mathbf{i}'.\mathbf{i}$$

which reduces to

$$\mathbf{i}.\mathbf{a}'+\mathbf{i}'.\mathbf{a}.$$

Thus the magnitude of the mutual moment is equal to the magnitude of the perpendicular distance between them multiplied by the sine of the angle between them. It is also equal to six times the volume of the tetrahedron subtended by representations of unit vectors in the two lines.

Example (1). A line passes through the point $\mathbf{r}_1$, in the direction of the unit vector $\mathbf{i}$. Show that its line co-ordinate '$\mathbf{a}$' is $\mathbf{r}_1\wedge\mathbf{i}$.

Example (2). Show that the line co-ordinates of the line joining the points $\mathbf{r}_1$ and $\mathbf{r}_2$ are

$$\mathbf{i}=\frac{\mathbf{r}_2-\mathbf{r}_1}{|\mathbf{r}_2-\mathbf{r}_1|}, \qquad \mathbf{a}=\frac{\mathbf{r}_1\wedge\mathbf{r}_2}{|\mathbf{r}_2-\mathbf{r}_1|}.$$

Example (3). Show that the line co-ordinates ($\mathbf{j}$, $\mathbf{b}$) of the common perpendicular to the lines of co-ordinates ($\mathbf{i}$, $\mathbf{a}$), ($\mathbf{i}'$, $\mathbf{a}'$), are

$$\mathbf{j}=\frac{\mathbf{i}\wedge\mathbf{i}'}{|\mathbf{i}\wedge\mathbf{i}'|}, \qquad \mathbf{b}=\frac{(\mathbf{i}\wedge\mathbf{i}')}{|\mathbf{i}\wedge\mathbf{i}'|^3}.[(\mathbf{i}.\mathbf{i}')(\mathbf{a}'\mathbf{i}+\mathbf{a}\mathbf{i}')-(\mathbf{a}\mathbf{i}+\mathbf{a}'\mathbf{i}')].$$

Example (4). Find formulae for the position vectors of the feet of the common perpendicular to two given lines ($\mathbf{i}$, $\mathbf{a}$), ($\mathbf{i}'$, $\mathbf{a}'$).

Example (5). Find the line co-ordinates of the line of intersection of the planes

$$\mathbf{r}.\mathbf{n}=\alpha, \qquad \mathbf{r}.\mathbf{n}'=\alpha',$$

where $\mathbf{n}$, $\mathbf{n}'$ are given unit vectors, α, α' given scalars, and $\mathbf{r}$ is a current position vector.

Clearly $\mathbf{n}$ and $\mathbf{n}'$ are the normals to the planes, α and α' the perpendiculars from the origin. The line of intersection, being normal to $\mathbf{n}$ and to $\mathbf{n}'$, has for its ' $\mathbf{i}$ ' co-ordinate

$$\mathbf{i}=\frac{\mathbf{n}\wedge\mathbf{n}'}{|\mathbf{n}\wedge\mathbf{n}'|}.$$

A point
$$\mathbf{r}=\mathbf{i}\wedge\mathbf{a}+\lambda\mathbf{i}$$

will then lie in the given planes if

$$\left[\frac{(\mathbf{n}\wedge\mathbf{n}')}{|\mathbf{n}\wedge\mathbf{n}'|}\wedge\mathbf{a}+\lambda\frac{\mathbf{n}\wedge\mathbf{n}'}{|\mathbf{n}\wedge\mathbf{n}'|}\right].\mathbf{n}=\alpha,$$

or
$$[(\mathbf{n}\wedge\mathbf{n}')\wedge\mathbf{a}].\mathbf{n}=\alpha|\mathbf{n}\wedge\mathbf{n}'|,$$

and similarly
$$[(\mathbf{n}\wedge\mathbf{n}')\wedge\mathbf{a}].\mathbf{n}'=\alpha'|\mathbf{n}\wedge\mathbf{n}'|.$$

Further, we must have $\mathbf{a}.(\mathbf{n}\wedge\mathbf{n}')=0.$

Expand the unknown vector $\mathbf{a}$ in the form

$$\mathbf{a}=x\mathbf{n}+y\mathbf{n}'+z(\mathbf{n}\wedge\mathbf{n}').$$

Then $z=0$. The two relations involving α and α' will then be found to reduce to

$$y=\frac{-\alpha}{|\mathbf{n}\wedge\mathbf{n}'|}, \qquad x=\frac{+\alpha'}{|\mathbf{n}\wedge\mathbf{n}'|}.$$

Hence
$$\mathbf{a}=\frac{\alpha'\mathbf{n}-\alpha\mathbf{n}'}{|\mathbf{n}\wedge\mathbf{n}'|}.$$

Example (6). If $\mathbf{i}$, $\mathbf{j}$ are unit vectors, and $\mathbf{a}$, $\mathbf{b}$ two other vectors satisfying

$$\mathbf{a}.\mathbf{i}=0, \qquad \mathbf{b}.\mathbf{j}=0, \qquad \mathbf{a}.\mathbf{j}+\mathbf{b}.\mathbf{i}=0,$$

prove that

$$(\mathbf{i}\wedge\mathbf{a})-(\mathbf{j}\wedge\mathbf{b})=(\mathbf{a}\wedge\mathbf{b}).\left[\frac{\mathbf{i}\mathbf{i}}{\mathbf{b}.\mathbf{i}}+\frac{\mathbf{j}\mathbf{j}}{\mathbf{a}.\mathbf{j}}\right].$$

Show that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{i}\wedge\mathbf{j}$ are linearly independent and then show that the scalar products of these with the two sides of the equation are equal. Alternatively, find the point of intersection of the intersecting lines $(\mathbf{i}, \mathbf{a})$, $(\mathbf{j}, \mathbf{b})$ in the two forms

$$\mathbf{r}=\mathbf{i}\wedge\mathbf{a}+\lambda\mathbf{i}, \qquad \mathbf{r}=\mathbf{j}\wedge\mathbf{b}+\lambda'\mathbf{j},$$

and equate the results.

Example (7). Find the perpendicular distance of the plane $\mathbf{r}.\mathbf{j}=p$ from the parallel line $(\mathbf{i}, \mathbf{a})$, where $\mathbf{i}.\mathbf{j}=0$.

CHAPTER VI

SYSTEMS OF LINE VECTORS

128. A line vector **P** has already been defined as the class of all representations AB of a given free vector **P** which lie in a given straight line l parallel to **P**. A system of line vectors, of which a typical member is **P**, will be denoted by (**P**).

Abstract statics is the theory of systems of line vectors in which the line vectors represent forces. Statics is concerned with such questions as whether a given rigid body, or system of rigid bodies, subject to a given system of forces is in equilibrium or not. Such a question may be treated by an appeal to dynamics, together with an appeal to a principle known as the *principle of the transmissibility of force.* This principle asserts that if a rigid body is in equilibrium under the action of a given system of forces, the equilibrium is unaffected by the application of any pair of equal and opposite forces at any two points of the body whose join is parallel to the applied forces. This in effect defines the word 'rigid' as applied in statics to a body under the action of a system of forces. For this reason the theory of the equilibrium of a 'rigid' body under the action of a system of forces is sometimes called 'rigid statics.'

But when we have completed the investigation of the equilibrium of a rigid body under the action of a certain system of forces, we have arrived at an attribute or property of the system of forces in question which is independent of the rigid body introduced for the forces to 'act' upon. The property of being 'in equilibrium' is a property of the body. But the relationships between the line vectors representing the system of forces do not depend on the rigid body introduced, and it is desirable therefore to analyse these relationships *per se.* Further, other systems of line vectors are encountered in applied mathematics besides systems of forces ; for example, the *momentum* of a system of particles is represented by a system of line vectors, and the small displacements of a system of particles are also represented by a system of line vectors. We need, therefore, the abstract properties of systems of line vectors independently of what the line vectors represent.

A system of line vectors is a sort of polarization of 'space,' of the space which is the seat of their lines of action. It is a geometrical entity, describable by the system of line vectors specified, but capable of being described by other systems of line vectors, which are then defined as being 'equivalent' to the original system. Just as the free vector is the

class of its representations, so there exists an entity which is the class of all systems of line vectors equivalent to a given system of line vectors. This entity is something more fundamental than any particular system which describes or represents it. Such entities are worthy of study for their own sakes, quite apart from particular applications. Just as a geometrical entity may be built up out of points, lines and planes, so the geometrical entities here in question are built up out of line vectors. The line vector is a geometrical unit or 'brick' of the structure, which is something essentially different from a point, a line or a plane: it is a combination of a *line*, a *length* and a *sense*. The same underlying entity may be built up out of these bricks in different arrangements. These we now proceed to analyse.

129. *Equivalent systems of line vectors.* Let S be a system of line vectors $(\mathbf{P})$. Let A be any point in the domain of S. Let Ω be a system of line vectors $(\mathbf{W})$, all passing through A, whose vector sum is zero, i.e. such that

$$\Sigma\mathbf{W}=0.$$

The system Ω or $(\mathbf{W})$ will be described as a nul-concurrent system; the simplest example of an Ω is a pair of equal and opposite line vectors $(\mathbf{X}, -\mathbf{X})$ in the same straight line.

Consider the system consisting of S and Ω together. It may be that a member $\mathbf{P}_1$ of S has the same line of action as a member $\mathbf{W}_1$ of Ω. If so, replace the line vectors $(\mathbf{P}_1)$, $(\mathbf{W}_1)$ by the single line vector consisting of $\mathbf{P}_1+\mathbf{W}_1$ in the common line of action of $\mathbf{P}_1$ and $\mathbf{W}_1$. If it happens that $\mathbf{P}_1+\mathbf{W}_1=0$, we agree to disregard it, so that a nul line vector may be removed from the system. The process may be repeated, and may be applied to any set of collinear members of S. Let S′ be the system of line vectors obtained at any stage of this procedure, consisting of the residual members of S and Ω together with the new line vectors of the type $\mathbf{P}_1+\mathbf{W}_1$. Then S′ is said to be *equivalent* to S. In particular, the aggregate $(\mathbf{P})+(\mathbf{W})$ is equivalent to S.

Any two systems of line vectors S_1 and S_2 are now said to be equivalent if S_2 can be derived from S_1 by a process of the kind described, i.e. by the addition of nul-concurrent sets and combinations of collinear members. The relationship of *equivalence* is clearly a symmetrical one, and we write it as $S_2\equiv S_1$.

130. *The addition of members of a concurrent set.* Let $\mathbf{P}$, $\mathbf{Q}$ be two line vectors in the same line l. Then clearly the system $(\mathbf{P}, \mathbf{Q})$ is equivalent to the single vector $\mathbf{P}+\mathbf{Q}$ in l. For we have only to add to the system $(\mathbf{P}, \mathbf{Q})$ the nul set $(\mathbf{P}, -\mathbf{P})$ in l, and combine appropriately.

Now let $\mathbf{P}$, $\mathbf{Q}$ be two line vectors in lines l_1, l_2 which intersect in a point O (Fig. 24). Let $\mathbf{R}$ be the vector sum of $\mathbf{P}$ and $\mathbf{Q}$, i.e. $\mathbf{R}=\mathbf{P}+\mathbf{Q}$. Add to the system $(\mathbf{P}, \mathbf{Q})$ the nul-concurrent set $(\mathbf{R}, -\mathbf{P}, -\mathbf{Q})$ concurrent

at O. Then, by combining collinear members and removing nul vectors, we see that the system (**P**, **Q**) is equivalent to the system (**R**) at O, i.e. is equivalent to the line vector **R** in a line l_3 through O parallel to the free vector **P**+**Q**. We write this result

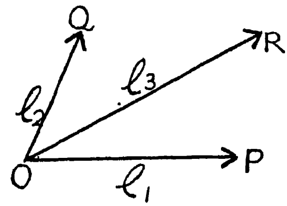

Fig. 24

$$(\mathbf{R}) \equiv (\mathbf{P}, \mathbf{Q}),$$

or

$$(\mathbf{R}) \equiv (\mathbf{P}) + (\mathbf{Q}),$$

and call (**R**) the *resultant* of (**P**) and (**Q**). The statement $(\mathbf{R}) \equiv (\mathbf{P}) + (\mathbf{Q})$ is to be carefully distinguished from the statement $\mathbf{R} = \mathbf{P} + \mathbf{Q}$; and the *resultant of concurrent line vectors* is to be carefully distinguished from the *sum of the corresponding free vectors.*

The process may be repeated with any number of concurrent line vectors, and their resultant (**R**) found, by combining say ($\mathbf{P}_1$) and ($\mathbf{P}_2$) to give ($\mathbf{R}_2$), and then ($\mathbf{R}_2$) and ($\mathbf{P}_3$) to give ($\mathbf{R}_3$), and so on.

It may happen that the final resultant (**R**) is a nul vector. In that case the system of concurrent line vectors is said to be *in equilibrium*, or to be *equivalent to zero.* Similarly if a system S is equivalent to a number of nul-concurrent sets (the different sets being possibly concurrent at different points), then S is said to be in equilibrium, or to be equivalent to zero ; and we write then $S \equiv 0$, or $(\mathbf{P}) \equiv 0$. Clearly if $(\mathbf{P}) \equiv (\mathbf{Q})$, then $(\mathbf{P}) + (-\mathbf{Q}) \equiv 0$, i.e. if two systems are equivalent, the system formed by either together with the other reversed is a system in equilibrium.

131. In order to find necessary and sufficient conditions for the equivalence of two systems of line vectors, it is necessary to pay attention both to their component vectors and to the lines of action of these vectors. Adequate information about the contribution of a given line vector to the system is usually contained in the statement of the *moment of the line vector about some given point.* This has been defined in § 122. We repeat here that the moment **M**(O) of a line vector (**F**) about a point O is given by

$$\mathbf{M}(O) = \mathbf{r} \wedge \mathbf{F},$$

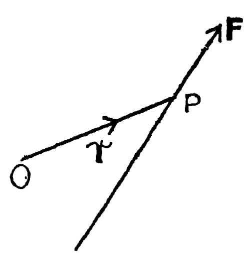

Fig. 25

where **r** is the position vector, with respect to O, of any point P on the line of action of (**F**) (Fig. 25). The moment **M**(O) has been shown to be independent of the position of P on the line of action of **F**.

132. *Moment of a set of concurrent vectors about any point.*

Theorem : The sum of the moments about any point O of any number of concurrent line vectors is equal to the moment of their resultant.

For, let ($\mathbf{P}_1$, $\mathbf{P}_2$, ... $\mathbf{P}_n$) be the system of line vectors concurrent at

A, and let $\mathbf{r}$ be the position vector of A with respect to O. Also let $\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_n$ be the position vectors of points on the lines of action of $\mathbf{P}_1, \mathbf{P}_2, \ldots \mathbf{P}_n$ respectively. Then the resultant ($\mathbf{R}$) is a line vector through A given by

$$\mathbf{R}=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n.$$

Multiplying each side of this equality vectorially by $\mathbf{r}$, we have

$$\mathbf{r}\wedge\mathbf{R}=\mathbf{r}\wedge\mathbf{P}_1+\mathbf{r}\wedge\mathbf{P}_2+\ldots+\mathbf{r}\wedge\mathbf{P}_n$$

by the distributive theorem for vector products (§ 25). Hence

$$\mathbf{r}\wedge\mathbf{R}=\mathbf{r}_1\wedge\mathbf{P}_1+\mathbf{r}_2\wedge\mathbf{P}_2+\ldots+\mathbf{r}_n\wedge\mathbf{P}_n.$$

This is the desired result. In $\mathbf{r}\wedge\mathbf{R}$, $\mathbf{r}$ may be replaced by the position vector $\boldsymbol{\rho}$ of any point on the line of action of $\mathbf{R}$.

Corollary. If $\mathbf{P}_1, \mathbf{P}_2, \ldots \mathbf{P}_n$ form a nul-concurrent set, then the sum of their moments about any point is zero. For the moment of their resultant is zero.

133. *Moment of a line vector about a line.* We now define the moment of a line vector $\mathbf{F}$ in l about another line l′ as the *component along* l′ *of the moment of* $\mathbf{F}$ *about any point* O *in* l′. Here we mean by 'component' the scalar value. Thus, if $\mathbf{i}'$ is a unit vector in l′, the moment of ($\mathbf{F}$) about l′ is

$$\mathbf{M}(\mathrm{O}).\mathbf{i}'$$

which is equal to $\mathbf{r}\wedge\mathbf{F}.\mathbf{i}'$,

where $\mathbf{r}$ is the position vector of any point A on l, the line of action of $\mathbf{F}$ (Fig. 26). If O′, A′ are any other points on l′, l respectively, then the scalar number

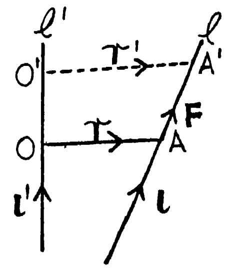

Fig. 26

$$\mathrm{O'A'}\wedge\mathbf{F}.\mathbf{i}'$$

is equal to $(\mathrm{O'O}+\mathrm{OA}+\mathrm{AA'})\wedge\mathbf{F}.\mathbf{i}'$.

But OO′ is parallel to $\mathbf{i}'$, and AA′ to $\mathbf{F}$. The contributions of these to the triple product therefore vanish, and we have

$$\mathrm{O'A'}\wedge\mathbf{F}.\mathbf{i}'=\mathrm{OA}\wedge\mathbf{F}.\mathbf{i}'.$$

Thus the moment of $\mathbf{F}$ in l about l′ is independent of the choice on l′ of the point O about which the moment of $\mathbf{F}$ is measured.

Clearly, if m is the mutual moment of the two lines l, l′ (§ 127), then the moment of $\mathbf{F}$ in l about l′ is $|\mathbf{F}|$m, provided the sense of $\mathbf{F}$ is the same as the specific sense of l.

It is particularly to be noted that the moment of a line vector about a *point* is a *vector*, the moment about a *line* a *scalar*. The concept of the moment of a line vector about a point is much more fundamental than the concept of the moment of a line vector about a line.

134. *Conditions satisfied by two equivalent systems of line vectors.*

Theorem : Let $(\mathbf{P}_1, \ldots \mathbf{P}_n)$, $(\mathbf{Q}_1, \ldots \mathbf{Q}_{n'})$ be two systems of line vectors,

and let $\mathbf{p}_1, \ldots \mathbf{p}_n, \mathbf{q}_1, \ldots \mathbf{q}_{n'}$ be the position vectors of points on their lines of action, respectively, with respect to a given point O. Then if the two systems are equivalent,

$$\text{(i)} \quad \sum_{r=1}^{n} \mathbf{P}_r = \sum_{r=1}^{n'} \mathbf{Q}_r,$$

$$\text{(ii)} \quad \sum_{r=1}^{n} \mathbf{p}_r \wedge \mathbf{P}_r = \sum_{r=1}^{n'} \mathbf{q}_r \wedge \mathbf{Q}_r.$$

In words, if $(\mathbf{P}) \equiv (\mathbf{Q})$, then their vector sums are equal and the sums of their moments about any point are equal.

For, since $(\mathbf{Q}) \equiv (\mathbf{P})$, there exists a set of points A_s and associated nul-concurrent sets of line vectors $(\mathbf{W})_s$ such that $(\mathbf{Q})$ can be derived from $(\mathbf{P})$ and the sets $(\mathbf{W})_s$ by combination and removal of nul vectors. Hence

$$\Sigma \mathbf{Q} = \Sigma \mathbf{P} + \sum_s (\mathbf{W})_s.$$

But each contribution $\Sigma \mathbf{W}$ vanishes separately. Hence the first part of the theorem. Again, since each system $(\mathbf{W})$ is a nul-concurrent set, the sum of the moments of the members of $(\mathbf{W})$ about O is zero. Hence the sum of the moments of the members of $(\mathbf{Q})$ about O, being the sum of the moments of the members of $(\mathbf{P})$ about O together with the sum of the moments of the members of the sets $(\mathbf{W})$, is equal to the sum of the moments of the members of $(\mathbf{P})$ about O.

The sum of the moments of the members of $(\mathbf{P})$ about O is called the moment of $(\mathbf{P})$ about O.

135. It follows from this theorem that the vectors $\Sigma \mathbf{P}$ and $\Sigma \mathbf{r} \wedge \mathbf{P}$ are the same for all equivalent systems of line vectors. The first vector depends only on the system; the second depends on the point O about which the moment is taken. We can therefore write

$$\mathbf{R} = \Sigma \mathbf{P}, \qquad \mathbf{G}(O) = \Sigma \mathbf{r} \wedge \mathbf{P}.$$

The question now suggests itself whether $\mathbf{R}$ and $\mathbf{G}(O)$ serve completely to characterize the system of line vectors in question; in other words, whether the converse of the theorem of § 134 is true.

136. The converse of this theorem is true, but is somewhat longer to prove. We have to show that given

$$\Sigma \mathbf{P} = \Sigma \mathbf{Q}, \qquad \Sigma \mathbf{r} \wedge \mathbf{P} = \Sigma \mathbf{r} \wedge \mathbf{Q},$$

then $(\mathbf{P}) \equiv (\mathbf{Q})$, i.e. that there exists a set of points A_s and associated nul-concurrent sets $(\mathbf{W})_s$ such that $(\mathbf{Q})$ can be derived from $(\mathbf{P})$ by aggregating $(\mathbf{P})$ with the sets $(\mathbf{W}_s)$, combining collinear members and removing nul vectors.

Instead of tackling directly the problem of locating a possible set of A_s's and $(\mathbf{W}_s)$'s, and so transforming $(\mathbf{P})$ into $(\mathbf{Q})$, we transform each into an equivalent simpler system and then show the equivalence of these simpler systems.

137. *Parallel line vectors.* Let **P**, **Q** be two *parallel* (or antiparallel) line vectors. We wish to investigate whether we can find a single line vector equivalent to them. Choose points A, B (Fig. 27) in the lines of action of **P** and **Q**, and introduce the nul-concurrent set (**X**, $-$**X**) acting along AB and BA. The resultant of $-$**X** and **P**, which meet at A, is, say, **Y** in the plane of the given line vectors **P**, **Q**; the resultant of $+$**X** and **Q** is, say, **Y′** at B, in the plane of **P** and **Q**. In general the lines of action of **Y** and **Y′** will intersect, say at D, a point in the plane of **P** and **Q**, and the resultant of **Y** and **Y′** will be a line vector **R**, through D, in the plane of **Y** and **Y′**, and so in the plane of **P** and **Q**. Since

$$(\mathbf{R}) \equiv (\mathbf{Y}, \mathbf{Y}') \equiv (\mathbf{P}, \mathbf{Q}),$$

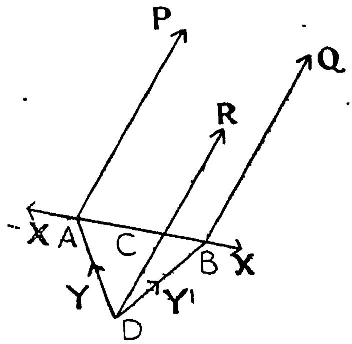

Fig. 27

we have by the preceding theorem

$$\mathbf{R} = \mathbf{P} + \mathbf{Q},$$

so that **R** is parallel to **P** and **Q**. Let the line of action of **R** meet AB in C. Then if O is any point whatever, we have (§ 134),

$$OC \wedge \mathbf{R} = OA \wedge \mathbf{P} + OB \wedge \mathbf{Q}.$$

Take O to coincide with C. Then

$$CA \wedge \mathbf{P} + CB \wedge \mathbf{Q} = 0.$$

Since **Q** and **P** are parallel vectors, we may put

$$\mathbf{Q} = \mu \mathbf{P}.$$

Hence
$$(CA + \mu CB) \wedge \mathbf{P} = 0.$$

But **P** is not parallel to CA or to CB, and, since CA and CB are parallel, **P** is therefore not parallel to CA$+\mu$CB. Hence

$$CA + \mu CB = 0,$$

or
$$-AC + \mu(AB - AC) = 0,$$

or
$$AC = \frac{\mu}{\mu + 1} AB.$$

This determines the position of C provided $\mu + 1 \neq 0$, i.e. provided $\mathbf{P} + \mathbf{Q} \neq 0$, i.e. provided that **P** and **Q** are not equal and opposite.

We notice that C divides AB so that

$$AC : CB = |\mathbf{Q}| : |\mathbf{P}|$$

if **P** and **Q** are in the same sense, whilst

$$AC : CB = -|\mathbf{Q}| : |\mathbf{P}|$$

if **P** and **Q** are in opposite senses. The line vector (**R**) is called the *resultant* of the parallel line vectors (**P**) and (**Q**).

138. *Couples.* We now consider the exceptional case $\mu = -1$.

A system of line vectors $(\mathbf{P}, \mathbf{P}')$ where $\mathbf{P}' = -\mathbf{P}$ but the lines of action of $\mathbf{P}$ and $\mathbf{P}'$ do not coincide is called a *couple*. Let O be any point whatever, A, A′ points on the lines of action of $\mathbf{P}$, $\mathbf{P}'$ respectively (Fig. 28). Then the moment of the system $(\mathbf{P}, \mathbf{P}')$ about O is

$$\begin{aligned} &\mathrm{OA}\wedge\mathbf{P}+\mathrm{OA}'\wedge\mathbf{P}' \\ &=(\mathrm{OA}-\mathrm{OA}')\wedge\mathbf{P} \\ &=\mathrm{A'A}\wedge\mathbf{P}. \end{aligned}$$

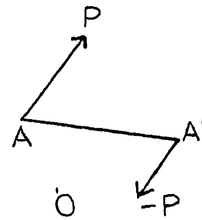

Fig. 28

Thus the moment of a couple about O is independent of the position of O. This vector $\mathrm{A'A}\wedge\mathbf{P}$ is clearly also independent of the positions of A, A′ on the lines of action of $\mathbf{P}$, $\mathbf{P}'$, and it is accordingly defined simply to be the *moment of the couple*, without point specified. The moment of a couple is a vector perpendicular to the plane of the two line vectors forming the couple.

139. *Equivalence of couples.* We shall now show that a couple is completely characterized by its moment—in other words, that in whatever way we construct a pair of equal and opposite line vectors having a given moment, all such pairs are equivalent.

Theorem: Two couples of equal moments are equivalent.

For, if the moments of the couples are equal, the couples themselves must lie in parallel planes. These planes may be distinct or coincident.

(i) Suppose first that the couples are in *coincident* planes. Let the couples be $(\mathbf{P}, \mathbf{P}')$, $(\mathbf{Q}, \mathbf{Q}')$. Either $\mathbf{P}$ is parallel to $\mathbf{Q}$ or it meets $\mathbf{Q}$.

(*a*) Suppose that the line of action of $\mathbf{P}$ meets that of $\mathbf{Q}$ in a point A, and that the line of action of $\mathbf{P}'$ meets that of $\mathbf{Q}'$ in A′ (Fig. 29). Then since the moments of the couples are equal,

$$\mathrm{A'A}\wedge\mathbf{P}=\mathrm{A'A}\wedge\mathbf{Q}$$

or $$\mathrm{A'A}\wedge(\mathbf{P}-\mathbf{Q})=0.$$

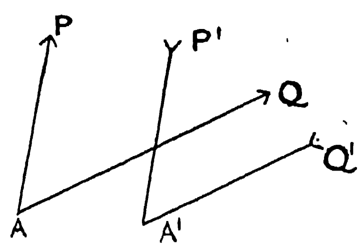

Fig. 29

Hence $(\mathbf{P}-\mathbf{Q})$ is parallel to A′A. To the system $(\mathbf{P}, \mathbf{P}')$ add the nul-concurrent vectors $(\mathbf{Q}-\mathbf{P})$ at A and $(\mathbf{P}-\mathbf{Q})$ or $(\mathbf{Q}'-\mathbf{P}')$ at A′, both of which act along AA′. The result is the system consisting of $\mathbf{Q}$ at A and $\mathbf{Q}'$ at A′. Hence $(\mathbf{Q}, \mathbf{Q}')\equiv(\mathbf{P}, \mathbf{P}')$.

(*b*) Suppose next that $\mathbf{P}$ is parallel to $\mathbf{Q}$, and so all four line vectors are parallel (Fig. 30). At least one member of the pair $(\mathbf{Q}, \mathbf{Q}')$ is such that $\mathbf{P}-\mathbf{Q}\neq 0$. Let it be $\mathbf{Q}$. Let a transversal meet $\mathbf{P}$, $\mathbf{P}'$; $\mathbf{Q}$, $\mathbf{Q}'$ in A, A′; B, B′ respectively. We are given that

$$\mathrm{A'A}\wedge\mathbf{P}=\mathrm{B'B}\wedge\mathbf{Q}.$$

We shall now prove that (**P**, **P**′) together with the reversal of (**Q**, **Q**′) are equivalent to zero.

For, since $\mathbf{P}-\mathbf{Q}\neq 0$, the parallel line vectors **P** at A and $-\mathbf{Q}$ at B possess a parallel resultant **R**, equal to $\mathbf{P}-\mathbf{Q}$, passing through a point O in the transversal such that

$$\mathrm{OA}\wedge\mathbf{P}+\mathrm{OB}\wedge(-\mathbf{Q})=0.$$

Similarly **P**′ or $-\mathbf{P}$ at A′ and $-\mathbf{Q}'$ or $+\mathbf{Q}$ at B′ possess a parallel resultant $\mathbf{R}'=\mathbf{P}'-\mathbf{Q}'=-\mathbf{P}+\mathbf{Q}=-\mathbf{R}$, passing through a point O′ in the transversal such that

$$\mathrm{O'A'}\wedge(-\mathbf{P})+\mathrm{O'B'}\wedge(+\mathbf{Q})=0.$$

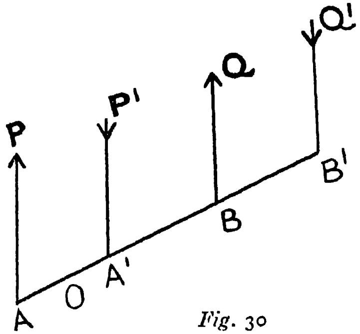

Fig. 30

Combining the three stated equations we have

$$(\mathrm{OA}+\mathrm{AA'}+\mathrm{A'O'})\wedge\mathbf{P}+(\mathrm{OB}+\mathrm{BB'}+\mathrm{B'O'})\wedge(-\mathbf{Q})=0,$$

or

$$\mathrm{OO'}\wedge(\mathbf{P}-\mathbf{Q})=0.$$

Since $\mathbf{P}-\mathbf{Q}\neq 0$ and since $\mathbf{P}-\mathbf{Q}$ is not parallel to the transversal, i.e. to OO′, we have $\mathrm{OO'}=0$. Hence **R** and **R**′ are along coincident lines, and $\mathbf{R}+\mathbf{R}'=0$. Hence $(\mathbf{R},\mathbf{R}')\equiv 0$. Hence $(\mathbf{P},\mathbf{P}')\equiv(\mathbf{Q},\mathbf{Q}')$.

(ii) Suppose secondly that the couples are in parallel *non-coincident* planes.

It follows from (i) that the line vectors forming a couple may be replaced by any pair of equal and opposite line vectors *in the same plane*, of the given moment. Let then the second couple be represented by a pair of line vectors (**Q**, **Q**′) in the plane of the second couple, and parallel to the pair (**P**, **P**′). Let **Q** be that member of the second couple which is such that $\mathbf{P}-\mathbf{Q}\neq 0$ (at least one of the members has this property). Let a plane meet **P**, **P**′ in A, A′ and **Q**, **Q**′ in B, B′ (Fig. 31). Then we are given that

$$\mathrm{A'A}\wedge\mathbf{P}=\mathrm{B'B}\wedge\mathbf{Q}.$$

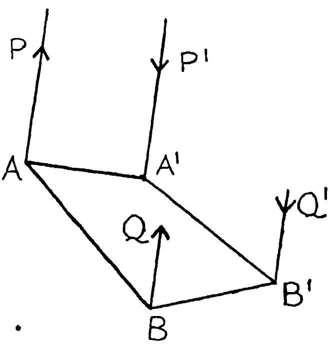

Fig. 31

We shall show that (**P**, **P**′) together with the reversal of (**Q**, **Q**′) are equivalent to zero. The parallel line vectors **P** at A and $-\mathbf{Q}$ at B have a parallel resultant **R**, equal to $\mathbf{P}-\mathbf{Q}$, passing through a point O in AB such that

$$\mathrm{OA}\wedge\mathbf{P}+\mathrm{OB}\wedge(-\mathbf{Q})=0.$$

The parallel line vectors **P**′ or $-\mathbf{P}$ at A′ and $-\mathbf{Q}'$ or $+\mathbf{Q}$ at B′ have a

resultant $\mathbf{R}'=\mathbf{P}'-\mathbf{Q}'=-\mathbf{P}+\mathbf{Q}=-\mathbf{R}$ passing through a point O′ in A′B′ such that

$$O'A'\wedge(-\mathbf{P})+O'B'\wedge(+\mathbf{Q})=0.$$

Combining the three stated equations we have

$$(OA+AA'+A'O')\wedge\mathbf{P}+(OB+BB'+B'O')\wedge(-\mathbf{Q})=0,$$

or

$$OO'\wedge(\mathbf{P}-\mathbf{Q})=0.$$

Since $\mathbf{P}-\mathbf{Q}\neq 0$ and OO′ lies in a plane meeting the four parallel line vectors, and so OO′ cannot be parallel to $\mathbf{P}-\mathbf{Q}$, we must have $OO'=0$. Hence $\mathbf{R}$ and $\mathbf{R}'$ are along coincident lines, and so $(\mathbf{R}, \mathbf{R}')\equiv 0$. Hence $(\mathbf{P}, \mathbf{P}')\equiv(\mathbf{Q}, \mathbf{Q}')$.

The reader should regard the foregoing theorem with some surprise. It is by no means obvious to common sense that couples of the same moment in distinct (parallel) planes are equivalent. The result is, of course, a consequence of the definition of equivalence adopted.

140. *Addition of couples.*

Theorem: A system consisting of two couples is equivalent to a single couple, whose moment is the sum of the moments of the given couples.

First, the combined system is equivalent to another couple.

For, if the two couples are in one plane, they may be represented by pairs of intersecting line vectors, which combine in pairs to form a new couple.

If the two couples are in distinct but parallel planes, one of the couples may be replaced by an equivalent couple in the plane of the other couple, and the members combined in pairs as above.

If the two couples are not coplanar, let their planes meet in a line l. Choose points O, O′ in l, and replace each couple by a pair of line vectors at O and O′, the pairs being in the planes of the respective couples. The two line vectors $(\mathbf{P}, \mathbf{Q})$ at O have a resultant $\mathbf{R}$, equal to $\mathbf{P}+\mathbf{Q}$, through O, and the two line vectors $(\mathbf{P}', \mathbf{Q}')$ at O′ have a resultant $\mathbf{R}'$, equal to $\mathbf{P}'+\mathbf{Q}'$, or to $-(\mathbf{P}+\mathbf{Q})$, at O′. The system $(\mathbf{R}, \mathbf{R}')$ forms a couple.

It now follows from the fundamental theorem of § 134, since the resultant couple is equivalent to the system consisting of the two given couples, that the moment of the resultant is equal to the sum of the moments of the given couples. This establishes the theorem.

It follows from the theorems of §§ 139, 140 that a couple is completely characterized by its *moment*, which is a *free vector*. Its contribution to any system of line vectors is a pair of antiparallel line vectors that can be constructed arbitrarily provided they have the given moment.

141. *Reduction of any system of line vectors.*

Theorem: Any system of line vectors is equivalent to a line vector passing through a prescribed point and a couple.

Let O be the prescribed point, ($\mathbf{P}$) any member of the system, A any point on the line of action of $\mathbf{P}$ (Fig. 32). Add to the system the nul-concurrent set ($\mathbf{P}$, $-\mathbf{P}$) at O. Repeat for every member ($\mathbf{P}$) of the system. Then the system ($\mathbf{P}_1$, ... $\mathbf{P}_n$) is equivalent to the system consisting of a set of concurrent line vectors $\mathbf{P}_1$, $\mathbf{P}_2$, ... $\mathbf{P}_n$ at O, together with a set of couples of the type $-\mathbf{P}$ at O, $+\mathbf{P}$ at A. The concurrent vectors $\mathbf{P}$ at O are equivalent to a single line vector at O, namely their resultant $\mathbf{R}=\Sigma\mathbf{P}$. The couples may be combined into a single equivalent couple, of moment $\mathbf{G}$ equal to the vector sum of the constituent moments, namely $\mathbf{G}=\Sigma \mathrm{OA}\wedge\mathbf{P}$. Thus

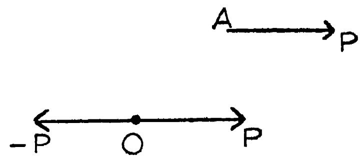

Fig. 32

$$(\mathbf{P}_1, \mathbf{P}_2, \ldots \mathbf{P}_n)\equiv(\mathbf{R}, \mathbf{G})$$

where
$$\mathbf{R}=\Sigma\mathbf{P}, \qquad \mathbf{G}=\Sigma \mathrm{OA}\wedge\mathbf{P}$$

and $\mathbf{R}$ passes through O.

It should be noted that the value of $\mathbf{R}$ is independent of O, but the line vector ($\mathbf{R}$) passes through O ; the value of $\mathbf{G}$ depends on O, but the couple $\mathbf{G}$ is a free vector.

142. *Sufficiency of conditions of equivalence.* We can now establish the converse of the theorem of § 134.

Theorem : If two systems of line vectors ($\mathbf{P}$), ($\mathbf{Q}$) are such that

$$\Sigma\mathbf{P}=\Sigma\mathbf{Q},$$
$$\Sigma\mathbf{r}\wedge\mathbf{P}=\Sigma\mathbf{r}\wedge\mathbf{Q},$$

where $\mathbf{r}$ is the position vector, reckoned from a fixed point O, of a typical point on any line vector, then

$$(\mathbf{P})\equiv(\mathbf{Q}).$$

For, let ($\mathbf{P}$) be reduced to a line vector $\mathbf{R}$ at O and a couple $\mathbf{G}$, ($\mathbf{Q}$) to a line vector $\mathbf{R}'$ at O and a couple $\mathbf{G}'$. Then, by the theorem of § 141 and the hypotheses of the present theorem,

$$\mathbf{R}\doteq\mathbf{R}'$$

and
$$\mathbf{G}=\mathbf{G}'.$$

Hence by proper choice of line vectors to represent $\mathbf{G}'$, ($\mathbf{R}'$, $\mathbf{G}'$) is identical with ($\mathbf{R}$, $\mathbf{G}$). But ($\mathbf{P}$) has been transformed into the equivalent system ($\mathbf{R}$, $\mathbf{G}$) by the addition of appropriate nul-concurrent sets, and ($\mathbf{Q}$) has been transformed into the equivalent system ($\mathbf{R}'$, $\mathbf{G}'$) by the addition of appropriate nul-concurrent sets. Hence ($\mathbf{P}$) may be transformed into ($\mathbf{Q}$) by the addition of appropriate nul-concurrent sets. Hence $(\mathbf{P})\equiv(\mathbf{Q})$.

Corollary. If $\Sigma\mathbf{P}=0$ and $\Sigma\mathbf{r}\wedge\mathbf{P}=0$, then $\mathbf{R}=0$ and $\mathbf{G}=0$, and so $(\mathbf{P})\equiv 0$, or ($\mathbf{P}$) is in equilibrium.

143. *Reduction to different bases.* When a system of line vectors $(\mathbf{P})$ is reduced to a line vector $\mathbf{R}$ at O and a couple $\mathbf{G}$, O is said to be the base point or base of the reduction. We proceed to examine the form of the reduction for different positions of the base point.

Let O′ be another base point, and let the system be reduced to a line vector $\mathbf{R}'$ at O′ and a couple $\mathbf{G}'$. Then

$$\mathbf{R}=\Sigma\mathbf{P}, \qquad \mathbf{G}=\Sigma \mathrm{OA}\wedge\mathbf{P},$$

$$\mathbf{R}'=\Sigma\mathbf{P}, \qquad \mathbf{G}'=\Sigma \mathrm{O'A}\wedge\mathbf{P}.$$

Hence
$$\mathbf{R}'=\mathbf{R},$$
and
$$\begin{aligned}\mathbf{G}'&=\Sigma(\mathrm{O'O}+\mathrm{OA})\wedge\mathbf{P}\\ &=\mathrm{O'O}\wedge\Sigma\mathbf{P}+\Sigma\mathrm{OA}\wedge\mathbf{P}\\ &=\mathrm{O'O}\wedge\mathbf{R}+\mathbf{G}.\end{aligned}$$

Thus, as remarked above, the value of the line vector $\mathbf{R}$ is independent of O, whilst the value of the free vector $\mathbf{G}$ depends on O.

We could have inferred the relation between $\mathbf{G}'$ and $\mathbf{G}$ directly from the equivalence of $(\mathbf{R}', \mathbf{G}')$ to $(\mathbf{R}, \mathbf{G})$. For since the moments about O′ are equal (Fig. 33),

$$\mathbf{G}'=\mathrm{O'O}\wedge\mathbf{R}+\mathbf{G}.$$

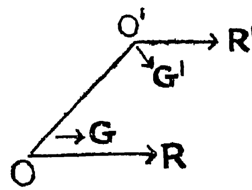

Fig. 33

Now consider the behaviour of $\mathbf{G}'$ as O′ varies. Put $\mathrm{OO}'=\mathbf{r}_0$. Then

$$\mathbf{G}'=\mathbf{G}-\mathbf{r}_0\wedge\mathbf{R}.$$

Hence
$$\mathbf{G}'.\mathbf{R}=\mathbf{G}.\mathbf{R}$$
or, more symmetrically
$$\mathbf{G}'.\mathbf{R}'=\mathbf{G}.\mathbf{R}.$$

This shows that $\mathbf{G}.\mathbf{R}$ is an invariant of the system of line vectors, possessing the same value for all equivalent systems. $\mathbf{R}^2$ is another (scalar) invariant.

As $\mathbf{r}_0$ varies, the direction of $\mathbf{G}'$ changes. But if $\mathbf{G}$ is not perpendicular to $\mathbf{R}$, $\mathbf{G}'$ can never be perpendicular to $\mathbf{R}'$; for if $\mathbf{G}.\mathbf{R}\neq 0$, we have always $\mathbf{G}'.\mathbf{R}'\neq 0$.

144. Let us now inquire whether a base $\mathbf{r}_0$ exists such that $\mathbf{G}'$ is parallel to $\mathbf{R}$. If so, we must have

$$\mathbf{G}'\wedge\mathbf{R}=0,$$

and so
$$(\mathbf{G}-\mathbf{r}_0\wedge\mathbf{R})\wedge\mathbf{R}=0,$$
or
$$\mathbf{r}_0\mathbf{R}^2-\mathbf{R}(\mathbf{r}_0\,.\mathbf{R})=\mathbf{R}\wedge\mathbf{G}.$$

The component of $\mathbf{r}_0$ along $\mathbf{R}$ is clearly not determined by the condition specified, since $\mathbf{r}_0$ occurs in the form $\mathbf{r}_0\wedge\mathbf{R}$. Hence the general solution is

$$\mathbf{r}_0=\frac{\mathbf{R}\wedge\mathbf{G}}{\mathbf{R}^2}+\lambda\mathbf{R},$$

where λ is arbitrary.

The locus of O′ is accordingly a straight line parallel to **R** (Fig. 34). The foot of the perpendicular from O to the locus of O′ is the point $(\mathbf{R}\wedge\mathbf{G})/\mathbf{R}^2$.

At any point O′ on this locus, **G** is parallel to **R**. Let the value of **G** at any point O′ on the locus be **Γ**. Then

$$\mathbf{\Gamma}=\mathbf{G}-\mathbf{r}_0\wedge\mathbf{R}=\mathbf{G}-\left[\frac{\mathbf{R}\wedge\mathbf{G}}{\mathbf{R}^2}+\lambda\mathbf{R}\right]\wedge\mathbf{R}=\mathbf{R}\frac{\mathbf{R}.\mathbf{G}}{\mathbf{R}^2}.$$

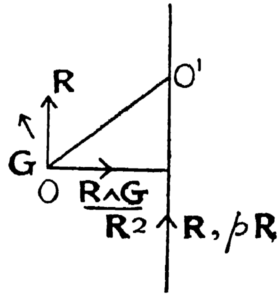

Fig. 34

Thus the value of **Γ** is independent of the position of O′ on the locus. Further $\mathbf{\Gamma}.\mathbf{R}=\mathbf{G}.\mathbf{R}$, a particular case of the invariance of **G.R.**

145. *Wrench. Central axis.* The locus of O′ such that **G** is parallel to **R** when O′ is base point is called the *central axis* of the system. The system consisting of a line vector **R** and a couple of moment **Γ** parallel to **R** is called a *wrench*. The central axis of the system consisting of a wrench is called simply the *axis* of the wrench. It follows that any system of line vectors is equivalent to a wrench.

146. *Pitch.* If the couple **Γ**, which is parallel to **R**, is put equal to p**R**, then p is called the *pitch* of the wrench. From § 144,

$$\mathrm{p}=\frac{\mathbf{G}.\mathbf{R}}{\mathbf{R}^2}.$$

This relation may be found more directly as follows. Suppose that (**R**, **G**) with base O is equivalent to the wrench (**R**, p**R**), whose axis passes through a point O′. Then equating moments about O,

$$\mathbf{G}=\mathrm{OO}'\wedge\mathbf{R}+\mathrm{p}\mathbf{R}.$$

Multiplying scalarly by **R** we have at once

$$\mathbf{G}.\mathbf{R}=\mathrm{p}\mathbf{R}^2,$$

which determines p as above. Multiplying vectorially by **R**, we have

$$\mathbf{G}\wedge\mathbf{R}=(\mathrm{OO}'\wedge\mathbf{R})\wedge\mathbf{R}=-\mathbf{R}^2\mathrm{OO}'+\mathbf{R}(\mathbf{R}.\mathrm{OO}').$$

This gives as before

$$\mathrm{OO}'=\frac{\mathbf{R}\wedge\mathbf{G}}{\mathbf{R}^2}+\lambda\mathbf{R},$$

which determines the axis.

147. *Uniqueness of wrench equivalent to a given system.* It follows from the above that the reduction of a given system of line vectors to a wrench can be made in one and only one way. For we have found a unique position for the axis of the wrench and a unique pitch p.

Alternatively, suppose that the wrench (**R**, p**R**), with axis through O, is equivalent to the wrench (**R**′, p′**R**′) with axis through O′. Then since

the two systems are equivalent, $\mathbf{R}=\mathbf{R}'$ and the moments about O are equal. Hence

$$\mathrm{p}\mathbf{R}=\mathrm{OO}'\wedge\mathbf{R}+\mathrm{p}'\mathbf{R}'.$$

Multiplying scalarly by $\mathbf{R}$ we have $\mathrm{p}=\mathrm{p}'$; and then it follows that $\mathrm{OO}'\wedge\mathbf{R}=0$, i.e. OO′ is parallel to $\mathbf{R}$. Thus O′ is another point on the axis of the same wrench.

148. *Particular cases.* If the system is such that $\mathbf{G}.\mathbf{R}=0$, whilst $|\mathbf{R}|\neq 0$, then the pitch p of the equivalent wrench vanishes, and the system is equivalent to a single line vector $\mathbf{R}$. In this case, $\mathbf{R}$ is called the *resultant* of the system. This usage is clearly consistent with our previous usage of the word in connection with a set of concurrent line vectors. Conversely, if the system reduces to a single line vector, then $\mathrm{p}=0$ and $\mathbf{G}.\mathbf{R}=0$.

If the system is such that $\mathbf{R}=0$, $\mathbf{G}\neq 0$, its moment about any point is $\mathbf{G}$, and the system reduces to a couple of the same moment $\mathbf{G}$ whatever base point is chosen. Thus if the invariant $\mathbf{G}.\mathbf{R}$ vanishes, and $\mathbf{R}\neq 0$, the system reduces to a single line vector; if the invariant $\mathbf{R}^2$ vanishes, it reduces to a couple.

149. *The moment of a given system of line vectors about a given line.* Let $(\mathbf{P})$ be a given system of line vectors, equivalent to $(\mathbf{R}, \mathbf{G})$ when O is base point. Let l be a given line, A any point on it (Fig. 35). Then the moment of each member of $(\mathbf{P})$ about l has been defined (§ 133) as the component along l, in a specified sense, of the moment of the member about A; the moment of the whole system $(\mathbf{P})$ about l is the sum of these components.

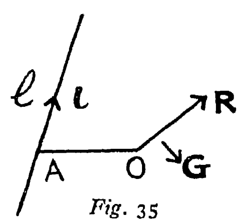

Fig. 35

Let l have line co-ordinates $(\mathbf{i}, \mathbf{a})$ with regard to O. Then the moment of $(\mathbf{P})$ about l is equal to

$$(\mathbf{G}+\mathrm{AO}\wedge\mathbf{R}).\mathbf{i}.$$

But by the definition of line co-ordinates (§ 124),

$$\mathrm{OA}=\mathbf{i}\wedge\mathbf{a}+\lambda\mathbf{i}.$$

Hence the moment of $(\mathbf{P})$ about l is

$$\mathbf{G}.\mathbf{i}-[(\mathbf{i}\wedge\mathbf{a}+\lambda\mathbf{i})\wedge\mathbf{R}].\mathbf{i}$$

$$=\mathbf{G}.\mathbf{i}+\mathbf{R}.\mathbf{a},$$

since $\mathbf{i}.\mathbf{a}=0$.

In particular, the moment about a line $(\mathbf{i}, \mathbf{a})$ of a single line vector $\mathbf{R}$ at O is $\mathbf{R}.\mathbf{a}$.

It follows from the above that if $(\mathbf{i}_1, \mathbf{a}_1), \ldots (\mathbf{i}_6, \mathbf{a}_6)$ are six lines such that the six-rowed determinant $||\mathbf{a}, \mathbf{i}||$ obtained by taking the components of the $\mathbf{a}$'s and the $\mathbf{i}$'s in any triad does not vanish, and if the moments of

(**P**) about these six lines vanish, then (**P**)$\equiv$o. For we must have **G**=o, **R**=o.

150. *Vanishing of the moment of a given line vector about a given line.* Since the moment of (**R**) at O about a line l, (**i**, **a**) is **R.a**, if **R.a**=o it follows that either |**R**|=o or **R** and **a** are perpendicular. But **a** is perpendicular to the plane through O and l (Fig. 36). Hence the line of action of **R** lies in this plane through O and l. Hence if the moment of a line vector (**R**) about a given line is zero, either **R**=o, or **R** meets l or **R** is parallel to l.

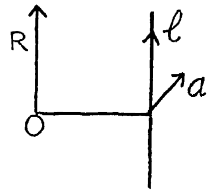

Fig. 36

151. *Systems of three line vectors.*

Theorem: If a system of three line vectors is equivalent to zero, the three line vectors must be coplanar and either concurrent or parallel.

For, let (**P**, **Q**, **R**) be the given system. Let AB be any line meeting (**P**) and (**Q**) in A and B. Since (**P**, **Q**, **R**)$\equiv$o, its moment about any line is zero. But the moments of (**P**) and (**Q**) about AB are zero. Hence the moment of (**R**) about AB is zero. Hence either **R**=o or (**R**) meets AB.

If **R**=o, (**P**, **Q**)$\equiv$o, and (**P**) and (**Q**) must be collinear and equal and opposite.

If **R**$\neq$o, (**R**) must meet similarly any second transversal AB_1, where B_1 is any other point on (**Q**). Hence (**R**) lies in the plane defined by A and (**Q**), and so (**Q**) and (**R**) are coplanar. It follows similarly that (**P**) lies in the plane defined by C, any point on (**R**), and (**Q**), and this plane contains both (**Q**) and (**R**). Hence (**P**), (**Q**) and (**R**) are coplanar. Let then (**P**) and (**Q**) meet in O. Then since (**P**, **Q**)$\equiv$(−**R**), it follows that (**R**) is equal and opposite to the resultant of (**P**) and (**Q**), and so passes through O. But if (**P**) and (**Q**) are parallel, then since **P**+**Q**+**R**=o, it follows that **R** is parallel to **P** and to **Q**. These are the required results.

It should be noted that (as for any number of line vectors) the vector sum of three line vectors can be zero without the system formed by them being equivalent to zero. If **P**+**Q**+**R**=o, and (**P**, **Q**, **R**)$\not\equiv$o, then (**P**, **Q**, **R**) must be equivalent to some couple **G**, of non-zero moment. For example three line vectors lying in the sides of a triangle, and proportional to the sides in order, clearly have vector sum zero, but are equivalent to a couple of moment perpendicular to the plane of the triangle and proportional to the area of the triangle. It should be noted further that there is a distinction between saying that three (or more) vectors are coplanar and saying that the three (or more) line vectors concerned are coplanar. Coplanarity means in the former case merely linear dependence.

152. *Reduction of a system to two line vectors.* Let (**P**) be a given system of line vectors, l some given line. We establish the following theorem.

Theorem: The system (**P**) may in general be reduced to two line vectors, of which one can be chosen so as to be along the given line l.

For, let the system be reduced to a line vector **R** at O, and a couple **G**. Let (**i**, **a**) be the line co-ordinates of the given line l with respect to O, and let (**i**′, **a**′) be the line co-ordinates of some other line l′. If possible, let the system be reduced to two line vectors, f**i** in l and f′**i**′ in l′. Then since

$$(\mathbf{R}, \mathbf{G}) \equiv (\mathrm{f}\mathbf{i}, \mathrm{f}'\mathbf{i}')$$

we must have, by § 124,

$$\mathrm{f}\mathbf{i}+\mathrm{f}'\mathbf{i}'=\mathbf{R}$$

$$\mathrm{f}\mathbf{a}+\mathrm{f}'\mathbf{a}'=\mathbf{G}.$$

We have in effect six scalar relations, and the unknowns are also six in number, namely the unknowns f and f′, and the four further unknowns comprised in **i**′, **a**′.

To solve these equations, we recollect that $\mathbf{a}'.\mathbf{i}'=0$. Hence

$$(\mathbf{R}-\mathrm{f}\mathbf{i}).(\mathbf{G}-\mathrm{f}\mathbf{a})=0,$$

or, since also $\mathbf{a}.\mathbf{i}=0$,

$$\mathrm{f}=\frac{\mathbf{R}.\mathbf{G}}{\mathbf{G}.\mathbf{i}+\mathbf{R}.\mathbf{a}}.$$

This determines the value of f provided $\mathbf{G}.\mathbf{i}+\mathbf{R}.\mathbf{a} \neq 0$, i.e. provided the moment of the system about the given line is not zero (§ 149). The value of f′ then follows from

$$\mathrm{f}'=|\mathbf{R}-\mathrm{f}\mathbf{i}|$$

since $|\mathbf{i}'|=1$; the value of **i**′ follows from

$$\mathbf{i}'=\frac{\mathbf{R}-\mathrm{f}\mathbf{i}}{\mathrm{f}'};$$

and **a**′ follows from

$$\mathbf{a}'=\frac{\mathbf{G}-\mathrm{f}\mathbf{a}}{\mathrm{f}'}.$$

These satisfy identically $\mathbf{a}'.\mathbf{i}'=0$. Thus the solution is in general determinate, and clearly it is unique if $\mathbf{G}.\mathbf{i}+\mathbf{R}.\mathbf{a} \neq 0$.

Further, by using $\mathbf{a}.\mathbf{i}=0$ we have as above

$$\mathrm{f}'=\frac{\mathbf{G}.\mathbf{R}}{\mathbf{G}.\mathbf{i}'+\mathbf{R}.\mathbf{a}'}.$$

Hence

$$\frac{\mathbf{i}}{\mathbf{G}.\mathbf{i}+\mathbf{R}.\mathbf{a}}+\frac{\mathbf{i}'}{\mathbf{G}.\mathbf{i}'+\mathbf{R}.\mathbf{a}'}=\frac{\mathbf{R}}{\mathbf{R}.\mathbf{G}},$$

and

$$\frac{\mathbf{a}}{\mathbf{G}.\mathbf{i}+\mathbf{R}.\mathbf{a}}+\frac{\mathbf{a}'}{\mathbf{G}.\mathbf{i}'+\mathbf{R}.\mathbf{a}'}=\frac{\mathbf{G}}{\mathbf{R}.\mathbf{G}}.$$

Multiplying the respective sides of these equalities scalarly together we find

$$\mathbf{i}'.\mathbf{a}+\mathbf{i}.\mathbf{a}'=\frac{(\mathbf{G}.\mathbf{i}+\mathbf{R}.\mathbf{a})(\mathbf{G}.\mathbf{i}'+\mathbf{R}.\mathbf{a}')}{\mathbf{R}.\mathbf{G}}.$$

The left-hand side is the mutual moment of the two lines. Thus, if a system (**R, G**) is reduced to two line vectors, the mutual moment of the lines of action of these two line vectors is equal to the product of the moments of the system (**R, G**) about the two lines divided by the invariant **R.G.**

Clearly, if $\mathbf{G.R} \neq 0$ and if $\mathbf{G.i}+\mathbf{R.a} \neq 0$, f is non-zero and f′ is non-zero. Hence $\mathbf{G.i}'+\mathbf{R.a}' \neq 0$. Hence $\mathbf{i}'.\mathbf{a}+\mathbf{i}.\mathbf{a}' \neq 0$, so that the two lines do not intersect.

A pair of lines such that a given system is equivalent to line vectors in them is said to be a *conjugate* pair with respect to the given system.

153. That the reduction of a given system to two line vectors, of which the line of action of one is given, is in general unique, may be proved otherwise as follows. If possible let the system be equivalent to ($\mathbf{P}_1$, $\mathbf{Q}_1$) and also to ($\mathbf{P}_2$, $\mathbf{Q}_2$), of which $\mathbf{P}_1$ and $\mathbf{P}_2$ are in l. Then the system ($\mathbf{P}_1$, $\mathbf{Q}_1$, $-\mathbf{P}_2$, $-\mathbf{Q}_2$) is equivalent to zero. That is, ($\mathbf{P}_1-\mathbf{P}_2$, $\mathbf{Q}_1$, $-\mathbf{Q}_2$)$\equiv 0$. Hence either $\mathbf{P}_1-\mathbf{P}_2=0$ and $(\mathbf{Q}_1)\equiv(\mathbf{Q}_2)$ or $\mathbf{P}_1-\mathbf{P}_2$, $\mathbf{Q}_1$ and $-\mathbf{Q}_2$ are coplanar. In that case $\mathbf{P}_1$ and $\mathbf{Q}_1$ are coplanar, and the system has a single resultant or reduces to a couple.

154. *Nul lines.* If (**j, b**) are the line co-ordinates of a line such that the moment of the system (**R, G**) at O about (**j, b**) is zero, then (**j, b**) is said to be a *nul line* of the system. In this case

$$\mathbf{R.b}+\mathbf{G.j}=0.$$

If such a nul line intersects a given line (**i, a**) then

$$\mathbf{i.b}+\mathbf{a.j}=0.$$

Hence
$$(\mathbf{R}-\mathrm{f}\mathbf{i}).\mathbf{b}+(\mathbf{G}-\mathrm{f}\mathbf{a}).\mathbf{j}=0,$$

or, in the notation of § 152,

$$\mathbf{i}'.\mathbf{b}+\mathbf{a}'.\mathbf{j}=0.$$

Hence if a nul line intersects a given line, it intersects also its conjugate with respect to the given system.

This is otherwise obvious. For (**R, G**) may be replaced by f**i** in the given line (**i, a**) and f′**i**′ in its conjugate (**i**′, **a**′). Since the given line meets the nul line, the moment of (**R, G**) about the nul line is equal to the moment of f′**i**′ in (**i**′, **a**′) about the nul line, and hence the latter moment is zero. Hence (**i**′, **a**′) meets the nul line. Thus the nul line meets both members of a conjugate pair.

Let P be any point, and let the system (**R, G**) at O be reduced to (**R, G**′) at P. If **j** is a unit vector in the direction of a variable line through P, then the moment of the system about this line is **G**′.**j**. Hence if **j** is perpendicular to **G**′, the line will be a nul line. Hence the locus of nul lines through P is a plane normal to the axis of **G**′. This is called the *nul plane* through P.

It follows that if l is now *any* line through P, its conjugate l′ must lie

in the nul plane of P ; for all lines through P in the plane, being nul lines, and meeting l, must meet l′.

If two lines intersect, so do their conjugates. For if they intersect in a point P, their conjugates lie in the nul plane of P, and so intersect.

Again, if l and l′ are conjugate, the nul plane of any point P on l passes through l′. Hence as P moves along l, the nul plane of P is the plane through P and l′ turning round l′.

155. Let $\mathbf{r}_0$ be the position vector, with regard to O, of a point P. Let a given system of line vectors be reduced to ($\mathbf{R}$, $\mathbf{G}$) with base point O. Take a given unit vector $\mathbf{i}$, and consider the locus of points P such that when the given system is reduced to ($\mathbf{R}$, $\mathbf{G}'$) with base point P, $\mathbf{G}'$ is parallel to $\mathbf{i}$. The required condition is

$$\mathbf{G}' \wedge \mathbf{i} = 0.$$

Now $\mathbf{G}'$, the moment of ($\mathbf{R}$, $\mathbf{G}$) about P, is given by

$$\mathbf{G}' = \mathbf{G} - \mathbf{r}_0 \wedge \mathbf{R}.$$

Hence $$(\mathbf{G} - \mathbf{r}_0 \wedge \mathbf{R}) \wedge \mathbf{i} = 0,$$

or $$\mathbf{G} \wedge \mathbf{i} + \mathbf{r}_0 (\mathbf{R}.\mathbf{i}) - \mathbf{R}(\mathbf{r}_0 . \mathbf{i}) = 0.$$

Since the component of $\mathbf{r}_0$ along $\mathbf{R}$ is clearly indeterminate, we have

$$\mathbf{r}_0 = \frac{\mathbf{i} \wedge \mathbf{G}}{\mathbf{R}.\mathbf{i}} + \lambda \mathbf{R}.$$

This locus is a straight line parallel to $\mathbf{R}$, i.e. to the central axis of the system and reducing to the central axis when $\mathbf{i}$ is parallel to $\mathbf{R}$. Call this locus L($\mathbf{i}$). Any line intersecting L($\mathbf{i}$) and perpendicular to $\mathbf{i}$ is clearly a nul line of the system, and so the nul planes of points on L($\mathbf{i}$) are perpendicular to $\mathbf{i}$. Given any plane whatever, of normal $\mathbf{i}$, it will be met by L($\mathbf{i}$) in one point, and it is the nul plane of this point. This point is called the *nul point* of the plane. An exception is given by planes parallel to $\mathbf{R}$; for, for such planes, $\mathbf{R}.\mathbf{i} = 0$, and the corresponding line is at infinity.

If the base point O is on the central axis, then $\mathbf{G} = p\mathbf{R}$, where p is the pitch of the equivalent wrench. Hence the position vector $\mathbf{r}_0$ of any point on L($\mathbf{i}$) reckoned from a point on the central axis is given by

$$\mathbf{r}_0 = p \frac{\mathbf{i} \wedge \mathbf{R}}{\mathbf{R}.\mathbf{i}} + \lambda \mathbf{R}.$$

The perpendicular distance from O to L($\mathbf{i}$) is accordingly

$$p \frac{\mathbf{i} \wedge \mathbf{R}}{\mathbf{R}.\mathbf{i}},$$

i.e. this is the perpendicular distance between the central axis and L($\mathbf{i}$). The modulus of this vector is p tan θ, where $\theta = \widehat{\mathbf{R}\mathbf{i}}$

The relation of L(**i**) to the central axis is shown in the accompanying diagram (Fig. 37), in which L(**i**) and the central axis lie in the plane of

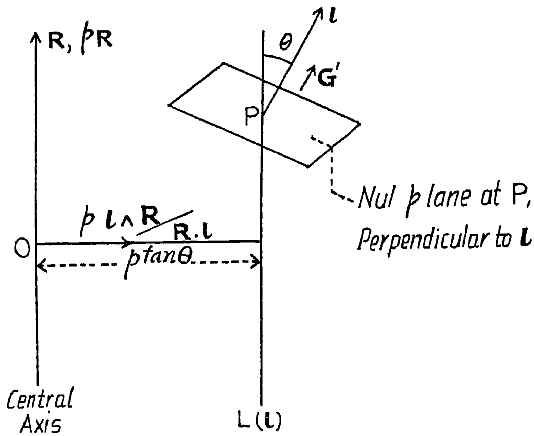

Fig. 37

the paper, and **i** lies in a plane through L perpendicular to the plane of the paper, **i** itself making an angle θ with L.

156. *Expressions for the invariants* $\mathbf{R}^2$ *and* **R.G** *of a system of line vectors.* Let $(\mathbf{P}_s)$ be any line vector of the system, O any origin, $\mathbf{r}_s$ the position vector with respect to O of any point on $(\mathbf{P}_s)$. Let the system be reduced to (**R, G**) when O is base point. Then

$$\mathbf{R}=\sum_s \mathbf{P}_s,$$

$$\mathbf{G}=\Sigma \mathbf{r}_s \wedge \mathbf{P}_s.$$

Hence
$$\mathbf{R}^2=\sum_s \mathbf{P}_s{}^2+2\sum_{s \neq t} \mathbf{P}_s.\mathbf{P}_t$$

and
$$\begin{aligned}\mathbf{G}.\mathbf{R}&=(\sum_s \mathbf{P}_s).(\sum_t \mathbf{r}_t \wedge \mathbf{P}_t)\\ &=\tfrac{1}{2}\sum_s\sum_t(\mathbf{r}_t \wedge \mathbf{P}_t.\mathbf{P}_s+\mathbf{r}_s \wedge \mathbf{P}_s.\mathbf{P}_t)\\ &=\tfrac{1}{2}\sum_s\sum_t \mathbf{P}_t \wedge \mathbf{P}_s.(\mathbf{r}_t-\mathbf{r}_s)\\ &=\tfrac{1}{2}\sum_s\sum_t(\mathbf{r}_{st} \wedge \mathbf{P}_t).\mathbf{P}_s\end{aligned}$$

where (Fig. 38), $\mathbf{r}_{st}=\mathbf{r}_t-\mathbf{r}_s$.

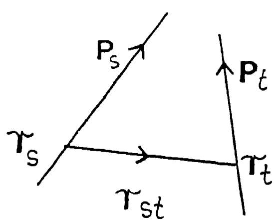

Fig. 38

This is the sum of the *mutual moments* of the lines of action of all pairs of vectors, multiplied by $|\mathbf{P}_t||\mathbf{P}_s|$. It is equal to six times the sum of the volumes of the tetrahedra subtended by representations of each pair of vectors $\mathbf{P}_s$, $\mathbf{P}_t$, with due regard to sign. This sum is thus the same for all equivalent systems of line vectors.

157. *Joint invariants of two systems of line vectors.* Consider two systems of line vectors, $(\mathbf{P})$ and $(\mathbf{Q})$, and evaluate the invariants of the combined system $(\mathbf{P})+(\mathbf{Q})$. We have

$$\mathbf{R}^2((\mathbf{P})+(\mathbf{Q}))=(\Sigma\mathbf{P}+\Sigma\mathbf{Q}).(\Sigma\mathbf{P}+\Sigma\mathbf{Q})$$
$$=(\Sigma\mathbf{P})^2+(\Sigma\mathbf{Q})^2+2(\Sigma\mathbf{P}).(\Sigma\mathbf{Q}).$$

But $(\Sigma\mathbf{P})^2$ and $(\Sigma\mathbf{Q})^2$ are invariants. Hence $(\Sigma\mathbf{P}).(\Sigma\mathbf{Q})$ is invariant. This result is obvious, and not of great interest. Consider, however, the invariant $I=\mathbf{R}.\mathbf{G}$ of the joint system. We have

$$I((\mathbf{P})+(\mathbf{Q}))=(\Sigma\mathbf{r}\wedge\mathbf{P}+\Sigma\mathbf{r}\wedge\mathbf{Q}).(\Sigma\mathbf{P}+\Sigma\mathbf{Q})$$
$$=I((\mathbf{P}))+I((\mathbf{Q}))+\sum_{s}\sum_{t}(\mathbf{r}_t\wedge\mathbf{P}_t).\mathbf{Q}_s+\sum_{s}\sum_{t}(\mathbf{r}_s\wedge\mathbf{Q}_s).\mathbf{P}_t.$$

But $I((\mathbf{P})+(\mathbf{Q}))$, $I((\mathbf{P}))$ and $I((\mathbf{Q}))$ are invariants, the same for all equivalent systems. Hence so is the sum of the remaining two terms on the right-hand side. This sum is equal to

$$\sum_{s\neq t}\mathbf{P}_t\wedge\mathbf{Q}_s.(\mathbf{r}_t-\mathbf{r}_s)$$
$$=\sum_{s\neq t}\mathbf{P}_t\wedge\mathbf{Q}_s.\mathbf{r}_{st}.$$

Thus the sum of the *mutual moments* of all vectors of $(\mathbf{P})$ with all vectors of $(\mathbf{Q})$ is invariant.

158. *Two wrenches.* We proceed to find the wrench equivalent to $(\mathbf{P}, p\mathbf{P})$ with axis through a point $\mathbf{a}$, and $(\mathbf{Q}, q\mathbf{Q})$ with axis through a point $\mathbf{b}$. Let the equivalent wrench be $(\mathbf{R}, \tilde{\omega}\mathbf{R})$ with axis through some point $\mathbf{c}$. Then by the conditions of equivalence,

$$\mathbf{R}=\mathbf{P}+\mathbf{Q}$$

and (by moments about O),

$$\tilde{\omega}\mathbf{R}+\mathbf{c}\wedge\mathbf{R}=p\mathbf{P}+\mathbf{a}\wedge\mathbf{P}+q\mathbf{Q}+\mathbf{b}\wedge\mathbf{Q}.$$

Multiplying the last equation scalarly by $\mathbf{R}$ or $(\mathbf{P}+\mathbf{Q})$, we have

$$\tilde{\omega}(\mathbf{P}+\mathbf{Q})^2=(p\mathbf{P}+q\mathbf{Q}).(\mathbf{P}+\mathbf{Q})+\mathbf{a}\wedge\mathbf{P}.\mathbf{Q}+\mathbf{b}\wedge\mathbf{Q}.\mathbf{P}$$
$$=p\mathbf{P}^2+q\mathbf{Q}^2+(p+q)\mathbf{P}.\mathbf{Q}+[(\mathbf{b}-\mathbf{a})\wedge\mathbf{Q}].\mathbf{P}.$$

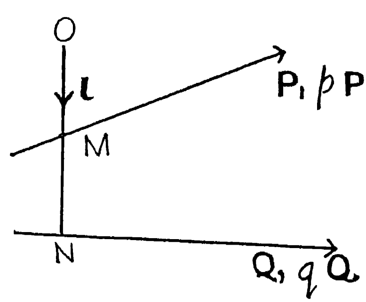

Fig. 39

This gives the pitch of the equivalent wrench. (The last term is the mutual moment of $\mathbf{Q}$ and $\mathbf{P}$.)

To determine the position of the axis of the equivalent wrench, we can multiply the equation expressing the equality of the moments vectorially by $\mathbf{R}$, or $(\mathbf{P}+\mathbf{Q})$, when we get an equation for $\mathbf{c}$. To interpret geometrically the resulting formula it is, however, simplest to begin by choosing a special origin O. Take MN (Fig. 39), the common perpen-

dicular to the axes of the given wrenches, and choose an arbitrary point O on MN for origin. If $\mathbf{i}$ is a unit vector along MN, let M be the point $\xi\mathbf{i}$, N the point $\eta\mathbf{i}$. Let X (not shown in the diagram) be the foot of the perpendicular from O to the axis of the equivalent wrench $(\mathbf{R}, \tilde{\omega}\mathbf{R})$. Then the equation expressing equality of moments about O is

$$\tilde{\omega}\mathbf{R}+OX\wedge\mathbf{R}=p\mathbf{P}+q\mathbf{Q}+\xi\mathbf{i}\wedge\mathbf{P}+\eta\mathbf{i}\wedge\mathbf{Q}.$$

But by construction $\mathbf{i}.\mathbf{P}=0$, $\mathbf{i}.\mathbf{Q}=0$ and so $\mathbf{i}.\mathbf{R}=0$. Hence, multiplying the last equation scalarly by $\mathbf{i}$, we have

$$OX\wedge\mathbf{R}.\mathbf{i}=0.$$

Hence OX, $\mathbf{R}$ and $\mathbf{i}$ are coplanar (free) vectors, and moreover OX is perpendicular to the axis of the equivalent wrench, i.e. to $\mathbf{R}$. But $\mathbf{i}$ is perpendicular to $\mathbf{R}$. Hence OX and $\mathbf{i}$ are parallel. Hence X lies in MN. Thus the axis of the equivalent wrench intersects MN, and is, of course, perpendicular to it.

If, now, $OX=x\mathbf{i}$, we have on multiplying the equation of moments vectorially by $\mathbf{R}$, i.e. by $(\mathbf{P}+\mathbf{Q})$,

$$(x\mathbf{i}\wedge\mathbf{R})\wedge\mathbf{R}=(p\mathbf{P}+q\mathbf{Q})\wedge(\mathbf{P}+\mathbf{Q})+(\mathbf{i}\wedge(\xi\mathbf{P}+\eta\mathbf{Q}))\wedge(\mathbf{P}+\mathbf{Q}),$$

or, since $\mathbf{i}.\mathbf{P}=0$, $\mathbf{i}.\mathbf{Q}=0$, $\mathbf{i}.\mathbf{R}=0$,

$$-x\mathbf{R}^2\mathbf{i}=(p-q)(\mathbf{P}\wedge\mathbf{Q})-\mathbf{i}[(\mathbf{P}+\mathbf{Q}).(\xi\mathbf{P}+\eta\mathbf{Q})].$$

Multiplying scalarly by $\mathbf{i}$,

$$x=\frac{(\mathbf{P}+\mathbf{Q}).(\xi\mathbf{P}+\eta\mathbf{Q})-(p-q)\mathbf{P}\wedge\mathbf{Q}.\mathbf{i}}{(\mathbf{P}+\mathbf{Q})^2},$$

which determines the point X in which the axis of the equivalent wrench meets MN. The pitch of the equivalent wrench follows as before by scalar multiplication of the equation of moments by $\mathbf{R}$, when we get

$$\tilde{\omega}=\frac{(p\mathbf{P}+q\mathbf{Q}).(\mathbf{P}+\mathbf{Q})+(\xi-\eta)\mathbf{i}.\mathbf{P}\wedge\mathbf{Q}}{(\mathbf{P}+\mathbf{Q})^2}.$$

That the axis of the equivalent wrench meets MN is easily seen directly. For MN is a nul line of the system, and it is perpendicular to the couple $\tilde{\omega}\mathbf{R}$. Hence the line vector $(\mathbf{R})$, having zero moment about it, must intersect it.

If the given wrenches reduce to two given forces, so that $p=0$, $q=0$, then

$$x=\frac{(\mathbf{P}+\mathbf{Q}).(\xi\mathbf{P}+\eta\mathbf{Q})}{(\mathbf{P}+\mathbf{Q})^2}.$$

Hence
$$\frac{x-\xi}{\eta-x}=\frac{(\eta-\xi)\mathbf{Q}.(\mathbf{Q}+\mathbf{P})}{(\eta-\xi)\mathbf{P}.(\mathbf{Q}+\mathbf{P})}=\frac{\mathbf{Q}.(\mathbf{Q}+\mathbf{P})}{\mathbf{P}.(\mathbf{Q}+\mathbf{P})}.$$

This gives the ratio in which the axis of the wrench equivalent to two line vectors divides their common perpendicular (Cambridge, Math. Trip. I, 1934).

Example (1). Find the line co-ordinates ($\mathbf{i}$, $\mathbf{a}$) of the central axis of the system ($\mathbf{R}$, $\mathbf{G}$) with base point the origin O.

We have seen that the central axis has for equation

$$\mathbf{r}=\frac{\mathbf{R}\wedge\mathbf{G}}{\mathbf{R}^2}+\lambda\mathbf{R}.$$

Hence $$\mathbf{i}=\mathbf{R}/|\mathbf{R}|.$$

The above locus must be the same as the locus

$$\mathbf{r}=\mathbf{i}\wedge\mathbf{a}+t\mathbf{i}, \qquad (\mathbf{a}.\mathbf{i}=0).$$

Hence values of λ, t exist such that

$$\frac{\mathbf{R}\wedge\mathbf{G}}{\mathbf{R}^2}+\lambda\mathbf{R}=\frac{\mathbf{R}\wedge\mathbf{a}}{|\mathbf{R}|}+t\frac{\mathbf{R}}{|\mathbf{R}|}.$$

To eliminate λ and t and at the same time solve for $\mathbf{a}$, multiply vectorially by $\mathbf{R}$. Since $\mathbf{a}.\mathbf{R}=0$ we get

$$\frac{(\mathbf{R}\wedge\mathbf{G})\wedge\mathbf{R}}{\mathbf{R}^2}=\mathbf{a}\frac{\mathbf{R}^2}{|\mathbf{R}|},$$

or $$\mathbf{a}=\frac{\mathbf{G}}{|\mathbf{R}|}-\mathbf{R}\frac{\mathbf{G}.\mathbf{R}}{|\mathbf{R}|^3}.$$

Example (2). Two systems of line vectors are equivalent to ($\mathbf{R}$, $\mathbf{G}$) and ($\mathbf{R}'$, $\mathbf{G}'$) respectively, when O is base point. Show that the condition that the axes of the equivalent wrenches intersect is

$$\mathbf{G}'.\mathbf{R}+\mathbf{G}.\mathbf{R}'=(\mathbf{R}.\mathbf{R}')(p+p'),$$

where p, p′ are their pitches.

The equations of the central axes of the systems are

$$\mathbf{r}=\frac{\mathbf{R}\wedge\mathbf{G}}{\mathbf{R}^2}+\lambda\mathbf{R}, \qquad \mathbf{r}'=\frac{\mathbf{R}'\wedge\mathbf{G}'}{\mathbf{R}'^2}+\lambda'\mathbf{R}'.$$

These intersect if values of λ, λ' can be found so that $\mathbf{r}$ and $\mathbf{r}'$ coincide, i.e. so that

$$\frac{\mathbf{R}\wedge\mathbf{G}}{\mathbf{R}^2}+\lambda\mathbf{R}=\frac{\mathbf{R}'\wedge\mathbf{G}'}{\mathbf{R}'^2}+\lambda'\mathbf{R}'.$$

This condition is equivalent to three scalar relations, and there are two unknowns λ and λ'. To eliminate λ and λ' simultaneously, multiply scalarly by $\mathbf{R}\wedge\mathbf{R}'$. We get

$$\frac{(\mathbf{G}\wedge\mathbf{R}).(\mathbf{R}\wedge\mathbf{R}')}{\mathbf{R}^2}+\frac{(\mathbf{G}'\wedge\mathbf{R}').(\mathbf{R}'\wedge\mathbf{R})}{\mathbf{R}'^2}=0.$$

Evaluating the triple products we have

$$(\mathbf{G}\wedge\mathbf{R}).(\mathbf{R}\wedge\mathbf{R}')=\mathbf{G}.(\mathbf{R}\wedge(\mathbf{R}\wedge\mathbf{R}'))=-\mathbf{R}^2(\mathbf{G}.\mathbf{R}')+(\mathbf{R}.\mathbf{R}')(\mathbf{G}.\mathbf{R}),$$
$$(\mathbf{G}'\wedge\mathbf{R}').(\mathbf{R}'\wedge\mathbf{R})=\mathbf{G}'.(\mathbf{R}'\wedge(\mathbf{R}'\wedge\mathbf{R}))=-\mathbf{R}'^2(\mathbf{G}'.\mathbf{R})+(\mathbf{R}.\mathbf{R}')(\mathbf{G}'.\mathbf{R}').$$

The relation obtained then gives

$$\mathbf{G}.\mathbf{R}'+\mathbf{G}'.\mathbf{R}=(\mathbf{R}.\mathbf{R}')\left[\frac{\mathbf{G}.\mathbf{R}}{\mathbf{R}^2}+\frac{\mathbf{G}'.\mathbf{R}'}{\mathbf{R}'^2}\right]$$
$$=(\mathbf{R}.\mathbf{R}')(\mathbf{p}+\mathbf{p}').$$

Example (3). (Lamb, *H.M.*)* Prove that a system of line vectors may be in general reduced to two line vectors, one of which passes through a given point whilst the other lies in a given plane.

Let O be the given point, π the given plane. Let the nul plane of the system at O meet π in a line l. Reduce the system to a line vector in l and a second line vector, say in l$'$. Then every line through O in the nul plane is a nul line, and it meets l. Hence it meets the line conjugate to l, namely l$'$. Hence either l$'$ passes through O or it lies in the plane of O and l. Thus the system has been reduced to two line vectors, in l and l$'$, of which l lies in the given plane π, and l$'$ either passes through O or lies in the plane of O and l. In the latter case, the given system reduces to a single resultant.

The whole point of the vector calculus is, however, that it affords an analytical method of solving problems of the above type, just as co-ordinate geometry affords an analytical method of solving problems of pure geometry. To construct a vectorial proof, take the given point O as origin, and let the given plane have for equation

$$\mathbf{r}=\mathbf{a}+x\mathbf{i}+y\mathbf{j} \qquad (\mathbf{a}.\mathbf{i}=0, \quad \mathbf{a}.\mathbf{j}=0)$$

where $\mathbf{r}$ is the position vector of an arbitrary point in the plane, $\mathbf{i}$ and $\mathbf{j}$ unit vectors parallel to the plane, and x, y parameters. Let the given system be equivalent to ($\mathbf{R}$, $\mathbf{G}$) when O is base point. We attempt to determine a line vector ($\mathbf{P}$) through O and a line vector ($\mathbf{Q}$) in the plane such that ($\mathbf{P}$, $\mathbf{Q}$) $\equiv$ ($\mathbf{R}$, $\mathbf{G}$). Let (x', y') be a point on ($\mathbf{Q}$) ; we shall have, since ($\mathbf{Q}$) is in the plane, $\mathbf{Q}.\mathbf{a}=0$. The systems will be equivalent if

$$\mathbf{R}=\mathbf{P}+\mathbf{Q},$$
$$\mathbf{G}=(\mathbf{a}+x'\mathbf{i}+y'\mathbf{j})\wedge\mathbf{Q}.$$

To solve the latter equation for $\mathbf{Q}$, multiply vectorially by $\mathbf{a}$. We get

$$\mathbf{G}\wedge\mathbf{a}=\mathbf{a}^2\mathbf{Q},$$

which determines $\mathbf{Q}$. Then $\mathbf{P}=\mathbf{R}-(\mathbf{G}\wedge\mathbf{a})/\mathbf{a}^2$. Multiplying the equation of moments scalarly by $\mathbf{a}$, we find

$$\mathbf{G}.\mathbf{a}=-x'(\mathbf{G}.\mathbf{i})-y'(\mathbf{G}.\mathbf{j})$$

and this linear relation between x' and y' determines the position of ($\mathbf{Q}$) in the plane. The equation of moments, being equivalent to three scalar

* The reference is to Lamb's *Higher Mechanics*.

relations, has now been fully used, the remaining two relations having contributed to the evaluation of $\mathbf{Q}$, which had two unknowns in it since $\mathbf{Q.a}=0$. The reduction is therefore accomplished.

Example (4). (*H. M.*) If a system consists of three couples of scalar moments L, M, N in three (oblique) co-ordinate planes, and α, β, γ are the angles between the axes, show that the plane of the resultant couple has for equation

$$\frac{\mathrm{Lx}}{\sin\alpha}+\frac{\mathrm{My}}{\sin\beta}+\frac{\mathrm{Nz}}{\sin\gamma}=0.$$

For, if $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors in the axes of co-ordinates, the couples have vector moments

$$\mathrm{L}\frac{\mathbf{j}\wedge\mathbf{k}}{\sin\alpha},\qquad \mathrm{M}\frac{\mathbf{k}\wedge\mathbf{i}}{\sin\beta},\qquad \mathrm{N}\frac{\mathbf{i}\wedge\mathbf{j}}{\sin\gamma}.$$

The moment of the equivalent couple is the vector sum of these. The line joining the origin to the point $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ will accordingly be perpendicular to the axis of the equivalent couple if

$$(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}).\left(\Sigma\mathrm{L}\frac{\mathbf{j}\wedge\mathbf{k}}{\sin\alpha}\right)=0,$$

or, since $\mathbf{i}\wedge\mathbf{j}.\mathbf{k}\neq 0$, if

$$\Sigma\frac{\mathrm{Lx}}{\sin\alpha}=0.$$

This is therefore the equation of a plane through the origin in the plane of the resultant couple.

Example (5). (Appel, *Méc. Rat. I.*) $(\mathbf{P})$ is a system of line vectors equivalent to zero, M_1, M_2, ... are the moments of a second system $(\mathbf{Q})$ about the lines of action of the members $(\mathbf{P}_1)$, $(\mathbf{P}_2)$, ... of $(\mathbf{P})$. Prove that

$$\Sigma|\mathbf{P}_s|\mathrm{M}_s=0.$$

For, if $\mathbf{r}_{st}$ is the vector joining given points on the lines of action of $(\mathbf{P}_s)$ and $(\mathbf{Q}_t)$, then

$$|\mathbf{P}_s|\mathrm{M}_s=\mathbf{P}_s.\sum_t(\mathbf{r}_{st}\wedge\mathbf{Q}_t).$$

Hence
$$\sum_s|\mathbf{P}_s|\mathrm{M}_s=\sum_s\sum_t\mathbf{P}_s.(\mathbf{r}_{st}\wedge\mathbf{Q}_t)=\sum_t\sum_s\mathbf{Q}_t.(\mathbf{r}_{ts}\wedge\mathbf{P}_s)$$
$$=\sum_t\mathbf{Q}_t.\sum_s(\mathbf{r}_{ts}\wedge\mathbf{P}_s)=0,$$

since $\sum_s\mathbf{r}_{ts}\wedge\mathbf{P}_s=0$, $(\mathbf{P})$ being equivalent to zero.

Example (6). (Cambridge, Intercoll., 1923.) A system of line vectors is equivalent to a line vector $\mathbf{R}$ at a variable point P of a given plane, and an associated couple $\mathbf{G}$(P). A line PP′ is constructed to represent the vector $\mathbf{G}$(P). Prove that the locus of P′ is a plane.

Let the given plane be $\mathbf{r}=x\mathbf{i}+y\mathbf{j}$, referred to an origin O in itself

and a pair of unit vectors $\mathbf{i}, \mathbf{j}$, and let $(\mathbf{R}, \boldsymbol{\Gamma})$ be the system when O is base point. If $\mathbf{r}$ is the position vector of P with respect to O,

$$G(P)=\boldsymbol{\Gamma}-\mathbf{r}\wedge\mathbf{R}$$

and P′ will have position vector $\mathbf{r}'$, where

$$\begin{aligned}\mathbf{r}' &= \mathbf{r}+\lambda G(P)\\ &= x\mathbf{i}+y\mathbf{j}+\lambda[\boldsymbol{\Gamma}-(x\mathbf{i}+y\mathbf{j})\wedge\mathbf{R}]\\ &= \lambda\boldsymbol{\Gamma}+x(\mathbf{i}-\lambda\mathbf{i}\wedge\mathbf{R})+y(\mathbf{j}-\lambda\mathbf{j}\wedge\mathbf{R}).\end{aligned}$$

This is a plane parallel to the fixed vectors $\mathbf{i}-\lambda\mathbf{i}\wedge\mathbf{r}$, $\mathbf{j}-\lambda\mathbf{j}\wedge\mathbf{R}$, and passing through the fixed point $\lambda\boldsymbol{\Gamma}$.

Example (7). (Routh, *Statics*.) A set of line vectors have representations $A_1A'_1$, $A_2A'_2$, ... and G, G′ are the centres of mass of equal particles at $A_1, A_2, \ldots, A'_1, A'_2, \ldots$, respectively. Prove that the central axis of the system is parallel to GG′. Also, if the line vectors meet any plane perpendicular to GG′ in $B_1, B_2, \ldots$ prove that the central axis meets this plane in D, where D is the centre of mass of particles at $B_1, B_2, \ldots$ whose masses are proportional to the resolved parts of the line vectors parallel to GG′.

For, let $\mathbf{R}$ be the vector sum of the line vectors $A_1A'_1, \ldots$. Then if O is any origin,

$$\begin{aligned}\mathbf{R} &= \Sigma AA'=\Sigma(OA'-OA)=nOG'-nOG\\ &= nGG',\end{aligned}$$

where n is the number of line vectors in the system.

For the second part, take a unit vector $\mathbf{i}$ normal to the plane, and unit vectors $\mathbf{j}, \mathbf{k}$ in the plane. Take D as origin, the point of intersection of the central axis with the plane, and put

$$DB_s=x_s\mathbf{j}+y_s\mathbf{k}.$$

Then if $p\mathbf{i}$ is the couple of the equivalent wrench,

$$p\mathbf{i}=\sum_s(x_s\mathbf{j}+y_s\mathbf{k})\wedge A_sA'_s.$$

Multiplying vectorially by $\mathbf{i}$, since $\mathbf{i}.\mathbf{j}=0=\mathbf{i}.\mathbf{k}$, we get

$$\sum_s(x_s\mathbf{j}+y_s\mathbf{k})(A_sA'_s.\mathbf{i})=0.$$

Hence the origin D is the centre of mass of particles of masses $A_sA'_s.\mathbf{i}$ at points B_s.

Example (8). Line vectors pass through the vertices of a tetrahedron, and are proportional to the opposite faces, with senses outwards. Prove that they form a system equivalent to zero.

Let O, A, B, C be the vertices of the tetrahedron. The line vectors are

at O, the vector $AB\wedge AC$; at A, the vector $OC\wedge OB$;

at B, the vector $OA\wedge OC$; at C the vector $OB\wedge OA$.

The *vector sum* of these has been shown to be identically zero in § 36. The *moment* of the line vectors about O is

$$\Sigma \mathrm{OA}\wedge(\mathrm{OC}\wedge\mathrm{OB}).$$

Expanding by the continued vector product theorem, we find this to be identically zero. Hence the given system is equivalent to zero.

Example (9). If the line vectors in the preceding example act at the centroids of the faces, they again form a system equivalent to zero.

For, the line vectors are now

at $\frac{1}{3}(\mathrm{OA}+\mathrm{OB}+\mathrm{OC})$, the vector $\mathrm{AB}\wedge\mathrm{AC}$;
at $\frac{1}{3}(\mathrm{OB}+\mathrm{OC})$, the vector $\mathrm{OC}\wedge\mathrm{OB}$;
at $\frac{1}{3}(\mathrm{OC}+\mathrm{OA})$, the vector $\mathrm{OA}\wedge\mathrm{OC}$;
at $\frac{1}{3}(\mathrm{OA}+\mathrm{OB})$, the vector $\mathrm{OB}\wedge\mathrm{OA}$.

The vector sum is zero as before. The moment about O is

$$\tfrac{1}{3}(\mathrm{OA}+\mathrm{OB}+\mathrm{OC})\wedge[(\mathrm{OB}-\mathrm{OA})\wedge(\mathrm{OC}-\mathrm{OA})]+\tfrac{1}{3}\sum_{\mathrm{A,B,C}}(\mathrm{OB}+\mathrm{OC})\wedge(\mathrm{OC}\wedge\mathrm{OB})$$

Put $$\mathbf{X}=\tfrac{1}{3}(\mathrm{OA}+\mathrm{OB}+\mathrm{OC}).$$

Then the moment is

$$\mathbf{X}\wedge[(\mathrm{OB}\wedge\mathrm{OC})+(\mathrm{OA}\wedge\mathrm{OB})+(\mathrm{OC}\wedge\mathrm{OA})]+\sum_{\mathrm{A,B,C}}(\mathbf{X}-\tfrac{1}{3}\mathrm{OA})\wedge(\mathrm{OC}\wedge\mathrm{OB})$$

which vanishes identically.

Example (10). Three straight lines are parallel to a plane. Prove that any straight line which meets them is parallel to a fixed plane.

Since the three given lines are parallel to a plane, any three free vectors in them are coplanar, and so linearly dependent, and we can choose line vectors $(\mathbf{X})$, $(\mathbf{Y})$, $(\mathbf{Z})$ in them so that $\mathbf{X}+\mathbf{Y}+\mathbf{Z}=0$. Let l be any line meeting $(\mathbf{X})$, $(\mathbf{Y})$, $(\mathbf{Z})$. Then the moment of $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ about l is zero. But since $\mathbf{X}+\mathbf{Y}+\mathbf{Z}=0$, $(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \mathbf{G}$, where $\mathbf{G}$ is a couple. Hence l is perpendicular to the axis of $\mathbf{G}$, and so parallel to the plane of $\mathbf{G}$, which is a fixed plane.

Example (11). (*H. M.*) Line vectors $X\mathbf{i}$, $Y\mathbf{j}$, $Z\mathbf{k}$ act in non-intersecting edges of a rectangular parallelepiped, the corresponding edges being of lengths a, b, c. Prove that they are equivalent to a single line vector if

$$\frac{a}{X}+\frac{b}{Y}+\frac{c}{Z}=0.$$

The three vectors may be taken to act as follows :

at $b\mathbf{j}$, the vector $X\mathbf{i}$; at $c\mathbf{k}$, the vector $Y\mathbf{j}$; at $a\mathbf{i}$, the vector $Z\mathbf{k}$;

(the position vectors being taken with respect to an origin at a vertex not lying on one of the given line vectors). If the system is equivalent to $(\mathbf{R}, \mathbf{G})$ with the origin as base point, then

$$\mathbf{R}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k},$$
$$\mathbf{G}=bX(\mathbf{j}\wedge\mathbf{i})+cY(\mathbf{k}\wedge\mathbf{j})+aZ(\mathbf{i}\wedge\mathbf{k}),$$

or $$\mathbf{G}=-(bX\mathbf{k}+cY\mathbf{i}+aZ\mathbf{j}).$$

We note that $\mathbf{R} \neq 0$, $\mathbf{G} \neq 0$. Hence the system will reduce to a single line vector if $\mathbf{G.R}=0$, i.e. if

$$(X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}).(bX\mathbf{k}+cY\mathbf{i}+aZ\mathbf{j})=0,$$

i.e. provided $$cXY+aYZ+bZX=0.$$

This is the required condition.

Example (12). Prove that the mid-points of the diagonals of a complete plane quadrilateral are collinear.

With the notation of the figure (Fig. 40), let L be the mid-point of AC, M of BD, N of EF. Consider the system of line vectors consisting of AB, AD, CB, CD. Then

$$AB+AD+CB+CD \equiv 2AM+2CM,$$

and so the system is equivalent to a single line vector passing through M. Similarly it is equivalent to a single line vector passing through L. But

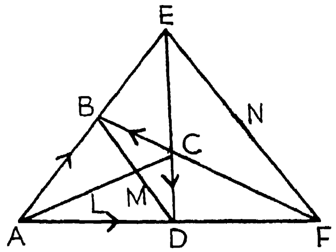

Fig. 40

$$\begin{aligned} AB+AC+CB+CD &\equiv (AE+EB)+(AF+FD)+(CF+FB)+(CE+ED) \\ &\equiv (AE+AF)+(CE+CF)+(EB+FB)+(FD+ED) \\ &\equiv 2AN+2CN+2NB+2ND. \end{aligned}$$

Hence the system is equivalent to a line vector through N. Hence the system is equivalent to a single line vector through L, M, N, which are therefore collinear.

Example (13). *Astatic centre.* A system of coplanar line vectors is given. A fixed point is taken on the line of action of each line vector, and the line vectors are turned about these points, in their plane, through an angle θ the same for all the vectors. Show that as θ varies, the resultant passes through a fixed point, and determine its position.

Let $(\mathbf{P})$ be a typical member of the system, $\mathbf{p}$ the position vector of the fixed point on $(\mathbf{P})$. Let $\mathbf{i}$ be a unit vector normal to the plane. Then $(\mathbf{P})$, on rotation through θ, becomes $(\mathbf{P}')$, where

$$\mathbf{P}'=\mathbf{P}\cos\theta+(\mathbf{i}\wedge\mathbf{P})\sin\theta.$$

The system, being coplanar, reduces to a single resultant or to a couple. Let it reduce to a single resultant $(\mathbf{R})$, which becomes $(\mathbf{R}')$ after the process described. Then by the conditions of equivalence

$$\Sigma\mathbf{P}'=\mathbf{R}', \qquad \Sigma\mathbf{p}\wedge\mathbf{P}'=\mathbf{r}\wedge\mathbf{R}',$$

where $\mathbf{r}$ is the position vector of any point on $(\mathbf{R}')$. We want to show that it is possible to choose $\mathbf{r}$ independent of θ. The stated conditions of equivalence give

$$\cos\theta\,\Sigma(\mathbf{p}\wedge\mathbf{P})+\sin\theta\,\Sigma\mathbf{p}\wedge(\mathbf{i}\wedge\mathbf{P})=\cos\theta\,\mathbf{r}\wedge\Sigma\mathbf{P}+\sin\theta\,\mathbf{r}\wedge(\mathbf{i}\wedge\Sigma\mathbf{P}).$$

This will be satisfied identically for all θ provided we can find an $\mathbf{r}$ satisfying the two relations

$$\mathbf{r}\wedge\Sigma\mathbf{P}=\Sigma(\mathbf{p}\wedge\mathbf{P}),$$

$$\mathbf{r}\wedge(\mathbf{i}\wedge\Sigma\mathbf{P})=\Sigma\mathbf{p}\wedge(\mathbf{i}\wedge\mathbf{P}).$$

Since $\mathbf{i}$ is perpendicular to all the other vectors concerned, the continued vector product theorem gives for the latter relation

$$\mathbf{r}.\Sigma\mathbf{P}=\Sigma(\mathbf{p}.\mathbf{P}).$$

Multiplying the former relation vectorially by $\Sigma\mathbf{P}$ and using the last relation we get

$$\mathbf{r}=\frac{(\Sigma\mathbf{P})(\Sigma\mathbf{p}.\mathbf{P})+(\Sigma\mathbf{P})\wedge\Sigma(\mathbf{p}\wedge\mathbf{P})}{(\Sigma\mathbf{P})^2}.$$

This may be written

$$\mathbf{r}=\frac{\mathbf{R}a+\mathbf{R}\wedge\mathbf{G}}{\mathbf{R}^2},$$

where $\mathbf{G}=\Sigma(\mathbf{p}\wedge\mathbf{P})$ is the moment of the system about the origin, and $a=\Sigma\mathbf{p}.\mathbf{P}$ is the *virial* of the system with respect to O.

If $\mathbf{R}=\Sigma\mathbf{P}=0$, then $\Sigma\mathbf{P}'=0$ and the system reduces to a couple for all θ. If $\mathbf{G}'$ is the moment of this couple when the angle of rotation is θ, then

$$\mathbf{G}'=\Sigma\mathbf{p}\wedge\mathbf{P}'=\cos\theta\,\Sigma\mathbf{p}\wedge\mathbf{P}+\sin\theta\,\Sigma\mathbf{p}\wedge(\mathbf{i}\wedge\mathbf{P})$$
$$=\mathbf{G}\cos\theta+a\sin\theta\,\mathbf{i}.$$

There is now one value of θ (or $\pi+\theta$) for which $\mathbf{G}'=0$, i.e. for which the system is equivalent to zero.

Example (14). If four line vectors are equivalent to zero, the invariant I ($=\mathbf{G}.\mathbf{R}$) of any two of them equals the invariant I of the other two; and the invariant I of any three is zero.

Example (15). A system of line vectors is replaced by three line vectors ($\mathbf{P}$), ($\mathbf{Q}$), ($\mathbf{R}$) acting at three given points A, B, C, the line of action of ($\mathbf{P}$) being given. Show that $\mathbf{P}$ is fixed in magnitude, and that the lines of action of ($\mathbf{Q}$) and ($\mathbf{R}$) lie in fixed planes.

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be the position vectors of A, B, C respectively. Let $\mathbf{i}$ be a unit vector in the given line of action of ($\mathbf{P}$), and put $\mathbf{P}=f\mathbf{i}$. Since C is a fixed point, the moment of the system about C is fixed. Hence

$$(\mathbf{b}-\mathbf{c})\wedge\mathbf{Q}+(\mathbf{a}-\mathbf{c})\wedge f\mathbf{i}=\text{const. vector}.$$

Multiply scalarly by $(\mathbf{b}-\mathbf{c})$. Then

$$f(\mathbf{b}-\mathbf{c}).(\mathbf{a}-\mathbf{c})\wedge\mathbf{i}=\text{const.}$$

Hence
$$f=\text{const.}$$

Hence
$$(\mathbf{b}-\mathbf{c})\wedge\mathbf{Q}=\text{const. vector}$$

or
$$\mathbf{Q}=\text{const. vector}+\lambda(\mathbf{b}-\mathbf{c}).$$

This shows that the line of action of $\mathbf{Q}$ lies in a fixed plane.

Example (16). If a system of line vectors has each of the six edges of a given tetrahedron for a nul line, the system is equivalent to zero.

Let OABC be the tetrahedron, $a\mathbf{i}$, $b\mathbf{j}$, $c\mathbf{k}$ the position vectors of A, B, C with respect to O ($\mathbf{i}, \mathbf{j}, \mathbf{k}$ being unit vectors). Let the system be equivalent to $\mathbf{R}$ at O and a couple $\mathbf{G}$. Since the moments about OA, OB, OC are zero, we have

$$\mathbf{G}.\mathbf{i}=0, \qquad \mathbf{G}.\mathbf{j}=0, \qquad \mathbf{G}.\mathbf{k}=0,$$

and hence, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ being linearly independent, we must have $\mathbf{G}=0$. But the moment of the system about BC is also zero. Hence

$$-b\mathbf{j}\wedge\mathbf{R}.(c\mathbf{k}-b\mathbf{j})=0$$

or
$$(\mathbf{j}\wedge\mathbf{k}).\mathbf{R}=0.$$

Similarly
$$(\mathbf{k}\wedge\mathbf{i}).\mathbf{R}=0, \qquad (\mathbf{i}\wedge\mathbf{j}).\mathbf{R}=0.$$

But $\mathbf{j}\wedge\mathbf{k}$, $\mathbf{k}\wedge\mathbf{i}$, $\mathbf{i}\wedge\mathbf{j}$ are linearly independent. Hence $\mathbf{R}=0$.

Example (17). (Cf. *H. M.*) Line vectors λOA, μOB, νOC act in the edges OA, OB, OC of a tetrahedron OABC ; and line vectors λ'BC, μ'CA, ν'AB act in the edges BC, CA, AB. Prove that they reduce to a single line vector or to a couple provided that

$$\lambda\lambda'+\mu\mu'+\nu\nu'=0,$$

and find the further conditions that they reduce to a couple.

The invariant I of the system, obtained by taking the sum of the volumes of the tetrahedra subtended by the different pair of vectors is clearly

$$\pm(\lambda\lambda'+\mu\mu'+\nu\nu')\times 6 \text{ vol (OABC)}.$$

Hence the given condition implies $\mathrm{I}=0$. The result follows.

Now suppose that the system reduces to ($\mathbf{R}$, $\mathbf{G}$) with base point O. Then

$$\mathbf{R}=\Sigma\lambda\mathrm{OA}+\Sigma\lambda'\mathrm{BC}=\Sigma(\lambda+\mu'-\nu')\mathrm{OA},$$

$$\mathbf{G}=\Sigma\mathrm{OB}\wedge\lambda'(\mathrm{OC}-\mathrm{OB})=\Sigma\lambda'\mathrm{OB}\wedge\mathrm{OC}.$$

Hence necessary conditions for $\mathbf{R}=0$ are

$$\lambda+\mu'-\nu'=0, \qquad \mu+\nu'-\lambda'=0, \qquad \nu+\lambda'-\mu'=0,$$

which imply in turn

$$\lambda+\mu+\nu=0, \qquad \lambda\lambda'+\mu\mu'+\nu\nu'=0.$$

The geometrical meaning of $\lambda+\mu+\nu=0$ is that the vector sum of the vectors at O is parallel to the plane ABC. Hence if the vector sums at any three of the four vertices are parallel to the opposite faces, the system reduces to a couple ; and conversely. If $\Sigma\lambda\lambda'=0$ but not all of the three additional conditions are satisfied, the system reduces to a single line vector.

Example (18). Three line vectors $(\mathbf{X})$, $(\mathbf{Y})$, $(\mathbf{Z})$ are given ; a common transversal meets their lines of action in A, B, C. Show that if

$$\Sigma BC(\mathbf{Y}\wedge\mathbf{Z}.\mathbf{i})=0,$$

where $\mathbf{i}$ is a unit vector along the transversal and BC, CA, AB are scalars with their proper signs, then the system can be reduced to a line vector along the common transversal and another line vector.

Take an origin O on the transversal, and let $x\mathbf{i}$, $y\mathbf{i}$, $z\mathbf{i}$ be the position vectors of A, B, C. The system will be equivalent to a line vector $f\mathbf{i}$ in the transversal and a line vector $g\mathbf{j}$ in a line of line co-ordinates $(\mathbf{j}, \mathbf{b})$ provided that

$$f\mathbf{i}+g\mathbf{j}=\mathbf{X}+\mathbf{Y}+\mathbf{Z}$$

and
$$g\mathbf{b}=\mathbf{i}\wedge(x\mathbf{X}+y\mathbf{Y}+z\mathbf{Z}).$$

The line co-ordinates $(\mathbf{j}, \mathbf{b})$ must satisfy

$$\mathbf{j}.\mathbf{b}=0.$$

Substituting for $\mathbf{b}$ and $\mathbf{j}$ from the conditions of equivalence we get

$$(\mathbf{X}+\mathbf{Y}+\mathbf{Z}-f\mathbf{i}).(\mathbf{i}\wedge(x\mathbf{X}+y\mathbf{Y}+z\mathbf{Z}))=0,$$

or
$$\Sigma\mathbf{i}\wedge(y\mathbf{Y}+z\mathbf{Z}).\mathbf{X}=0,$$

or
$$\Sigma\mathbf{i}.[z(\mathbf{Z}\wedge\mathbf{X})-y(\mathbf{X}\wedge\mathbf{Y})]=0,$$

or
$$\Sigma(\mathbf{i}.\mathbf{Y}\wedge\mathbf{Z})(y-z)=0.$$

If this condition of consistency is satisfied, the six unknowns corresponding to $\mathbf{j}$, $\mathbf{b}$ and f, g are determinate.

Example (19). Unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ form a positive orthogonal triad, and lines parallel to $\mathbf{i}, \mathbf{j}, \mathbf{k}$ pass through points of position vectors $\beta\mathbf{j}+\gamma\mathbf{k}$, $\gamma\mathbf{k}+\alpha\mathbf{i}$, $\alpha\mathbf{i}+\beta\mathbf{j}$ respectively. Show that their line co-ordinates are

$$(\mathbf{i},\ -\beta\mathbf{k}+\gamma\mathbf{j}),\qquad(\mathbf{j},\ -\gamma\mathbf{i}+\alpha\mathbf{k}),\qquad(\mathbf{k},\ -\alpha\mathbf{j}+\beta\mathbf{i}).$$

Deduce that they intersect in pairs, and illustrate with a figure ($\alpha>0$, $\beta>0$, $\gamma>0$). Obtain the line co-ordinates of lines parallel to $\mathbf{i}, \mathbf{j}, \mathbf{k}$ through points of position vectors $\beta\mathbf{j}-\gamma\mathbf{k}$, $\gamma\mathbf{k}-\alpha\mathbf{i}$, $\alpha\mathbf{i}-\beta\mathbf{j}$, and show that they form a non-intersecting set. If line vectors $(X\mathbf{i})$, $(Y\mathbf{j})$, $(Z\mathbf{k})$ act along the members of the non-intersecting set, reduce them to $(\mathbf{R}, \mathbf{G})$ with base point O, and evaluate $\mathbf{G}.\mathbf{R}$ and $\mathbf{G}\wedge\mathbf{R}$. Determine the line co-ordinates of the axis of the equivalent wrench.

(The above examples have been selected to illustrate the variety of problem that may be treated by vector methods, and the power of the vector calculus in solving them. The reader is recommended to try other problems out of current textbooks and examination papers.)

CHAPTER VII

STATICS OF RIGID BODIES

159. *Forces as vectors.* It is possible to introduce the entity known in mechanics as a *force* either conceptually or as derived from kinematic experiments. When we come to discuss the foundations of dynamics, the latter procedure can be adopted. Here, to discuss the equilibrium of a body or of systems of bodies, it is sufficient to introduce force as an undefined concept, i.e. one understood by the reader. In accordance with this concept, a force is described by its magnitude, direction and point of application. Further, it is supposed to 'act on' the particle, or small element of matter, at its point of application. A force acting on a particle P may then be represented by a position vector **R** associated with the point P.

For the purposes of statics it is then sufficient to assume as an experimental fact that a particle of matter is in equilibrium when the vector sum of the vectors representing the forces acting on it is zero, and conversely, that when the vector sum of the force vectors is zero, the particle is in equilibrium.

160. *Measurement of force.* The above statements do not become precise, in the positivist sense, until methods are stated for measuring forces. *For the purposes of statics*, force-measurement may be supposed to be based on gravitational attraction.

Let a standard volume of a standard material under standard conditions be taken as the *unit of mass*. Then m such standard specimens are said to have mass m. When m is an integer, this involves the judgment that the specimens are 'equal,' the judgment being made on the basis of length measures, here assumed defined already. It is supposed that tests of congruence have been formed, and that by the employment of 'dividers,' i.e. superposable rigid length measures, given lengths may be subdivided into aliquot parts. This permits the evaluation of mass measures of specimens of the standard substance when m is not an integer.

Let then a mass m of standard substance be placed in the pan of a spring-balance, which is then allowed to come to equilibrium in a given gravitational field, for example the earth's field at a given place at a given height above the ground. Let d be the resulting deflection recorded by the spring-balance. Then if a certain quantity of some other substance, placed in the same pan of the same spring-balance under the same circumstances and allowed to come to equilibrium, gives the same deflection d, it is said to have the same mass m.

The 'external' forces acting on the two substances are then said to be equal, and each equal to mg, where g is a certain multiplier. From Galileo's experiments on falling bodies, and from countless more accurate experiments, that two specimens of equal mass m undergo equal downward constant accelerations when released at a given place in the earth's field, so as to fall freely, an acceleration which is independent of the value of m. The number chosen for g is the value of this constant acceleration. But for the purposes of statics the value chosen for g (at any fixed place) is immaterial. The quantity mg is called the *weight* of the body of mass m.

It is an experimental fact that if two masses are equal, as judged by the same spring-balance at a given place, then they are also equal as judged by any other spring-balance at any other place, although the values of g may be different. The verification of this fact involves the immediate judgment (comparable to that involved in the congruence of rigid length measures under transportation) that the two specimens of matter have not undergone any relevant alterations in the course of transport to the new place; for example, that no evaporation has occurred.

If the same spring-balance, supporting the same quantity of the given material, is moved to a different position on the earth, its deflection may alter. Let the original deflection be restored by changing the mass of material supported, increasing or decreasing it by a known amount measured as above. Then, if we make an immediate judgment that no relevant change has occurred in the spring-balance, the latter is said to be exerting the same force F as before. Let m′ be the new mass. Then if we put

$$F = mg = m'g',$$

it is an experimental fact that g′ is equal to the acceleration under gravity at the new place. It then follows, in accordance with our definition of force, that the force exerted by the spring-balance on the original mass was $F' = mg'$. The facts expressed in the equations $F = mg$, $F' = mg'$ are usually stated in the form that *inertial* and *gravitational* mass are equal. But the use here of the word 'inertial' indicates that other, dynamical considerations have gone towards the framing of the assertion. These are irrelevant to statics.

The ideal experiments above described enable us to assign a numerical value to the force exerted by the spring-balance for any deflection d, each deflection d corresponding to some mass m and a force $F = mg$. At any other place the spring-balance may be supposed to exert a force given by its reading, so that the spring-balance may be taken to provide a transportable scale of force.*

The direction of the force corresponding to the earth's pull on a given particle of matter is taken to be the direction of a fine string supporting

* In principle, corrections may ultimately have to be applied to allow for the possibly varying action of gravity on the scale-pan itself.

the particle. Later dynamical considerations distinguish between the gravitational pull of the earth and the resultant pull when the latter is corrected for the 'centrifugal' force associated with the earth's rotation. But these considerations are irrelevant to the formulation of statics. The direction of the force exerted by the spring-balance on the material it supports is defined to be the opposite of the downward direction of a fine string supporting a particle at the same locality. This direction may then be related to a frame of reference fixed with regard to the spring-balance itself, and this allows us to assign a direction to the force exerted by the spring-balance when placed in other orientations relative to the earth.

The orientation and deflection of a spring-balance now provide a measure of the direction and magnitude of the force it exerts under any circumstances. Such a force may be described by a line vector, which 'acts on' the particle exposed to the force.

It is now found that if a particle of material is maintained in equilibrium under the action of a number of spring-balances, differently inclined to one another, then the corresponding vectors have a vector sum zero. Any sub-set of these vectors has accordingly a vector sum equal and opposite to the vector sum of the remainder. Either of these is said to measure the *resultant force* exerted on the particle by the corresponding sub-set.

161. *Reactions.* A particle of matter is usually part of a system of bodies, and it may be acted on by forces other than those arising from the action of the earth's pull or of applied spring-balances. For example, forces may arise from contacts with other particles, or from contiguous elements of material. Such indeed occur in the very use of a spring-balance itself, as we see when we analyse it into its constituent actions: for example, there are the force exerted by the scale-pan on the material particle it supports, and the force exerted by the suspension on the scale-pan. The existence of such forces may be recognized by verifying that the forces which can be identified as 'external' do not possess a vector sum equal to zero, even though the particle is in equilibrium. These other forces are called *reactions*. It is reasonable to suppose that reactions thus detected when combined with the external forces acting on the particle amount to vector sum zero. This can be verified experimentally in circumstances in which it is possible to replace such an internal action by an external, measurable force. It is found, however, that if we introduce a force $\mathbf{R}$ to represent the forces arising from a set of contacts of other bodies on a given particle, then the equilibrium of other portions of the system requires the introduction of another set of forces equivalent to a resultant $-\mathbf{R}$ acting on these other bodies.

This is the content of Newton's 'Third Law,' that *action* and *reaction* are equal and opposite. It is regarded above as a fact capable of experimental verification. But when statics is regarded as a particular case of

dynamics, the third law may be introduced in a different way, as an axiom equivalent to a definition of the equality of two forces. This other method of introducing the third law, due to Mach, is perhaps logically preferable. But if, as here, we wish to treat statics as a separate subject, and thereby introduce an independent measure of force like a spring-balance, we are compelled to regard the law of equality of action and reaction as a verifiable statement. In any analysis of reality in terms of propositions, it is to a certain extent arbitrary how many of the propositions are statements *defining* certain entities, and how many of them are statements of properties the verification of which in nature permits the recognition in nature of objects corresponding to the entities so defined.

The choice of a direction for a particular action or reaction depends on the circumstances. The actions inside a stressed solid are analysed in the most general way into forces per unit area of an arbitrary plane. The action of a smooth surface on a particle in contact with it is taken to be normal to the surface at the point concerned. The action of a rough surface is analysed into the action the surface would exert if smooth, together with a *frictional* action opposite to the direction of the initial motion which would ensue if the surface were replaced by a smooth one. The action between contiguous portions of a string is taken to be in the direction of the tangent to the string.

162. The justification of these statements may be supposed to be either experimental (as relating to the actual apparatus concerned) or as part of the definition of the apparatus regarded as an abstraction. It perhaps hardly needs to be stated that the whole of the applications of theoretical mechanics may be regarded as approximations close to what is actually realized in nature and the external world, or as exact results referring to idealizations or abstractions, representing (or standing for) the actual occurrences in nature. In theoretical mechanics we emphasize the abstract side ; the problems investigated usually refer to conventional objects, such as ' smooth ' joints and surfaces, ' inextensible,' ' mass-less ' strings, and ' rigid ' bodies. Each successive branch of mechanics—elasticity, hydrostatics, hydrodynamics of frictionless fluids, hydrodynamics of viscous fluids—is an attempt to treat with increased approximation the facts of nature.

163. *Statics of a system of particles.* Let the forces on any particle P of the system be divided into internal and external forces, the internal ones being reactions between different members of the system. Let $\mathbf{F}_i$ be a typical internal force, $\mathbf{F}_e$ a typical external force, on a particle P of the system. Then, for each particle, by the principle already enunciated,

$$\Sigma\mathbf{F}_i + \Sigma\mathbf{F}_e = 0.$$

Hence for the sum over all the particles of the system,

$$\sum_P(\Sigma\mathbf{F}_i + \Sigma\mathbf{F}_e) = 0.$$

But the internal forces occur in sets represented by equal and opposite vector sums. Hence

$$\sum_{P}(\Sigma \mathbf{F}_i)=0.$$

Hence $$\sum_{P}(\Sigma \mathbf{F}_e)=0.$$

Further, if $\mathbf{r}_P$ is the position vector of P,

$$\mathbf{r}_P \wedge (\Sigma \mathbf{F}_i + \Sigma \mathbf{F}_e)=0.$$

Hence, summing over all the particles of the system,

$$\sum_{P}(\mathbf{r}_P \wedge \Sigma \mathbf{F}_i)+\sum_{P}(\mathbf{r}_P \wedge \Sigma \mathbf{F}_e)=0.$$

But the internal forces $\mathbf{F}_i$ form a system of line vectors equivalent to zero. Hence (§ 134),

$$\sum_{P}(\mathbf{r}_P \wedge \Sigma \mathbf{F}_i)=0.$$

Hence $$\sum_{P}(\mathbf{r}_P \wedge \Sigma \mathbf{F}_e)=0.$$

The latter may now be written concisely as

$$\Sigma \mathbf{r} \wedge \mathbf{F}_e = 0,$$

where $\mathbf{r}$ is the position vector of the point of application of each external force $\mathbf{F}_e$. The former of the two conditions may similarly be written

$$\Sigma \mathbf{F}_e = 0.$$

These two conditions are *necessary* conditions for the equilibrium of a system of particles under external forces. They imply that the system of external forces is equivalent to zero. They are not, of course, *sufficient* conditions.

The same relations hold good for any *portion* of the system considered by itself ; but some of the forces previously considered as internal may now have to be considered as external.

164. *Statics of a rigid body.* A rigid body is defined as a collection of particles whose mutual distances are invariable. This definition involves, as a prior concept, the concept of the rigid transportable length-scale. A force applied to any *particle* of the body is said to be applied to the *body* at that point. By § 161, it is *necessary* that if a rigid body is in equilibrium, the system of external forces be equivalent to zero. This condition can be seen also to be *sufficient* if we introduce as an additional property of a rigid body the following axiom.

Two equal and opposite forces, applied to a rigid body in equilibrium, do not disturb its equilibrium.

This axiom is sometimes called the *Principle of the Transmissibility of force.*

Applying this axiom, we can now suppress or add pairs of forces, equal and opposite, without disturbing the equilibrium of the rigid body.

We can also suppress or add any nul-concurrent systems of forces in the same way. We can therefore move the point of application of any external force to any other point in its line of action without affecting the equilibrium, for this is accomplished by introducing a suitable pair of equal and opposite forces. It follows that only the line of action and the magnitude of each force are relevant to the equilibrium of the rigid body on which it acts ; its point of application is irrelevant. Hence, as far as concerns the equilibrium of a rigid body subject to a system of external forces, the given system may be represented by any system of line vectors *equivalent* to the given system, in the sense of Chapter VI.

A rigid body is now in equilibrium if the system of line vectors representing the forces acting on it is equivalent to zero. For, if so, we may introduce such nul-concurrent sets of forces as will ensure that each particle of the rigid body is acted on by a nul-concurrent set, and so is in equilibrium. The whole body is therefore in equilibrium.

In the course of suppressing or adding nul-concurrent sets of forces, we, of course, alter the internal actions.

The principle of the transmissibility of force may be considered either as an experimental law, approximately obeyed by bodies considered as rigid, or as an additional property defined to be possessed by a rigid body. In the latter case its verification allows the recognition in nature of the occurrence of entities realizing the abstract notion of a rigid body.

If a *system* of rigid bodies is in equilibrium, it must be possible to introduce such action and reactions between them as will satisfy the conditions of equilibrium for each rigid body separately.

165. *Applications.* The conditions of equilibrium of each rigid body occur in the form of two vector equations, which express the fact that the systems of line vectors representing the forces are equivalent to zero.

In treatises on statics, it is usually recommended that unknown forces such as reactions be eliminated by taking moments about suitable *lines*. In practice, whilst it is often clear that a number of forces in which we are not interested act through the same *point*, it is by no means obvious about which *lines* moments should be taken. Further, the mental labour of taking moments about chosen lines in three-dimensional situations is often serious. It is therefore often preferable to write down the vector equation of moments about one or more suitable points, and then derive scalar relations from them by suitable choice of scalar multiplication. The statical facts are thus written down compactly in two vector statements, and the subsequent manipulation is only a question of vector algebra ; the required operations are suggested by the forms of the vector equations. Awkward mental operations are thus avoided, and no *a priori* decision is required as to what co-ordinate system is to be adopted, as is the case in the traditional method of solution of three-dimensional statical problems. In effect, vector methods permit the use of oblique

systems of axes, if required, with the same ease as orthogonal systems; for we can choose primitive vectors in any convenient directions.

Vector methods are less appropriate for two-dimensional statical problems, though they can always be employed in such contexts by introducing a unit vector normal to the plane. In three-dimensional problems, vector methods often give the solution with the minimum of mental strain, with little demand on spatial intuition, and without the necessity of special care in determining the senses of turning moments.

The object of a mathematical technique, apart from its æsthetic beauty, is to save labour, to avoid the risk of error and to minimize mental strain. The following examples will illustrate these points.

Example (1). The line of hinges of a uniform rectangular door makes an angle α with the vertical, the upper hinge overhanging the lower one. If the door is opened an angle θ, show that the couple necessary to maintain it in this position is $Wa \sin\alpha \sin\theta$, where W is the weight of the door, a the distance of the centre of mass from the line of hinges. Discuss the couple exerted by the reactions at the hinges about the mid-point of the line of hinges.

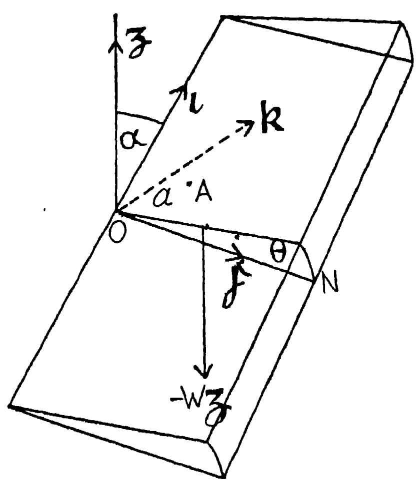

Fig. 41

Let O (Fig. 41) be the mid-point of the line of hinges, A the mass-centre of the opened door. Take a unit vector $\mathbf{z}$ vertically upwards, a unit vector $\mathbf{i}$ in the line of hinges (upwards) and a unit vector $\mathbf{j}$ perpendicular to $\mathbf{i}$ in the plane of the *unopened* door, drawn from O towards the undisplaced position of A. Put $\mathbf{k}=\mathbf{i}\wedge\mathbf{j}$.

Let the reactions of the hinges together with the applied external couple be equivalent to a force $\mathbf{R}$ at O together with a couple $\mathbf{G}$. The position vector of A with respect to O is $a(\mathbf{j}\cos\theta+\mathbf{k}\sin\theta)$, where $OA=a$. The moment of the weight $-W\mathbf{z}$ about O is thus $a(\mathbf{j}\cos\theta+\mathbf{k}\sin\theta)\wedge(-W\mathbf{z})$. Hence the equation of moments about O is

$$\mathbf{G}+a(\mathbf{j}\cos\theta+\mathbf{k}\sin\theta)\wedge(-W\mathbf{z})=0.$$

But
$$\mathbf{z}=\mathbf{i}\cos\alpha-\mathbf{j}\sin\alpha.$$

Hence
$$\mathbf{G}-aW(-\mathbf{k}\cos\theta\cos\alpha+\mathbf{j}\sin\theta\cos\alpha+\mathbf{i}\sin\theta\sin\alpha)=0.$$

This gives the components of $\mathbf{G}$ along the directions of $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

The reactions of the hinges all intersect the line of hinges, and hence

have zero moment component along **i**. Consequently the external couple is **G.i**, which is

$$aW \sin\theta \sin\alpha.$$

The moment of the system of forces exerted by the hinges about the line OA is

$$\mathbf{G}.(\mathbf{j}\cos\theta+\mathbf{k}\sin\theta),$$

or $\quad aW(\mathbf{i}\sin\theta\sin\alpha+\mathbf{j}\sin\theta\cos\alpha-\mathbf{k}\cos\theta\cos\alpha).(\mathbf{j}\cos\theta+\mathbf{k}\sin\theta),$

which is identically zero. Hence the *couple* exerted by the hinges is normal to OA. (The hinges also exert a total force equal to W**z**.) The magnitude of the couple exerted by the hinges, when O is taken as base point, is

$$aW|\mathbf{j}\sin\theta\cos\alpha-\mathbf{k}\cos\theta\cos\alpha|$$
$$=aW\cos\alpha,$$

and so is independent of θ.

Example (2). (Routh, *Statics*.) XYZU is a vertical wall of finite height; O is a point on the ground, outside it, and the vertical plane through O perpendicular to the wall meets the wall in XU, X being in the ground. A uniform rod OB has one end at O and rests in an oblique position lying against the upper edge of the wall at A. If slipping (coefficient of friction μ) is on the point of occurring at A, show that $\sin\theta=\mu\cot\beta$, where $\theta=A\hat{O}U$, $\beta=X\hat{O}U$. Determine the normal reaction at A.

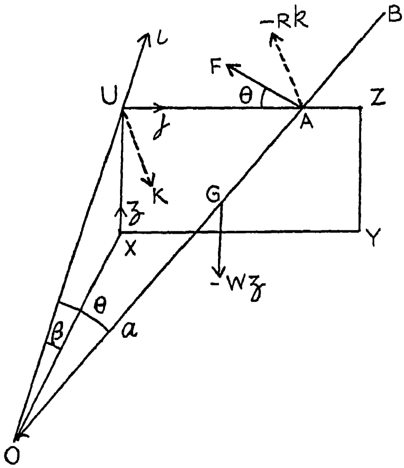

Fig. 42

Take unit vectors **z** vertically upwards, **i** along OU, **j** along UZ, **k** making a positive orthogonal triad with **i** and **j** (Fig. 42). Let OG=a, OA=l.

Then a unit vector along the rod OA is

$$\mathbf{i}\cos\theta+\mathbf{j}\sin\theta,$$

and $$\mathbf{z}=\mathbf{i}\sin\beta-\mathbf{k}\cos\beta.$$

The forces acting on the rod, other than those at O, are:

at $a(\mathbf{i}\cos\theta+\mathbf{j}\sin\theta)$, the force $-W\mathbf{z}$;
at $l(\mathbf{i}\cos\theta+\mathbf{j}\sin\theta)$, the normal reaction $-R\mathbf{k}$;
at $l(\mathbf{i}\cos\theta+\mathbf{j}\sin\theta)$, the limiting friction μR, perpendicular to OA in the plane OAU, and accordingly represented by the vector $-\mu R(\mathbf{j}\cos\theta-\mathbf{i}\sin\theta)$.

(The position vectors have been taken with O as origin.) Hence the equation of moments about O is

$$\mathrm{Wa}(\mathbf{i}\cos\theta+\mathbf{j}\sin\theta)\wedge(-\mathbf{z})+\mathrm{l}(\mathbf{i}\cos\theta+\mathbf{j}\sin\theta)\wedge$$
$$[-\mathrm{R}\mathbf{k}-\mu\mathrm{R}(\mathbf{j}\cos\theta-\mathbf{i}\sin\theta)]=0.$$

To obtain a relation independent of W, we must multiply scalarly by $\mathbf{z}$. This gives

$$(\mathbf{i}\cos\theta+\mathbf{j}\sin\theta)\wedge(\mathbf{k}+\mu\cos\theta\mathbf{j}-\mu\sin\theta\mathbf{i}).\mathbf{z}=0.$$

or
$$(-\cos\theta\mathbf{j}+\sin\theta\mathbf{i}+\mu\mathbf{k}).\mathbf{z}=0.$$

But
$$\mathbf{z}.\mathbf{j}=0,\qquad \mathbf{z}.\mathbf{i}=\sin\beta,\qquad \mathbf{z}.\mathbf{k}=-\cos\beta.$$

Hence
$$\sin\theta=\mu\cot\beta.$$

To determine R, the equation of moments may be multiplied scalarly by $\mathbf{i}$ or $\mathbf{j}$. Multiplying scalarly by $\mathbf{i}$, we have

$$\mathrm{Wa}\sin\theta(-\mathbf{k}.\mathbf{z})-\mathrm{Rl}\sin\theta(\mathbf{j}\wedge\mathbf{k}.\mathbf{i})=0$$

or
$$\mathrm{R}=(\mathrm{Wa}/\mathrm{l})\cos\beta.$$

Notice that the equation of moments is equivalent to only *two* scalar relations; for it gives zero identically on scalar multiplication by $(\mathbf{i}\cos\theta+\mathbf{j}\sin\theta)$. This corresponds to the fact that every force acting intersects OAB, and so the force system has necessarily zero moment about the line OAB.

Example (3). (Manchester, 1926.) A smooth uniform rod OP of weight W is pivoted freely at a fixed point O, and rests against a smooth horizontal rail not vertically above O. The rod is prevented from slipping by a force F parallel to the rail applied at the end P beyond the rail. Determine the magnitude of F.

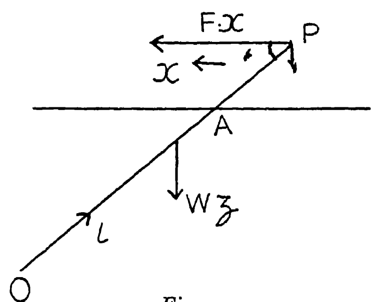

Fig. 43

Take a unit vector $\mathbf{i}$ in the direction OP, a unit vector $\mathbf{x}$ parallel to the rail in the direction of F and a unit vector $\mathbf{z}$ vertically downwards (Fig. 43). The reactions at the point of contact A, being normal to the rod and to the rail, is of the form $\lambda(\mathbf{i}\wedge\mathbf{x})$. If OP$=l, OA=$r, the equation of moments about O is

$$(\tfrac{1}{2}\mathrm{l}\mathbf{i})\wedge\mathrm{W}\mathbf{z}+\mathrm{l}\mathbf{i}\wedge\mathrm{F}\mathbf{x}+\mathrm{r}\mathbf{i}\wedge\lambda(\mathbf{i}\wedge\mathbf{x})=0.$$

To eliminate λ, multiply scalarly by $(\mathbf{i}\wedge\mathbf{x})$. We get

$$\tfrac{1}{2}\mathrm{W}(\mathbf{i}\wedge\mathbf{z}).(\mathbf{i}\wedge\mathbf{x})+\mathrm{F}(\mathbf{i}\wedge\mathbf{x})^2=0.$$

But
$$(\mathbf{i}\wedge\mathbf{z}).(\mathbf{i}\wedge\mathbf{x})=[(\mathbf{i}\wedge\mathbf{x})\wedge\mathbf{i}].\mathbf{z}=-(\mathbf{i}.\mathbf{z})(\mathbf{i}.\mathbf{x}),$$

since $\mathbf{x}.\mathbf{z}=0$. Hence

$$\mathrm{F}=\tfrac{1}{2}\mathrm{W}\frac{(\mathbf{i}.\mathbf{z})(\mathbf{i}.\mathbf{x})}{1-(\mathbf{i}.\mathbf{x})^2}.$$

Thus, if β is the inclination of the rod to the vertical, α the angle between rod and rail, $\mathbf{i.x}=-\cos\alpha$, $\mathbf{i.z}=-\cos\beta$, and

$$F=\tfrac{1}{2}W\frac{\cos\alpha\cos\beta}{\sin^2\alpha}.$$

(This example illustrates the point that in any problem it is most direct to introduce first any number of convenient *vectors*, and then to ascertain what combinations of them are required, rather than to attempt *a priori* to see what angles will finally be required.)

Example (4). (Manchester, 1927.) A uniform, solid, smooth cone of weight W can turn freely about its vertex O, which is fixed. It rests against a smooth horizontal rail (not vertically above O) which it touches at a point P of one of its generators. It is prevented from slipping by a horizontal force F, perpendicular to the rail, applied at the centre Q of the base of the cone. If $OP=\lambda OQ$, and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{x}$, $\mathbf{y}$ are unit vectors parallel to OQ, OP, the rail and F, prove that

$$(\mathbf{a.b})(\mathbf{b.x})=\mathbf{a.x}$$

and that the reaction at P is $k(\mathbf{b}\wedge\mathbf{x})$, where

$$k=\tfrac{3}{4}\frac{W}{\lambda}\,\frac{\mathbf{a.b}}{\mathbf{b.y}}.$$

Example (5). A uniform rod OA rests with one end O on very rough ground, and with the other end A against a rough vertical wall. If N is the foot of the perpendicular from O to the wall, if $NOA=\alpha$ and if θ is the inclination of NA to the vertical, prove that when slipping is on the point of occurring at A, $\mu=\tan\alpha\tan\theta$, μ being the coefficient of friction at A. Discuss the normal reaction at A.

Example (6). A rectangular box with uniform horizontal lid is tilted so that the line of hinges makes an angle α with the horizontal. Show that the couple necessary to hold the lid opened by an angle θ is $Wa\cos\alpha\cos\theta$, where W is the weight of the lid, and a the distance of the centre of gravity from the line of hinges.

Example (7). (Routh, *Statics*.) A solid heavy cone, with a generating line in contact with a rough vertical wall, can turn freely about its vertex, which is fixed. It is acted on by a couple whose moment is L and whose plane is parallel to the base. Prove that the inclination to the vertical, θ, of the contact generator, is given by $L=\tfrac{3}{4}Wh\sin\theta\tan\alpha$, where α is the semi-vertical angle of the cone, h its altitude. If the rim only of the cone is rough, prove that the least value of the coefficient of friction is $2\tan\theta\operatorname{cosec}2\alpha$.

Let O be the vertex, OA the contact generator, G the centre of mass (Fig. 44). Take unit vectors $\mathbf{z}$ vertically downwards, $\mathbf{i}$ in OG, $\mathbf{j}$ in OA. Then clearly

$$\mathbf{i}=\cos\alpha\,\mathbf{j}+\sin\alpha\,\frac{\mathbf{j}\wedge\mathbf{z}}{\sin\theta},$$

$$\mathbf{i.j}=\cos\alpha,\qquad \mathbf{j.z}=\cos\theta.$$

The forces acting, other than those at O, are:

the applied couple $L\mathbf{i}$;
the weight $W\mathbf{z}$ at the point $\frac{3}{4}h\mathbf{i}$;
the reactions and frictions in OA.

Hence the equation of moments about OA is

$$\mathbf{j}.[L\mathbf{i}+\tfrac{3}{4}h\mathbf{i}\wedge W\mathbf{z}]=0,$$

which gives

$$L\cos\alpha=-\tfrac{3}{4}Wh(\mathbf{i}\wedge\mathbf{z}).\mathbf{j}=-\tfrac{3}{4}Wh\mathbf{i}.(\mathbf{z}\wedge\mathbf{j})$$

$$=\tfrac{3}{4}Wh\sin\alpha\frac{(\mathbf{j}\wedge\mathbf{z})^2}{\sin\theta}=\tfrac{3}{4}Wh\sin\alpha\sin\theta.$$

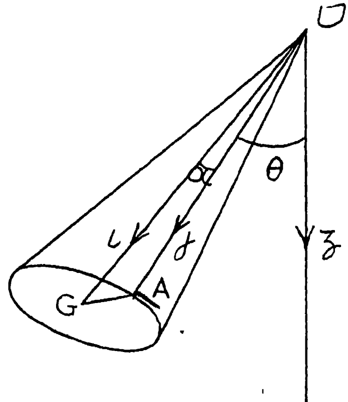

Fig. 44

When the rim only is rough, the reactions pass through A, whose position vector is $h\sec\alpha\,\mathbf{j}$. If R is the magnitude of the normal reaction, the vector representing it is $R\dfrac{\mathbf{j}\wedge\mathbf{z}}{\sin\theta}$, and the frictional force is $\mu R(\mathbf{j}\cot\theta-\mathbf{z}\operatorname{cosec}\theta)$. The equation of moments about O is

$$L\mathbf{i}+\tfrac{3}{4}h\mathbf{i}\wedge W\mathbf{z}+R\frac{h\sec\alpha}{\sin\theta}\mathbf{j}\wedge[(\mathbf{j}\wedge\mathbf{z})+\mu(\mathbf{j}\cos\theta-\mathbf{z})]=0.$$

To eliminate L and W, multiply scalarly by $\mathbf{i}\wedge(\mathbf{i}\wedge\mathbf{z})$. We get

$$\mathbf{j}\wedge[(\mathbf{j}\wedge\mathbf{z})-\mu\mathbf{z}].[\mathbf{i}\wedge(\mathbf{i}\wedge\mathbf{z})]=0.$$

On reducing this, and using the value of $\mathbf{i}$ in terms of $\mathbf{j}$ and $\mathbf{z}$, we find

$$\mu=\frac{\tan\theta}{\sin\alpha\cos\alpha}.$$

Example (8). P is a point vertically above the corner A of a horizontal square heavy lamina ABCD hinged to fixed supports along AB. The lamina is held in an inclined position ABC′D′ by a string PC′, C′ being diagonally opposite A. If W is the weight of the lamina, show that the tension in the string is

$$\tfrac{1}{2}W\frac{PC'}{AP}.$$

CHAPTER VIII

THE DISPLACEMENT OF A RIGID BODY

166. *Definition of the displacement of a set of particles as a rigid body.* Consider a system of particles P_0, P_1, P_2 ..., of position vectors $\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2, \ldots$ with respect to some origin. Let them be displaced to P_0', P_1', P_2', ..., of position vectors $\mathbf{r}_0', \mathbf{r}_1', \mathbf{r}_2', \ldots$ with respect to the same origin. Then the displacement is said to be that of *a rigid body* provided that for every pair of particles P_m, P_n

$$(\mathbf{r}_n-\mathbf{r}_m)^2=(\mathbf{r}_n'-\mathbf{r}_m')^2 \qquad (n, m=0, 1, 2, \ldots).$$

If the displacement of one given point is zero, say that of P_0, we may choose P_0 for origin, and the conditions become

$$\mathbf{r}_n^{\,2}=\mathbf{r}_n'^{\,2} \qquad (n=1, 2, \ldots)$$

$$(\mathbf{r}_n-\mathbf{r}_m)^2=(\mathbf{r}'_n-\mathbf{r}_m')^2 \qquad (n, m=1, 2, \ldots).$$

The displacement is then said to be *a displacement about* P_0.

167. *Analysis of the displacement.*

Theorem: Any displacement of a rigid body may be reduced to a translation together with a displacement about any chosen particle of the system.

For let P_0 be the chosen particle, and put

$$\mathbf{r}_0'-\mathbf{r}_0=\mathbf{d}.$$

Write also $\qquad \mathbf{r}_n-\mathbf{r}_0=\mathbf{R}_n, \qquad \mathbf{r}_n'-\mathbf{r}_0'=\mathbf{R}_n'.$

Then $\qquad \mathbf{r}_n'-\mathbf{r}_n=\mathbf{d}+\mathbf{R}_n'-\mathbf{R}_n.$

Hence the displacement $\mathbf{r}_n \rightarrow \mathbf{r}_n'$ is the result of the operations defined by the constant displacement $\mathbf{d}$ followed by the displacement $\mathbf{R}_n \rightarrow \mathbf{R}_n'$. But by the conditions of rigidity

$$\mathbf{R}_n^{\,2}=(\mathbf{r}_n-\mathbf{r}_0)^2=(\mathbf{r}_n'-\mathbf{r}_0')^2=\mathbf{R}_n'^{\,2},$$

and $\qquad (\mathbf{R}_m-\mathbf{R}_n)^2=(\mathbf{r}_m-\mathbf{r}_n)^2=(\mathbf{r}_m'-\mathbf{r}_n')^2=(\mathbf{R}_m'-\mathbf{R}_n')^2.$

Hence the displacements $\mathbf{R}_n \rightarrow \mathbf{R}_n'$ are those of a rigid body about P_0. The constant displacement $\mathbf{d}$, the same for all particles, is called a *translation.*

In virtue of this theorem it is sufficient to consider the displacement of a set of particles as a rigid body *about one of its members.*

168. *Transformation of the conditions of rigidity.* When P_0 is taken

as origin, and the particle P_0 is unchanged in position, the second form of the equations of rigidity (§ 166) may be written

$$(\mathbf{r}_n'-\mathbf{r}_n)\cdot(\mathbf{r}_n'+\mathbf{r}_n)=0,$$

$$[(\mathbf{r}_n-\mathbf{r}_m)-(\mathbf{r}_n'-\mathbf{r}_m')]\cdot[(\mathbf{r}_n-\mathbf{r}_m)+(\mathbf{r}_n'-\mathbf{r}_m')]=0.$$

Put
$$\mathbf{r}_n'-\mathbf{r}^n=\sigma_n$$

$$\tfrac{1}{2}(\mathbf{r}_n+\mathbf{r}^{n\prime})=\rho_n.$$

Then the conditions of rigidity become

$$\sigma_n\cdot\rho_n=0,$$

$$(\sigma_n-\sigma_m)\cdot(\rho_n-\rho_m)=0,$$

which latter reduces to

$$\sigma_m\cdot\rho_n+\sigma_n\cdot\rho_m=0.$$

Conversely the conditions

$$\sigma_n\cdot\rho_n=0, \qquad \sigma_m\cdot\rho_n+\sigma_n\cdot\rho_m=0$$

imply the conditions of rigidity.

Example. If ρ_1, ρ_2, ρ_3, σ_1, σ_2, σ_3 are six vectors such that

$$\rho_1\cdot\sigma_1=0, \qquad \rho_2\cdot\sigma_2=0, \qquad \rho_3\cdot\sigma_3=0,$$

$$\rho_2\cdot\sigma_3=-\rho_3\cdot\sigma_2\neq 0, \qquad \rho_3\cdot\sigma_1=-\rho_1\cdot\sigma_3\neq 0, \qquad \rho_1\cdot\sigma_2=-\rho_2\cdot\sigma_1\neq 0,$$

then either σ_1, σ_2, σ_3 are coplanar or ρ_1, ρ_2, ρ_3 are coplanar.

For, since

$$(\sigma_1\wedge\sigma_2)\cdot(\rho_1\wedge\rho_2)=[(\rho_1\wedge\rho_2)\wedge\sigma_1]\cdot\sigma_2=(\rho_1\cdot\sigma_2)^2\neq 0,$$

it follows that σ_1, σ_2 and $\rho_1\wedge\rho_2$ are linearly independent, forming in fact a positive triad. Hence σ_3 may be expressed in the form

$$\sigma_3=A\sigma_1+B\sigma_2+C(\rho_1\wedge\rho_2).$$

Multiplying in turn scalarly by ρ_1, ρ_2 and ρ_3 we find

$$\sigma_3\cdot\rho_1=B(\sigma_2\cdot\rho_1), \qquad \sigma_3\cdot\rho_2=A(\sigma_1\cdot\rho_2),$$

$$0=A(\sigma_1\cdot\rho_3)+B(\sigma_2\cdot\rho_3)+C(\rho_1\wedge\rho_2\cdot\rho_3).$$

Substituting for A and B and using the given conditions we have

$$C(\rho_1\wedge\rho_2\cdot\rho_3)=0.$$

Hence either $C=0$ or $\rho_1\wedge\rho_2\cdot\rho_3=0$. In the former case, σ_3 is coplanar with σ_1 and σ_2 ; in the latter case, ρ_1, ρ_2, ρ_3 are coplanar.

The interpretation of this example will become clear to the reader as we proceed.

169. *The rotation of a rigid body.* Let the particle P undergo the displacement $P\rightarrow P'$ about O as a rigid body. Then $|OP|=|OP'|$. Let M (Fig. 45) be the mid-point of PP′, and let OX be a line through O in some direction perpendicular to PP′. Let N be the foot of the perpen-

dicular from M on OX, and let $\mathbf{i}$ be a unit vector in OX. Then PP′, being perpendicular to ON and to OM, is perpendicular to the plane NOM, and so to NM. Hence $\mathrm{P\hat{N}M} = \mathrm{M\hat{N}P'} = \frac{1}{2}\theta$, say.

If $\mathrm{OP} = \mathbf{r}$, $\mathrm{OP'} = \mathbf{r}'$, then

$$\overrightarrow{\mathrm{OM}} = \tfrac{1}{2}(\mathbf{r}+\mathbf{r}') = \boldsymbol{\rho}$$
$$\overrightarrow{\mathrm{PP'}} = \mathbf{r}' - \mathbf{r} = \boldsymbol{\sigma}.$$

Further, PM, being perpendicular to OM and to ON, is parallel to $\mathbf{i}\wedge\frac{1}{2}(\mathbf{r}+\mathbf{r}')$. And the magnitude of PM is NM tan $\frac{1}{2}\theta$, where $\mathrm{NM} = \mathrm{OM} \sin \mathrm{N\hat{O}M} = |\mathbf{i}\wedge\frac{1}{2}(\mathbf{r}+\mathbf{r}')|$. Hence

$$\tfrac{1}{2}(\mathbf{r}'-\mathbf{r}) = [\mathbf{i}\wedge\tfrac{1}{2}(\mathbf{r}'+\mathbf{r})] \tan \tfrac{1}{2}\theta$$

or $$\mathbf{r}'-\mathbf{r} = (2\mathbf{i} \tan \tfrac{1}{2}\theta)\wedge\tfrac{1}{2}(\mathbf{r}'+\mathbf{r})$$

or $$\boldsymbol{\sigma} = \boldsymbol{\epsilon}\wedge\boldsymbol{\rho}$$

where $$\boldsymbol{\epsilon} = 2\mathbf{i} \tan \tfrac{1}{2}\theta.$$

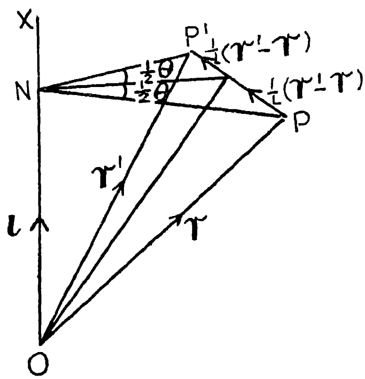

Fig. 45

It will be seen that P can be displaced to P′ by *rotating* NP about the line OX through the angle θ. We now *define* a *finite rotation* $\boldsymbol{\epsilon}$ about O as such that it brings any particle $\mathbf{r}$ to $\mathbf{r}'$, where

$$\boldsymbol{\sigma} = \boldsymbol{\epsilon}\wedge\boldsymbol{\rho}$$

and $\boldsymbol{\sigma} = \mathbf{r}' - \mathbf{r}$, $\quad \boldsymbol{\rho} = \frac{1}{2}(\mathbf{r}+\mathbf{r}')$.

The direction of the vector $\boldsymbol{\epsilon}$ is called the *axis of rotation*, and the angle of rotation θ is such that $2 \tan \frac{1}{2}\theta = |\boldsymbol{\epsilon}|$. More generally, we can specify an arbitrary sense for the axis of rotation, and if then $\mathbf{i}$ is a unit vector in this sense, the angle of rotation θ (positive or negative) is such that $\boldsymbol{\epsilon} = 2\mathbf{i} \tan \frac{1}{2}\theta$.

170. *A displacement about a point as a rigid body is always a rotation.* We now prove the following theorem.

Theorem : If O, P_1, P_2 are three particles of a rigid body, and if the body is displaced about O, then the body has undergone a rotation about O determined by the displacements of P_1 and P_2.

We have to show that we can find a vector $\boldsymbol{\epsilon}$ such that if $\mathbf{r}_1 \to \mathbf{r}_1'$, $\mathbf{r}_2 \to \mathbf{r}_2'$ and if

$$\boldsymbol{\rho}_1 = \tfrac{1}{2}(\mathbf{r}_1'+\mathbf{r}_1), \qquad \boldsymbol{\sigma}_1 = \mathbf{r}_1' - \mathbf{r}_1$$
$$\boldsymbol{\rho}_2 = \tfrac{1}{2}(\mathbf{r}_2'+\mathbf{r}_2), \qquad \boldsymbol{\sigma}_2 = \mathbf{r}_2' - \mathbf{r}_2,$$

then $$\boldsymbol{\sigma}_1 = \boldsymbol{\epsilon}\wedge\boldsymbol{\rho}_1, \qquad \boldsymbol{\sigma}_2 = \boldsymbol{\epsilon}\wedge\boldsymbol{\rho}_2\ ;$$

and further that if $\mathbf{r}$ is any other particle of the rigid body, and if

$$\boldsymbol{\rho} = \tfrac{1}{2}(\mathbf{r}'+\mathbf{r}), \qquad \boldsymbol{\sigma} = \mathbf{r}' - \mathbf{r},$$

then the same finite rotation

$$\boldsymbol{\sigma} = \boldsymbol{\epsilon}\wedge\boldsymbol{\rho}$$

displaces $\mathbf{r}$ to $\mathbf{r}'$ in such a way that the displacement $\mathbf{r}\to\mathbf{r}'$ satisfies the conditions for a rigid body displacement.

For, by the conditions of rigidity satisfied by the displacements $\mathbf{r}_1\to\mathbf{r}_1'$, $\mathbf{r}_2\to\mathbf{r}_2'$, we have

$$\sigma_1\cdot\rho_1=0, \qquad \sigma_2\cdot\rho_2=0, \qquad \sigma_1\cdot\rho_2+\sigma_2\cdot\rho_1=0.$$

Now if ϵ exists, σ_1 and σ_2 must be perpendicular to ϵ, and therefore ϵ must be of the form $\epsilon=c(\sigma_1\wedge\sigma_2)$. The required conditions therefore require the consistency of the relations

$$\sigma_1=c(\sigma_1\wedge\sigma_2)\wedge\rho_1, \qquad \sigma_2=c(\sigma_1\wedge\sigma_2)\wedge\rho_2$$

or

$$\sigma_1=-c\sigma_1(\sigma_2\cdot\rho_1), \qquad \sigma_2=c\sigma_2(\rho_2\cdot\sigma_1)$$

or

$$c=-\frac{1}{\sigma_2\cdot\rho_1}=+\frac{1}{\sigma_1\cdot\rho_2}.$$

These are consistent by the conditions of rigidity, and c is determined provided $\sigma_2\cdot\rho_1\neq 0$. Assuming this condition to be fulfilled, we have for the finite rotation ϵ

$$\epsilon=\frac{\sigma_1\wedge\sigma_2}{\sigma_1\cdot\rho_2}=\frac{\sigma_2\wedge\sigma_1}{\sigma_2\cdot\rho_1}.$$

If now $\mathbf{r}$ is any arbitrary particle of the rigid body, and if, with the above value of ϵ,

$$\sigma=\epsilon\wedge\rho,$$

then

$$\sigma\cdot\rho=0,$$

$$\sigma\cdot\rho_1=(\epsilon\wedge\rho)\cdot\rho_1=-(\epsilon\wedge\rho_1)\cdot\rho=-\sigma_1\cdot\rho,$$

$$\sigma\cdot\rho_2=(\epsilon\wedge\rho)\cdot\rho_2=-(\epsilon\wedge\rho_2)\cdot\rho=-\sigma_2\cdot\rho.$$

The point $\mathbf{r}$ is therefore rigidly connected to O, P_1 and P_2. In the same way it can be shown that $\mathbf{r}$ is rigidly connected with any other particle **R** of the system undergoing the same finite rotation.

Example. Verify that the same value is obtained for ϵ whatever pair of particles is chosen to determine it, i.e. verify that

$$\frac{\sigma_1\wedge\sigma_3}{\sigma_1\cdot\rho_3}=\frac{\sigma_1\wedge\sigma_2}{\sigma_1\cdot\rho_2}.$$

Assume that ρ_1, ρ_2, ρ_3 are not coplanar. Then by the example of § 168, σ_1, σ_2, σ_3 must be coplanar, i.e. σ_1 is a linear function of σ_2 and σ_3. Putting

$$\sigma_1=A\sigma_2+B\sigma_3,$$

we have

$$\sigma_1\cdot\rho_3=A\sigma_2\cdot\rho_3$$

$$\sigma_1\cdot\rho_2=B\sigma_3\cdot\rho_2$$

whence multiplying the condition of coplanarity in turn by σ_2 and σ_3 vectorially,

$$\sigma_1\wedge\sigma_2=\frac{\sigma_1\cdot\rho_2}{\sigma_3\cdot\rho_2}(\sigma_3\wedge\sigma_2), \qquad \sigma_1\wedge\sigma_3=\frac{\sigma_1\cdot\rho_3}{\sigma_2\cdot\rho_3}(\sigma_2\wedge\sigma_3).$$

The desired verification now follows from the conditions of rigidity. The case when $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$, $\boldsymbol{\rho}_3$ are coplanar may be obtained by a limiting process in which a vector $\boldsymbol{\rho}_3'$, non-coplanar with $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$, tends to $\boldsymbol{\rho}_3$.

171. *Expressions for* $\mathbf{r}'$ *in terms of* $\mathbf{r}$. The equation

$$\boldsymbol{\sigma}=\boldsymbol{\epsilon}\wedge\boldsymbol{\rho},$$

or

$$\mathbf{r}'-\mathbf{r}=[\mathbf{i}\wedge(\mathbf{r}'+\mathbf{r})] \tan \tfrac{1}{2}\theta, \tag{1}$$

may be solved for $\mathbf{r}'$ as follows. We can write it

$$\mathbf{r}'-(\mathbf{i}\wedge\mathbf{r}') \tan \tfrac{1}{2}\theta=\mathbf{r}+(\mathbf{i}\wedge\mathbf{r}) \tan \tfrac{1}{2}\theta. \tag{2}$$

Multiply vectorially by $\mathbf{i}$. Then

$$\mathbf{i}\wedge\mathbf{r}'-[-\mathbf{r}'+\mathbf{i}(\mathbf{i}.\mathbf{r}')] \tan \tfrac{1}{2}\theta=\mathbf{i}\wedge\mathbf{r}+[-\mathbf{r}+\mathbf{i}(\mathbf{i}.\mathbf{r})] \tan \tfrac{1}{2}\theta.$$

But (1) gives $\quad \mathbf{i}.\mathbf{r}'=\mathbf{i}.\mathbf{r}.$

Hence

$$(\mathbf{i}\wedge\mathbf{r}')+\mathbf{r}' \tan \tfrac{1}{2}\theta=\mathbf{i}\wedge\mathbf{r}-\mathbf{r} \tan \tfrac{1}{2}\theta+2\mathbf{i} \tan \tfrac{1}{2}\theta(\mathbf{i}.\mathbf{r}). \tag{3}$$

Eliminate the term $\mathbf{i}\wedge\mathbf{r}'$ between (2) and (3), by multiplying (3) by $\tan \frac{1}{2}\theta$ and adding to (2). We get

$$\mathbf{r}' \sec^2 \tfrac{1}{2}\theta=\mathbf{r}(1-\tan^2 \tfrac{1}{2}\theta)+2(\mathbf{i}\wedge\mathbf{r}) \tan \tfrac{1}{2}\theta+2\mathbf{i} \tan^2 \tfrac{1}{2}\theta(\mathbf{i}.\mathbf{r}),$$

or

$$\mathbf{r}'=\mathbf{r} \cos \theta+(\mathbf{i}\wedge\mathbf{r}) \sin \theta+(\mathbf{i}.\mathbf{r})\mathbf{i}(1-\cos \theta). \tag{4}$$

Since, by a result of § 69, we have

$$\mathbf{i}\wedge\mathbf{r}=(\mathbf{U}\wedge\mathbf{i}).\mathbf{r},$$

and since (§ 52)

$$\mathbf{i}(\mathbf{i}.\mathbf{r})=(\mathbf{ii}).\mathbf{r},$$

the solution may be written as an inner product of a tensor and $\mathbf{r}$ in the form

$$\mathbf{r}'=[\cos \theta\ \mathbf{U}+\sin \theta\ \mathbf{U}\wedge\mathbf{i}+(1-\cos \theta)\mathbf{ii}].\mathbf{r}. \tag{5}$$

Formula (4) gives explicitly the position vector $\mathbf{r}'$ to which a given particle $\mathbf{r}$ is displaced by a rotation through an angle θ about an axis $\mathbf{i}$. Formula (5) expresses $\mathbf{r}'$ operationally in terms of $\mathbf{r}$.

172. *Use of tensor operators.* These formulæ can be expressed otherwise. We can write the relation

$$\mathbf{r}'-\mathbf{r}=\boldsymbol{\epsilon}\wedge\tfrac{1}{2}(\mathbf{r}+\mathbf{r}')$$

in the form

$$\mathbf{r}'-\boldsymbol{\eta}\wedge\mathbf{r}'=\mathbf{r}+\boldsymbol{\eta}\wedge\mathbf{r},$$

where

$$\boldsymbol{\eta}=\tfrac{1}{2}\boldsymbol{\epsilon}=\mathbf{i} \tan \tfrac{1}{2}\theta.$$

Hence

$$(\mathbf{U}-\mathbf{U}\wedge\boldsymbol{\eta}).\mathbf{r}'=(\mathbf{U}+\mathbf{U}\wedge\boldsymbol{\eta}).\mathbf{r}$$

or

$$\begin{aligned}\mathbf{r}'&=(\mathbf{U}-\mathbf{U}\wedge\boldsymbol{\eta})^{-1}.(\mathbf{U}+\mathbf{U}\wedge\boldsymbol{\eta}).\mathbf{r}\\&=\mathbf{T}.\mathbf{r},\end{aligned}$$

say, where

$$\mathbf{T}=(\mathbf{U}-\mathbf{U}\wedge\boldsymbol{\eta})^{-1}.(\mathbf{U}+\mathbf{U}\wedge\boldsymbol{\eta}).$$

Now the value of $\mathbf{r}'$ may be obtained directly by multiplying the second equation of this section vectorially by $\boldsymbol{\eta}$, using $\boldsymbol{\eta}.\mathbf{r}'=\boldsymbol{\eta}.\mathbf{r}$, and

eliminating $\eta\wedge\mathbf{r}'$ between the two equations so obtained. The work is similar to that of § 171, and we obtain

$$\mathbf{r}'=\frac{\mathbf{r}(1-\eta^2)+2\eta\wedge\mathbf{r}+2\eta(\eta.\mathbf{r})}{1+\eta^2}$$

$$=\frac{(1-\eta^2)\mathbf{U}+2\mathbf{U}\wedge\eta+2\eta\eta}{1+\eta^2}.\mathbf{r}.$$

Hence $$(\mathbf{U}-\mathbf{U}\wedge\eta)^{-1}.(\mathbf{U}+\mathbf{U}\wedge\eta)\equiv\frac{(1-\eta^2)\mathbf{U}+2\mathbf{U}\wedge\eta+2\eta\eta}{1+\eta^2}.$$

This expression for $\mathbf{T}$ may also be obtained by evaluating $(\mathbf{U}-\mathbf{U}\wedge\eta)^{-1}$ in the form

$$(\mathbf{U}-\mathbf{U}\wedge\eta)^{-1}=\frac{\mathbf{U}+\mathbf{U}\wedge\eta+\eta\eta}{1+\eta^2}$$

and then operating on $(\mathbf{U}+\mathbf{U}\wedge\eta)$.

173. *Combination of two finite rotations.* Let a system of particles undergo a finite rotation ϵ_1 (or $2\eta_1$) about O, and then let the system undergo a second finite rotation ϵ_2 (or $2\eta_2$) about the same particle O. By the theorem of § 170, this succession of operations is equivalent to some rotation ϵ_3 (or $2\eta_3$) about O. Hence, if $\mathbf{r}$ is a typical particle, $\mathbf{r}'$ its position vector after the first displacement, $\mathbf{r}''$ its position vector after the second displacement, $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$ the tensors associated with the displacements η_1, η_2, η_3, we shall have

$$\mathbf{r}'=\mathbf{T}_1.\mathbf{r},\qquad \mathbf{r}''=\mathbf{T}_2.\mathbf{r}',\qquad \mathbf{r}''=\mathbf{T}_3.\mathbf{r},$$

whence $$\mathbf{T}_3=\mathbf{T}_2.\mathbf{T}_1.$$

The tensor $\mathbf{T}_3$ can be found by carrying out the inner multiplication of $\mathbf{T}_2$ and $\mathbf{T}_1$.

174. *Small rotations.* By direct inner multiplication it is easily verified that $\mathbf{T}_2.\mathbf{T}_1\neq\mathbf{T}_1.\mathbf{T}_2$. Hence the resultant rotation ϵ_3 equivalent to two successive rotations about a particle O depends on the *order* in which the rotations are carried out. If, however, η^2 is neglected, we have from § 172, approximately,

$$\mathbf{T}=\mathbf{U}+2\mathbf{U}\wedge\eta=\mathbf{U}+\mathbf{U}\wedge\epsilon,$$

whence
$$\begin{aligned}\mathbf{T}_1.\mathbf{T}_2&=(\mathbf{U}+\mathbf{U}\wedge\epsilon_1).(\mathbf{U}+\mathbf{U}\wedge\epsilon_2)\\&=\mathbf{U}+\mathbf{U}\wedge(\epsilon_1+\epsilon_2)\\&=\mathbf{T}_2.\mathbf{T}_1\end{aligned}$$

on neglecting the product of ϵ_1 and ϵ_2. The result of two successive *small* rotations is therefore approximately a small rotation $\epsilon_3=\epsilon_1+\epsilon_2$ which is independent of the order in which the rotations are performed.

This may be seen more simply direct from formula (4) of § 171. Neglecting θ^2 in this we have approximately

$$\mathbf{r}' = \mathbf{r} + \boldsymbol{\epsilon} \wedge \mathbf{r},$$

where now approximately $\qquad \boldsymbol{\epsilon} = \theta \mathbf{i}.$

Hence for two successive small rotations $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$

$$\mathbf{r}' = \mathbf{r} + \boldsymbol{\epsilon}_1 \wedge \mathbf{r},$$

$$\mathbf{r}'' = \mathbf{r}' + \boldsymbol{\epsilon}_2 \wedge \mathbf{r}',$$

whence approximately $\qquad \mathbf{r}'' = \mathbf{r} + (\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2) \wedge \mathbf{r}.$

To the same approximation, if $\boldsymbol{\epsilon}_3$ is the resultant small rotation, we have

$$\mathbf{r}'' = \mathbf{r} + \boldsymbol{\epsilon}_3 \wedge \mathbf{r}.$$

Since these hold good for all $\mathbf{r}$, we have to the same approximation

$$\boldsymbol{\epsilon}_3 = \boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2.$$

It follows that any number of *small* rotations of angles θ_1, θ_2, ... about axes $\mathbf{i}_1$, $\mathbf{i}_2$, ..., if applied successively, may be combined by the law of vector addition of the rotations

$$\boldsymbol{\epsilon}_1 = \theta_1 \mathbf{i}_1, \qquad \boldsymbol{\epsilon}_2 = \theta_2 \mathbf{i}_2,$$

etc. ; and the order of the operations is then immaterial.

Finite rotations do not combine according to the law of vector addition : although each finite rotation can be fully described by a vector, the operation $\mathbf{r} \to \mathbf{r}'$ requires for its description a tensor which is non-linear in the rotation.

175. *The abstract theory of small displacements.* Since a small displacement $\mathbf{r} \to \mathbf{r} + \Delta\mathbf{r}$ is *approximately* given by

$$\Delta\mathbf{r} = \boldsymbol{\epsilon} \wedge \mathbf{r},$$

it is convenient to introduce the differential $d\mathbf{r}$ defined by

$$d\mathbf{r} = \boldsymbol{\epsilon} \wedge \mathbf{r},$$

and to speak of this as the 'small displacement' $d\mathbf{r}$. A rigorous theory of small displacements then becomes formally possible, though it must always be remembered that the 'small displacements' thus discussed represent only approximately the corresponding actual displacements. However, the formal theory is a sufficient basis for the concept of the work of a system of line vectors, and it leads to the continuous motion of a rigid body in a natural way, as we shall see later.

176. *Definition of a screw.* From the preceding, any small displacement of a set of rigidly connected particles (which may be spoken of collectively as a *rigid body*) can be reduced to a small uniform translation,

$\mathbf{u}$, and a small rotation $\boldsymbol{\epsilon}$ about some given particle O. When $\mathbf{u}$ and $\boldsymbol{\epsilon}$ are parallel vectors, the displacement is said to constitute a *screw*. If $\mathbf{u}=p\boldsymbol{\epsilon}$, p is said to be the *pitch* of the screw.

177. *Reduction of any small displacement of a rigid body to a screw.*

Theorem: Any small displacement of a rigid body can be reduced to a screw.

For, let the displacement be specified as a small uniform translation $\mathbf{u}$, and a small rotation $\boldsymbol{\epsilon}$ about some particle O. The displacement $d\mathbf{r}$ of any particle, of position vector $\mathbf{r}$ with regard to O, is then given by

$$d\mathbf{r}=\mathbf{u}+\boldsymbol{\epsilon}\wedge\mathbf{r}.$$

This displacement can alternatively be specified as a small rotation $\boldsymbol{\epsilon}_1$ about a particle $\mathbf{r}_1$, together with a translation $p\boldsymbol{\epsilon}_1$ parallel to $\boldsymbol{\epsilon}_1$, provided a vector $\mathbf{r}_1$ can be found so that, for all $\mathbf{r}$,

$$d\mathbf{r}=p\boldsymbol{\epsilon}_1+\boldsymbol{\epsilon}_1\wedge(\mathbf{r}-\mathbf{r}_1).$$

Equating the two expressions for $d\mathbf{r}$, we have

$$\mathbf{u}+\boldsymbol{\epsilon}\wedge\mathbf{r}=p\boldsymbol{\epsilon}_1+\boldsymbol{\epsilon}_1\wedge(\mathbf{r}-\mathbf{r}_1).$$

This is to be true for all $\mathbf{r}$. A necessary condition is

$$\boldsymbol{\epsilon}_1=\boldsymbol{\epsilon},$$

and then

$$\mathbf{u}=p\boldsymbol{\epsilon}-\boldsymbol{\epsilon}\wedge\mathbf{r}_1.$$

The component of $\mathbf{r}_1$ along $\boldsymbol{\epsilon}$ is clearly not determined by this equation, and hence we may choose $\mathbf{r}_1$ so that

$$\boldsymbol{\epsilon}.\mathbf{r}_1=0.$$

Multiplying in turn scalarly and vectorially by $\boldsymbol{\epsilon}$ we find

$$p=\frac{\mathbf{u}.\boldsymbol{\epsilon}}{\boldsymbol{\epsilon}^2},$$

$$\mathbf{r}_1=\frac{\mathbf{u}\wedge\boldsymbol{\epsilon}}{\boldsymbol{\epsilon}^2}.$$

The locus of possible points $\mathbf{r}_1$ is now given by

$$\mathbf{r}_1=\frac{\mathbf{u}\wedge\boldsymbol{\epsilon}}{\boldsymbol{\epsilon}^2}+\lambda\boldsymbol{\epsilon}.$$

This is a straight line parallel to the vector $\boldsymbol{\epsilon}$. It is called the *axis* of the screw.

178. *Representation of a set of small displacements as a system of line vectors.* Suppose we specify a set of small displacements to which a rigid body is successively subjected. Let a typical displacement be the uniform translation $\mathbf{u}_n$ and the small rotation $\boldsymbol{\epsilon}_n$ about a point of position vector $\mathbf{r}_n$. Then the result of the successive small displacements will be

equivalent to a single uniform translation $\mathbf{u}$ and a single small rotation $\boldsymbol{\epsilon}$ about the origin O provided that for all $\mathbf{r}$,

$$\mathbf{u}+\boldsymbol{\epsilon}\wedge\mathbf{r}=\Sigma\mathbf{u}_n+\Sigma\boldsymbol{\epsilon}_n\wedge(\mathbf{r}-\mathbf{r}_n).$$

This requires

$$\boldsymbol{\epsilon}=\Sigma\boldsymbol{\epsilon}_n,$$

$$\mathbf{u}=\Sigma\mathbf{u}_n+\Sigma\mathbf{r}_n\wedge\boldsymbol{\epsilon}_n.$$

It now follows that *two* sets of successive small displacements, of which typical members are $\mathbf{u}_n$ and $\boldsymbol{\epsilon}_n$ about $\mathbf{r}_n$, $\mathbf{u}_m'$ and $\boldsymbol{\epsilon}_m'$ about $\mathbf{r}_m'$, yield the same final position of the rigid body provided that

$$\sum_n\boldsymbol{\epsilon}_n=\sum_m\boldsymbol{\epsilon}_m',$$

$$\sum_n\mathbf{u}_n+\sum_n\mathbf{r}_n\wedge\boldsymbol{\epsilon}_n=\sum_m\mathbf{u}_m'+\sum_m\mathbf{r}_m'\wedge\boldsymbol{\epsilon}_m'.$$

But these relations show that if we represent the rotation $\boldsymbol{\epsilon}_n$ about $\mathbf{r}_n$ as a *line vector* $\boldsymbol{\epsilon}_n$ through $\mathbf{r}_n$, and if we represent the uniform translation $\mathbf{u}_n$ by a *couple* of moment $\mathbf{u}_n$, then the conditions of equivalence of the two sets of displacements are precisely the conditions of equivalence of the two sets of corresponding line vectors. It follows that a line vector $\boldsymbol{\epsilon}_n$ through a particle $\mathbf{r}_n$ completely represents the small rotation $\boldsymbol{\epsilon}_n$ about $\mathbf{r}_n$, and a couple of moment $\mathbf{u}_n$ completely represents a uniform translation $\mathbf{u}_n$. Just as two systems of line vectors are equivalent if the introduction of nul-concurrent sets of line vectors allows one to be transformed into the other, so two sets of successive small displacements are equivalent if one can be transformed into the other by the introduction of sets of small rotations about specified points such that the vector sum of each such set is zero.

Every theorem in the theory of systems of line vectors has an immediate restatement in the theory of the small displacements of a rigid body ; the words 'small rotation' replace the words 'line vector' (or 'force'), the word 'translation' replaces 'couple,' the word 'screw' replaces 'wrench,' and the axis of the screw equivalent to a given set of small displacements replaces the central axis of the system of line vectors.

For example, the theorem (§ 152) that a system of line vectors may be replaced by two line vectors, of which the line of action of one may be specified beforehand, becomes the theorem that any system of successive small displacements is equivalent to two small rotations, of which the axis of one may be specified beforehand.

The student should scrutinize the formulæ for $\boldsymbol{\epsilon}$ and $\mathbf{u}$, given above, to make clear to himself why, in the calculation of a small displacement from a small rotation $\boldsymbol{\epsilon}$ in the form $\boldsymbol{\epsilon}\wedge\mathbf{r}$, the position vector $\mathbf{r}$ comes *second*, whilst in the moment of a line vector $\mathbf{F}$, in the form $\mathbf{r}\wedge\mathbf{F}$, the position vector $\mathbf{r}$ comes first.

179. *A uniform translation as a couple.* From the above correspondence between small displacements and line vectors, since a couple is equivalent

to two equal antiparallel line vectors, we may anticipate that a uniform translation is equivalent to two equal small rotations about parallel axes in opposite directions. This is readily verified. If the small rotations are $\boldsymbol{\epsilon}$ about the particle $\mathbf{r}_1$, and $-\boldsymbol{\epsilon}$ about the particle $\mathbf{r}_2$, then the successive small displacements of a particle r are

$$d_1\mathbf{r}=\boldsymbol{\epsilon}\wedge(\mathbf{r}-\mathbf{r}_1)$$

$$d_2\mathbf{r}=-\boldsymbol{\epsilon}\wedge(\mathbf{r}-\mathbf{r}_2)$$

whence
$$d\mathbf{r}=d_1\mathbf{r}+d_2\mathbf{r}=\boldsymbol{\epsilon}\wedge(\mathbf{r}_2-\mathbf{r}_1).$$

But $\boldsymbol{\epsilon}\wedge(\mathbf{r}_2-\mathbf{r}_1)$ is a constant vector independent of $\mathbf{r}$. Hence the resultant displacement is a uniform translation. Moreover, the vector representing the uniform translation is equal to the vector representing the moment of the couple composed of the two given parallel and opposite small rotations ; for the latter couple is $\mathbf{r}_2\wedge(-\boldsymbol{\epsilon})+\mathbf{r}_1\wedge\boldsymbol{\epsilon}=\boldsymbol{\epsilon}\wedge(\mathbf{r}_2-\mathbf{r}_1)$.

180. To summarize this correspondence, the vector sum of the line vectors *representing* the given small displacements represents the *equivalent small rotation*, and the moment of the line vectors about any *given point* represents the associated *uniform translation* when the equivalent small rotation is chosen to be about this given point. The equivalent small rotation $\boldsymbol{\epsilon}$ is independent of the point chosen ; the associated uniform translation depends on the point chosen. The vector $\boldsymbol{\epsilon}$ is the analogue of the vector $\mathbf{R}$ in the theory of line vectors, the displacement vector $\mathbf{u}(O)$ is the analogue of the couple $\mathbf{G}(O)$ giving the moment of the system of line vectors about O. It follows that $\boldsymbol{\epsilon}^2$ and $\boldsymbol{\epsilon}.\mathbf{u}$ are invariants of the system of displacements, independent of O and the same for all equivalent systems.

181. *Nul lines with respect to a given small displacement.* Let a small displacement be specified as a small translation $\mathbf{u}$ and a small rotation $\boldsymbol{\epsilon}$ about a particle O. Consider the points of the rigid body lying in a given line of line co-ordinates $\mathbf{i}$, $\mathbf{a}$ with respect to O. Any point on this line has, with respect to O, a position vector $\mathbf{r}$ of the form

$$\mathbf{r}=\mathbf{i}\wedge\mathbf{a}+\lambda\mathbf{i}, \qquad (\mathbf{i}.\mathbf{a}=0)$$

and its small displacement is given by

$$d\mathbf{r}=\mathbf{u}+\boldsymbol{\epsilon}\wedge(\mathbf{i}\wedge\mathbf{a}+\lambda\mathbf{i}).$$

This small displacement will be perpendicular to the given line if $d\mathbf{r}.\mathbf{i}=0$, i.e. if

$$\mathbf{u}.\mathbf{i}+\boldsymbol{\epsilon}.\mathbf{a}=0.$$

When this condition is satisfied the line is said to be a *nul line* for the given small displacement $\boldsymbol{\epsilon}$, $\mathbf{u}$. Clearly the moment of the system of line vectors $(\mathbf{R}, \mathbf{G})$, where $\mathbf{R}$ (at O)$=\boldsymbol{\epsilon}$ and $\mathbf{G}=\mathbf{u}$, about the line $(\mathbf{i}, \mathbf{a})$ is zero. Thus nul lines defined in the above way are nul lines of any system of

line vectors representing the displacement, and their further properties follow from the discussion given in Chapter VI.

Example (1). A rigid body is displaced about a particle O so that the particle $\mathbf{r}_1$ undergoes the displacement $d\mathbf{r}_1$, the particle $\mathbf{r}_2$ the displacement $d\mathbf{r}_2$. Show that the small rotation $\boldsymbol{\epsilon}$ about O is given by

$$\boldsymbol{\epsilon}=\frac{d\mathbf{r}_1\wedge d\mathbf{r}_2}{d\mathbf{r}_1 . \mathbf{r}_2}=\frac{d\mathbf{r}_2\wedge d\mathbf{r}_1}{d\mathbf{r}_2 . \mathbf{r}_1}.$$

Example (2). Show that the result of successive small rotations about and proportional to the three sides of a plane triangle is a uniform translation.

CHAPTER IX

THE WORK OF A SYSTEM OF LINE VECTORS

182. *Definition of work.* Let a particle $\mathbf{r}$ on the line of action of a line vector $\mathbf{P}$ undergo a displacement $\Delta\mathbf{r}$. Then the *work* of $\mathbf{P}$ in the displacement $\Delta\mathbf{r}$ is defined to be the scalar product

$$\mathbf{P}.\Delta\mathbf{r}.$$

183. *Work of the resultant of a system of concurrent line vectors.*

Theorem: If $(\mathbf{P})$ is a system of concurrent line vectors passing through the point O, $(\mathbf{R})$ their resultant, then the work of the system $(\mathbf{P})$ in any displacement of the particle $\mathbf{r}$ is equal to the work of their resultant $(\mathbf{R})$.

For we have

$$\mathbf{R}=\Sigma\mathbf{P},$$

whence
$$\mathbf{R}.\Delta\mathbf{r}=(\Sigma\mathbf{P}).\Delta\mathbf{r}=\Sigma(\mathbf{P}.\Delta\mathbf{r}).$$

184. *Small displacement as a rigid body.* Let $\mathbf{r}_1$, $\mathbf{r}_2$ be the position vectors of two particles on the line of action of a line vector $\mathbf{P}$. Let the two particles undergo small displacements $d\mathbf{r}_1$, $d\mathbf{r}_2$ as a rigid body.

Theorem: The work of $(\mathbf{P})$ in the small displacement $d\mathbf{r}_1$ is equal to the work of $(\mathbf{P})$ in the small displacement $d\mathbf{r}_2$. (Here $d\mathbf{r}_1$ and $d\mathbf{r}_2$ are to be understood in the usual sense of differentials.)

For, since the displacement is that of a rigid body, we have

$$(\mathbf{r}_1-\mathbf{r}_2)^2=\text{const.},$$

whence
$$(\mathbf{r}_1-\mathbf{r}_2).(d\mathbf{r}_1-d\mathbf{r}_2)=0.$$

But $\mathbf{P}$ is parallel to $\mathbf{r}_1-\mathbf{r}_2$. Hence

$$\mathbf{P}.(d\mathbf{r}_1-d\mathbf{r}_2)=0$$

or
$$\mathbf{P}.d\mathbf{r}_1=\mathbf{P}.d\mathbf{r}_2.$$

185. *Work of a system of line vectors.* Let $(\mathbf{P})$ be a system of line vectors, $\mathbf{r}_n$ a particle on the line of action of a typical member $(\mathbf{P}_n)$. Let the particles $\mathbf{r}_n$ be subject to the displacements $\Delta\mathbf{r}_n$. Then the *work* of $(\mathbf{P})$ in this set of displacements is

$$\sum_n \mathbf{P}_n.\Delta\mathbf{r}_n.$$

It should be particularly noted that this definition is to be taken strictly as it stands. It implies no suggestion that the line vectors $(\mathbf{P}_n)$ themselves have had their lines of action displaced, or that the line vector $(\mathbf{P}_n)$ has

continued to 'act' on the particle $\mathbf{r}_n$ during the displacement. Thus the 'work' of $(\mathbf{P})$ in the given set of displacements is a conventional construct, obtained by multiplying each member $\mathbf{P}_n$ by a specified displacement and summing. The displacements may be of any character.

186. *Work of a system of line vectors in a small rigid body displacement.* A particular type of displacement is one in which the set of points $\mathbf{r}_n$ remain rigidly connected during the displacement. Now choose another set of points $\mathbf{r}_n'$ on the lines of action of the line vectors, and let this second set of points be displaced as a rigid body itself rigidly connected to the first set of points. Then by the theorem of § 184 we have the following theorem.

Theorem: The work of a given system of line vectors in a given small rigid body displacement of a set of points on their lines of action is equal to that in any small displacement of any other set of points on their lines of action rigidly connected with the first set.

187. *Evaluation of the work of a system of line vectors.* Let the system of line vectors $(\mathbf{P})$ be equivalent to a line vector $\mathbf{R}$ at O and a couple $\mathbf{G}$. Then by the conditions of equivalence

$$\mathbf{R}=\Sigma\mathbf{P}_n,$$
$$\mathbf{G}=\Sigma\mathbf{r}_n\wedge\mathbf{P}_n.$$

Let the given small rigid body displacement be equivalent to a uniform translation $\mathbf{u}$ and a small rotation $\boldsymbol{\epsilon}$ about O. Then

$$d\mathbf{r}_n=\mathbf{u}+\boldsymbol{\epsilon}\wedge\mathbf{r}_n.$$

Hence the work of $(\mathbf{P})$ in the displacement $(\mathbf{u}, \boldsymbol{\epsilon})$ is

$$\begin{aligned}&\sum_n\mathbf{P}_n.(\mathbf{u}+\boldsymbol{\epsilon}\wedge\mathbf{r}_n)\\ &=\mathbf{u}.\Sigma\mathbf{P}_n+\boldsymbol{\epsilon}.\Sigma\mathbf{r}_n\wedge\mathbf{P}_n\\ &=\mathbf{u}.\mathbf{R}+\boldsymbol{\epsilon}.\mathbf{G}.\end{aligned}$$

Thus the work is determinate in terms of $\mathbf{R}$, $\mathbf{G}$, $\mathbf{u}$, $\boldsymbol{\epsilon}$. But all equivalent systems have the same $\mathbf{R}$, $\mathbf{G}$ when O is base point. Hence we have the following theorem:

Theorem: The work done by a system of line vectors in a given small rigid body displacement is equal to that done by any equivalent system of line vectors in the same displacement.

188. *Systems equivalent to zero.* We can now obtain conditions for the equilibrium of a system of line vectors in terms of their work in arbitrary small displacements, as follows:

Theorem: If a system of line vectors is equivalent to zero, the work done in any small rigid body displacement of points on their lines of action is zero; and conversely, if the work done in all small rigid body displacements is zero, then the system of line vectors is equivalent to zero.

For the work done in the displacement $(\mathbf{u}, \boldsymbol{\epsilon})$ is $\mathbf{u}.\mathbf{R}+\boldsymbol{\epsilon}.\mathbf{G}$, and this

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vanishes when $\mathbf{R}=0$, $\mathbf{G}=0$. Conversely, if $\mathbf{u}.\mathbf{R}+\boldsymbol{\epsilon}.\mathbf{G}=0$ for all $(\mathbf{u}, \boldsymbol{\epsilon})$, then $\mathbf{R}=0$, $\mathbf{G}=0$.

It will be noticed that the expression for the work done, $\mathbf{u}.\mathbf{R}+\boldsymbol{\epsilon}.\mathbf{G}$, contains no reference to the particular points $\mathbf{r}_n$ on the lines of action of the members of $(\mathbf{P})$. All we need to know is the values of $\mathbf{u}$ and $\boldsymbol{\epsilon}$, i.e. the rigid body displacement. It would be misleading, of course, to suppose that the work is done 'on' any such rigid body. The rigid body serves no other purpose than to illustrate the meaning of $\mathbf{u}$ and $\boldsymbol{\epsilon}$, just as in statics we can use a rigid body to illustrate the meaning of $\mathbf{R}$ and $\mathbf{G}$. We can if we like identify the two rigid bodies mentioned. The essential point is that the work done is a function of two systems of line vectors, namely ($\mathbf{R}$ at O, $\mathbf{G}$) and ($\mathbf{u}$, $\boldsymbol{\epsilon}$ at O).

189. *The work of a couple.* Since $\mathbf{R}$ is a line vector through O, and since the displacement of the particle O itself is just $\mathbf{u}$, the term $\mathbf{u}.\mathbf{R}$ in the expression for the work is the contribution due to the line system consisting of $\mathbf{R}$ alone, at O. It follows that $\boldsymbol{\epsilon}.\mathbf{G}$ must be the work of the couple $\mathbf{G}$ in the small rotation $\boldsymbol{\epsilon}$ about O. This is readily verified.

For the system consisting of $\mathbf{P}$ at $\mathbf{r}_1$ and $-\mathbf{P}$ at $\mathbf{r}_2$ is equivalent to the couple $\mathbf{G}$ where

$$\mathbf{G}=(\mathbf{r}_1-\mathbf{r}_2)\wedge\mathbf{P}.$$

The displacements of the particles $\mathbf{r}_1$ and $\mathbf{r}_2$ in the displacement $(\mathbf{u}, \boldsymbol{\epsilon})$ being

$$d\mathbf{r}_1=\mathbf{u}+\boldsymbol{\epsilon}\wedge\mathbf{r}_1, \qquad d\mathbf{r}_2=\mathbf{u}+\boldsymbol{\epsilon}\wedge\mathbf{r}_2,$$

the work of the couple is

$$\begin{aligned}&\mathbf{P}.(\mathbf{u}+\boldsymbol{\epsilon}\wedge\mathbf{r}_1)+(-\mathbf{P}).(\mathbf{u}+\boldsymbol{\epsilon}\wedge\mathbf{r}_2)\\ &=\boldsymbol{\epsilon}.(\mathbf{r}_1-\mathbf{r}_2)\wedge\mathbf{P}=\boldsymbol{\epsilon}.\mathbf{G}.\end{aligned}$$

Thus the work of a couple in a small displacement depends only on the moment of the couple and the small rotation part of the displacement, and is independent of the location of the line vectors constituting the couple or the translation part of the displacement.

190. *The work of a wrench in a screw.* The system $(\mathbf{R}, \mathbf{G})$ with base point O is equivalent to some wrench. The displacement $(\mathbf{u}, \boldsymbol{\epsilon})$ about O is equivalent to some screw. By the general theory, the work of the wrench in a small displacement specified by the screw is

$$\mathbf{u}.\mathbf{R}+\boldsymbol{\epsilon}.\mathbf{G}.$$

Now let the system $(\mathbf{R}, \mathbf{G})$ with base point O be equivalent to the wrench $f\mathbf{i}$, $pf\mathbf{i}$, in the line of line co-ordinates $(\mathbf{i}, \mathbf{a})$. Let the displacement $(\mathbf{u}, \boldsymbol{\epsilon})$ about O be equivalent to the screw $(g\mathbf{j}, p'g\mathbf{j})$ with axis in the line of line co-ordinates $(\mathbf{j}, \mathbf{b}.)$ Then by the conditions of equivalence

$$\mathbf{R}=f\mathbf{i}, \qquad \mathbf{G}=pf\mathbf{i}+f\mathbf{a},$$
$$\boldsymbol{\epsilon}=g\mathbf{j}, \qquad \mathbf{u}=p'g\mathbf{j}+g\mathbf{b}.$$

Hence the work done, namely $\mathbf{u}.\mathbf{R}+\boldsymbol{\epsilon}.\mathbf{G}$, is equal to

$$fg[(p+p')\mathbf{i}.\mathbf{j}+(\mathbf{a}.\mathbf{j}+\mathbf{b}.\mathbf{i})].$$

Now $\mathbf{i}.\mathbf{j}=\cos\theta$, where θ is the angle between the axes of the wrench and the screw ; and $\mathbf{a}.\mathbf{j}+\mathbf{b}.\mathbf{i}$ is the mutual moment m of the two axes, which is equal to $\pm h\sin\theta$, where h is the perpendicular distance between the two axes. Hence the work of the wrench along the screw may be written in the form

$$|\mathbf{R}|\,|\epsilon|[(p+p')\cos\theta+m]$$

or

$$|\mathbf{R}|\,|\epsilon|[(p+p')\cos\theta\pm h\sin\theta].$$

The upper or lower sign is to be taken according as the mutual moment of the lines, with their proper senses, is positive or negative.

191. *The Principle of Virtual Work.* Suppose a given system of rigid bodies is in equilibrium under the action of a given system of forces. (Particular members of the system of bodies may reduce to particles.) Then each rigid body is separately in equilibrium under the action of certain of the given forces together with certain reactions. Now let the set of rigid bodies undergo *small* displacements compatible with the geometrical conditions of attachment, contact, etc., between the different rigid parts of the system. It follows that if $(\mathbf{P})$ is the given system of forces, $(\mathbf{X})$ the system of reactions, $\mathbf{r}_n$ the point of application of a typical applied force $\mathbf{P}_n$, ρ_n the point of application of a typical reaction $\mathbf{X}_n$, then

$$\Sigma\mathbf{P}_n.d\mathbf{r}_n+\Sigma\mathbf{X}_n.d\rho_n=0.$$

It can now be shown that under certain circumstances

$$\Sigma\mathbf{X}_n.d\rho_n=0.$$

The proof of this requires the consideration of the different possible types of connection between the different members of the system. Assuming this to be established, it follows that

$$\sum_n\mathbf{P}_n.d\mathbf{r}_n=0.$$

This result is known as the *Principle of Virtual Work* for a statical system. The differential scalar $\sum_n\mathbf{P}_n.d\mathbf{r}_n$ is called the Virtual Work of the system in the given displacement.

By considering displacements in which the given constraints are violated, the reactions caused by the constraints may be transferred to the sum $\Sigma\mathbf{P}_n.d\mathbf{r}_n$. In this way internal reactions may be determined.

192. To examine the value of $\Sigma\mathbf{X}_n.d\rho_n$, divide the reactions into two classes :

(1) those exerted by fixed external bodies ;

(2) those exerted between different parts of the system.

(1) If $\mathbf{X}$ is a reaction due to an external contact, and if the contact is *smooth*, then $\mathbf{X}$ is perpendicular to the surfaces in contact. If the contact is preserved during the displacement, the displacement $d\rho$ must be a

vector lying in the plane of contact, for the only permissible displacement is one lying in the tangent plane. Hence $\mathbf{X}$ and $d\rho$ are perpendicular, or $\mathbf{X}.d\rho=0$. If the contact is *rough*, and if only rolling motion is allowed, the *actual* displacement $\delta\rho$ of the point of contact, i.e. of the particle on which the reaction is acting, is of the second order in the small quantities defining the displacement, and the *differential* $d\rho$ itself is zero. Hence $\mathbf{X}.d\rho$ again vanishes.

(2) If $\mathbf{X}$ is an internal reaction, it may arise as the tension of a string, or as a reaction between a particle and an internal surface, or as a reaction between two internal surfaces in contact.

If it arises as the tension of a string, there will be two particles, $\mathbf{P}_1$ and $\mathbf{P}_2$, on which the reactions $\mathbf{X}$ and $-\mathbf{X}$ act. In any displacement which conserves the length of the string, $\mathbf{P}_1$ and $\mathbf{P}_2$ behave as if rigidly connected, and so, as in § 184,

$$\mathbf{X}.d\mathbf{r}_1+(-\mathbf{X}).d\mathbf{r}_2=0,$$

and the contributions of $(\mathbf{X}, -\mathbf{X})$ to the virtual work of the small displacement is zero.

If $\mathbf{X}$ arises as a reaction between a particle P and an internal smooth surface with which it is in contact, then in any displacement compatible with this constraint, the *relative* displacement of the particle P and the original point of contact Q will be in the tangent plane. The two contributions to the virtual work will thus add to

$$\mathbf{X}.(\mathbf{u}+d'\mathbf{r}_1)+(-\mathbf{X}).(\mathbf{u}+d'\mathbf{r}_2),$$

where $d'\mathbf{r}_1-d'\mathbf{r}_2$ is the relative displacement of P with respect to Q. This is equal to

$$\mathbf{X}.(d'\mathbf{r}_1-d'\mathbf{r}_2),$$

which vanishes since, if the contact is smooth, the relative displacement is perpendicular to the reaction.

If $\mathbf{X}$ arises as the reaction between two surfaces in contact, the relative displacement of the two original particles in contact will, in a rolling displacement, be of the second order of the small quantities describing the displacement. The reactions at the point of contact being equal and opposite, their contribution to the virtual work is zero. For *smooth* contacts, the contribution of the reaction to the virtual work will be zero also for a rolling and slipping relative displacement.

193. When the external forces on the system are given, together with the natures of the constraints, the Principle of Virtual Work suffices, by choice of suitable virtual displacements, to determine the configuration of the system. The Principle of Virtual Work can also be used, as mentioned above, to determine internal reactions by severing a constraint and choosing a displacement in which the forces of reaction do work.

194. *Sufficiency of the Principle of Virtual Work.* It can be shown that the Principle of Virtual Work is also *sufficient* to ensure the equilibrium

of a system ; i.e. if the virtual work of the applied forces is zero for any displacement, then the system is in equilibrium.

The proof of this requires an appeal to dynamics, and so should really occur at a later stage of our exposition. But it is convenient to insert it here. We establish it by proving that if the system is not in equilibrium, then there exists a displacement for which the virtual work of the forces is not zero.

For, if the system is not in equilibrium, it will begin to set itself in motion, and each particle of the system will possess some acceleration, in general not zero. If $\mathbf{r}$ is the position vector of a typical particle, $\ddot{\mathbf{r}}_0$ its initial acceleration, then by the laws of dynamics,

$$\mathbf{F}=m\ddot{\mathbf{r}}_0,$$

where $\mathbf{F}$ is the force acting on the particle, m its mass. Further the initial small motion $\delta\mathbf{r}$ is given by

$$\delta\mathbf{r}=\tfrac{1}{2}\ddot{\mathbf{r}}_0 t^2.$$

The initial small motions form a displacement compatible with the constraints. Take then as a virtual displacement the set of displacements

$$d\mathbf{r}=\lambda\ddot{\mathbf{r}}_0,$$

where $\lambda(>0)$ is small. Then

$$\Sigma\mathbf{F}.d\mathbf{r}=\lambda\Sigma m\ddot{\mathbf{r}}_0{}^2>0.$$

But the internal reactions form a system of line vectors equivalent to zero. Hence their virtual work in the displacement is zero. Hence the virtual work of the external forces in this displacement is greater than zero.

This contradicts the hypothesis that the virtual work of the external forces is zero in all displacements. Hence the system must be in equilibrium, since all the $\ddot{\mathbf{r}}_0$'s must be zero.

Part III. Dynamics

CHAPTER X

KINEMATICS

195. *Scope of the chapter.* In this chapter we examine the relations which must exist between the various quantities describing a particle or system of particles in motion. Such systems of particles will include rigid bodies. In particular we shall find it convenient to use as systems of reference rigid bodies which are themselves in motion. This is sometimes referred to as the use of systems of 'moving axes,' though what is important is not the motion of the axes themselves, but of the rigid body in which the axes are fixed. Actually most problems of dynamics can be solved by the use of *moving vectors* only, without using a moving rigid body as frame of reference for the vectors; and for a considerable portion of this chapter our work will avoid the equivalent of the use of *moving axes.* Nevertheless, the motion of a rigid body is fundamental in what follows, and it is therefore convenient to begin with a description of the motion of a rigid body.

196. *Methods of approach to the motion of a rigid body.* A description of the motion of a rigid body *can* be obtained from the theory of the small displacements of a rigid body, by proceeding to a limit. This method will in fact be used for the sake of illustration, but the method is not satisfactory, inasmuch as the theory of the *small* displacements of a rigid body is essentially an *approximate* theory only. It has a specific significance only when all displacements are understood as differentials. It is more satisfactory to derive the fundamental theorems concerning the motion of a rigid body *ab initio*, beginning not with the theory of small displacements but with the rigid body in its actual state of motion.

This alternative development is indeed essential. For whilst a meaning can be attached (as we shall see) to saying that a rigid body has *simultaneously* two or more angular velocities (by which we mean that a body A has an angular velocity Ω in a rigid frame of reference S, and that *at the same time* the frame S has an angular velocity Ω_1 relative to some frame S_1, and so on), no meaning can be attached to saying that a rigid body undergoes simultaneously two or more small displacements; for the order in which the displacements are undergone is material to the final position of the rigid body. It is true that we can describe such displacements in

terms of a small displacement $(\mathbf{u}, \boldsymbol{\epsilon})$ relative to a frame of reference S, and then a small displacement $(\mathbf{u}_1, \boldsymbol{\epsilon}_1)$ of S relative to a frame S_1 ; but the final position is then not given exactly by the sum $(\mathbf{u}+\mathbf{u}_1, \boldsymbol{\epsilon}+\boldsymbol{\epsilon}_1)$, considered as a single displacement relative to S_1. On the other hand, the rigid body in actual motion has actually the angular velocity $\boldsymbol{\Omega}+\boldsymbol{\Omega}_1$ relative to S_1. There is not therefore a complete analogy between angular velocities and small rigid body displacements. In the following section we use the theory of small displacements in a purely exploratory way.

197. *Motion of a rigid body about a fixed point. Use of theory of small displacements.* Consider a rigid body, or set of particles rigidly connected. which are in motion about one particle O of the system, which is fixed, During an interval of time (t, t+dt) the displacements of the members of the system of particles will be rigid body displacements as defined in Chapter VIII. It follows that if during (t, t+dt) a particle of position vector $\mathbf{r}$ relative to O is displaced to $\mathbf{r}'$, then there exists a vector $\boldsymbol{\epsilon}$, given by

$$\boldsymbol{\epsilon}=2\mathbf{i}\tan\tfrac{1}{2}\theta,$$

which is such that

$$\mathbf{r}'-\mathbf{r}=\boldsymbol{\epsilon}\wedge\tfrac{1}{2}(\mathbf{r}+\mathbf{r}').$$

Let
$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=\lim_{dt\to o}\frac{\mathbf{r}'-\mathbf{r}}{dt}.$$

Dividing both sides of the displacement relation by dt and proceeding to the limit, we have

$$\mathbf{v}=\boldsymbol{\Omega}\wedge\mathbf{r},$$

where
$$\boldsymbol{\Omega}=\lim_{dt\to o}\mathbf{i}\,\frac{\theta}{dt}.$$

This limit vector $\boldsymbol{\Omega}$ is called the angular velocity of the rigid body about O. It is clear that $|\boldsymbol{\Omega}|$ or $\lim\limits_{dt\to o}\theta/dt$ is the rate of turning of the rigid body, instantaneously, about the axis $\mathbf{i}$, i.e. about an axis in the direction of $\boldsymbol{\Omega}$. It is also clear that $\boldsymbol{\Omega}$ is not a differential coefficient. For there is no variable θ which is a function of the time, whose differential coefficient would be $|\boldsymbol{\Omega}|$, for any value of t ; for the value of the angle θ turned through by the rigid body in any finite interval (t_1, t_2) depends on both t_1 and t_2, and is not simply additive for change of t_1.

If $\mathbf{r}_1$, $\mathbf{r}_2$ are two given particles of the system, $\mathbf{v}_1$ and $\mathbf{v}_2$ their velocities, then $\boldsymbol{\Omega}$ can be expressed in terms of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{r}_1$, $\mathbf{r}_2$. For, by the formulæ of § 170, we derive

$$\boldsymbol{\Omega}=\frac{\mathbf{v}_1\wedge\mathbf{v}_2}{\mathbf{v}_1.\mathbf{r}_2}=\frac{\mathbf{v}_2\wedge\mathbf{v}_1}{\mathbf{v}_2.\mathbf{r}_1}.$$

We proceed to a rigorous derivation of these formulæ, and of the existence of the angular velocity vector, directly from consideration of the angular body in motion.

198. *Existence of an angular velocity.* Let $\mathbf{r}_1$, $\mathbf{r}_2$ be the position vectors at time t of two given particles of a rigid body in motion about a particle O, from which $\mathbf{r}_1$ and $\mathbf{r}_2$ are measured ; and let $\dot{\mathbf{r}}_1$, $\dot{\mathbf{r}}_2$ be their velocities.

Theorem : There exists a unique vector $\mathbf{\Omega}$ with the properties

$$\mathbf{\Omega}\wedge\mathbf{r}_1=\dot{\mathbf{r}}_1, \qquad \mathbf{\Omega}\wedge\mathbf{r}_2=\dot{\mathbf{r}}_2$$

provided $\mathbf{r}_1$ and $\mathbf{r}_2$ are not parallel ; and further, if $\mathbf{r}$ is any other particle of the system, rigidly connected with $\mathbf{r}_1$ and $\mathbf{r}_2$, then

$$\mathbf{\Omega}\wedge\mathbf{r}=\dot{\mathbf{r}}.$$

For, the conditions of rigidity are

$$\mathbf{r}_1{}^2=\text{const.}, \qquad \mathbf{r}_2{}^2=\text{const.}, \qquad (\mathbf{r}_1-\mathbf{r}_2)^2=\text{const.}$$

Differentiating these with respect to the time t we have

$$\mathbf{r}_1.\dot{\mathbf{r}}_1=0, \qquad \mathbf{r}_2.\dot{\mathbf{r}}_2=0,$$

and $$(\mathbf{r}_1-\mathbf{r}_2).(\dot{\mathbf{r}}_1-\dot{\mathbf{r}}_2)=0,$$

whence $$\mathbf{r}_1.\dot{\mathbf{r}}_2+\mathbf{r}_2.\dot{\mathbf{r}}_1=0.$$

These constitute the relations between position and velocity for the two particles considered.

Now, if there existed two distinct vectors $\mathbf{\Omega}$ and $\mathbf{\Omega}'$ with the above properties, then we should have

$$(\mathbf{\Omega}-\mathbf{\Omega}')\wedge\mathbf{r}_1=0, \qquad (\mathbf{\Omega}-\mathbf{\Omega}')\wedge\mathbf{r}_2=0.$$

Hence either $\mathbf{\Omega}-\mathbf{\Omega}'$ is parallel to both $\mathbf{r}_1$ and $\mathbf{r}_2$, or $\mathbf{\Omega}-\mathbf{\Omega}'=0$. Since $\mathbf{r}_1$ and $\mathbf{r}_2$ are given not to be parallel, we must have $\mathbf{\Omega}=\mathbf{\Omega}'$, and the solution is unique, if it exists at all. If one solution $\mathbf{\Omega}$ exists, $\mathbf{\Omega}$ must be perpendicular to both $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$ and therefore of the form

$$\mathbf{\Omega}=\lambda(\dot{\mathbf{r}}_1\wedge\dot{\mathbf{r}}_2).$$

The required properties then hold provided

$$\dot{\mathbf{r}}_1=\lambda(\dot{\mathbf{r}}_1\wedge\dot{\mathbf{r}}_2)\wedge\mathbf{r}_1=-\lambda\dot{\mathbf{r}}_1(\dot{\mathbf{r}}_2.\mathbf{r}_1)$$

and $$\dot{\mathbf{r}}_2=\lambda(\dot{\mathbf{r}}_1\wedge\dot{\mathbf{r}}_2)\wedge\mathbf{r}_2=+\lambda\dot{\mathbf{r}}_2(\dot{\mathbf{r}}_1.\mathbf{r}_2),$$

where we have used the conditions of rigidity $\dot{\mathbf{r}}_1.\mathbf{r}_1=0=\dot{\mathbf{r}}_2.\mathbf{r}_2$. These relations require the relations

$$\lambda=-\frac{1}{\dot{\mathbf{r}}_2.\mathbf{r}_1}=+\frac{1}{\dot{\mathbf{r}}_1.\mathbf{r}_2},$$

and these are self-consistent by the condition of rigidity $\mathbf{r}_1.\dot{\mathbf{r}}_2+\mathbf{r}_2.\dot{\mathbf{r}}_1=0$. The solution is therefore of the form

$$\mathbf{\Omega}=\frac{\dot{\mathbf{r}}_1\wedge\dot{\mathbf{r}}_2}{\dot{\mathbf{r}}_1.\mathbf{r}_2}=\frac{\dot{\mathbf{r}}_2\wedge\dot{\mathbf{r}}_1}{\dot{\mathbf{r}}_2.\mathbf{r}_1},$$

and this is immediately verified to be an actual solution.

The conditions of rigidity between the particles $\mathbf{r}_1$, $\mathbf{r}_2$ and any third particle $\mathbf{r}$ are

$$\mathbf{r}.\dot{\mathbf{r}}=0, \qquad \mathbf{r}.\dot{\mathbf{r}}_1+\mathbf{r}_1.\dot{\mathbf{r}}=0, \qquad \mathbf{r}.\dot{\mathbf{r}}_2+\mathbf{r}_2.\dot{\mathbf{r}}=0.$$

Hence if we put $$\mathbf{X}=\boldsymbol{\Omega}\wedge\mathbf{r},$$

we have, using the conditions of rigidity,

$$(\mathbf{X}-\dot{\mathbf{r}}).\mathbf{r}=0,$$
$$(\mathbf{X}-\dot{\mathbf{r}}).\mathbf{r}_1=(\boldsymbol{\Omega}\wedge\mathbf{r}).\dot{\mathbf{r}}_1-\dot{\mathbf{r}}.\mathbf{r}_1=-(\boldsymbol{\Omega}\wedge\mathbf{r}_1).\mathbf{r}+\dot{\mathbf{r}}_1.\mathbf{r}=0,$$
$$(\mathbf{X}-\dot{\mathbf{r}}).\mathbf{r}_2=(\boldsymbol{\Omega}\wedge\mathbf{r}).\mathbf{r}_2-\dot{\mathbf{r}}.\mathbf{r}_2=-(\boldsymbol{\Omega}\wedge\mathbf{r}_2).\mathbf{r}+\dot{\mathbf{r}}_2.\mathbf{r}=0.$$

Hence, provided $\mathbf{r}$, $\mathbf{r}_1$, and $\mathbf{r}_2$ are linearly independent, $\mathbf{X}-\dot{\mathbf{r}}=0$, or

$$\boldsymbol{\Omega}\wedge\mathbf{r}=\dot{\mathbf{r}}.$$

If $\mathbf{r}$ is coplanar with $\mathbf{r}_1$ and $\mathbf{r}_2$ then

$$\mathbf{r}=\alpha\mathbf{r}_1+\beta\mathbf{r}_2,$$

whence $$\dot{\mathbf{r}}=\alpha\dot{\mathbf{r}}_1+\beta\dot{\mathbf{r}}_2,$$

and $$\boldsymbol{\Omega}\wedge\mathbf{r}=\alpha(\boldsymbol{\Omega}\wedge\mathbf{r}_1)+\beta(\boldsymbol{\Omega}\wedge\mathbf{r}_2)=\alpha\dot{\mathbf{r}}_1+\beta\dot{\mathbf{r}}_2=\dot{\mathbf{r}},$$

as before.

It now follows that if $\mathbf{r}_n$ is any particle of the system, then $\dot{\mathbf{r}}_n=\boldsymbol{\Omega}\wedge\mathbf{r}_n$, whence $\boldsymbol{\Omega}$ might equally have been derived from *any* pair of particles, $\mathbf{r}_n$ and $\mathbf{r}_m$. It is thus independent of the pair originally chosen.

199. *Kinematical meaning of* $\boldsymbol{\Omega}$. The vector $\boldsymbol{\Omega}$ so isolated is called the *angular velocity* of the rigid body about O. To justify this phrase, consider the particular case of a motion in which $\boldsymbol{\Omega}$ is constant in time. Then

$$\frac{d}{dt}(\boldsymbol{\Omega}.\mathbf{r})=\boldsymbol{\Omega}.\dot{\mathbf{r}}=\boldsymbol{\Omega}.(\boldsymbol{\Omega}\wedge\mathbf{r})=0,$$

whence $$\boldsymbol{\Omega}.\mathbf{r}=\text{const.}$$

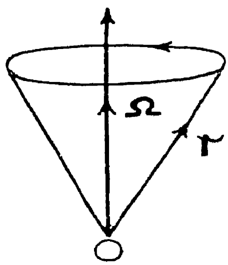

Fig. 46

Hence, since $|\mathbf{r}|=\text{const.}$ and $|\boldsymbol{\Omega}|=\text{const.}$, we must have $\widehat{\boldsymbol{\Omega}\mathbf{r}}=\text{const.}$ Hence the vector $\mathbf{r}$ describes a cone about $\boldsymbol{\Omega}$ as axis, and since $|\mathbf{r}|=\text{const.}$, the particle of position vector $\mathbf{r}$ describes a circle lying on this cone. Moreover

$$\dot{\mathbf{r}}^2=\boldsymbol{\Omega}^2\mathbf{r}^2-(\boldsymbol{\Omega}.\mathbf{r})^2=\text{const.},$$

and hence the particle $\mathbf{r}$ describes the circle with uniform speed. The time of a complete revolution is

$$\frac{\text{circumference of circle}}{\text{speed}}=\frac{2\pi|\mathbf{r}\wedge\boldsymbol{\Omega}|}{|\boldsymbol{\Omega}|\,|\dot{\mathbf{r}}|}=\frac{2\pi}{|\boldsymbol{\Omega}|}.$$

Hence $|\boldsymbol{\Omega}|$ is the rate of description of the arc of the circle measured in angle (Fig. 46).

When $\boldsymbol{\Omega}$ is not constant in time, the direction of $\boldsymbol{\Omega}$ is called the *instantaneous axis* of the rigid body. The motions of all particles of the

rigid body are perpendicular to the instantaneous direction of $\mathbf{\Omega}$, and their speeds are proportional to their perpendicular distances from this axis.

200. *Rigid body in motion in any manner.* Let O be a fixed origin (Fig. 47). Let O_0 be any particle of the body, of vector position $\mathbf{r}_0$ with respect to O. Let $\mathbf{r}$ be the vector position of any particle with respect to O, $\mathbf{r}'$ its position vector with respect to O_0. Then

$$\mathbf{r}=\mathbf{r}_0+\mathbf{r}', \qquad \dot{\mathbf{r}}=\dot{\mathbf{r}}_0+\dot{\mathbf{r}}'.$$

By the conditions of rigidity, the aggregate of vectors of the type $\mathbf{r}'$, $\dot{\mathbf{r}}'$ define a rigid body motion relative to O_0, and hence there exists a unique angular velocity $\mathbf{\Omega}$ about O_0 such that for all $\mathbf{r}'$,

$$\dot{\mathbf{r}}'=\mathbf{\Omega}\wedge\mathbf{r}'.$$

Hence
$$\dot{\mathbf{r}}=\dot{\mathbf{r}}_0+\mathbf{\Omega}\wedge\mathbf{r}',$$
$$=\dot{\mathbf{r}}_0+\mathbf{\Omega}\wedge(\mathbf{r}-\mathbf{r}_0).$$

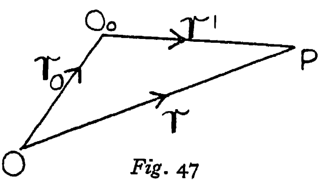

Fig. 47

Hence the most general motion of a rigid body can be represented as a rate of translation combined with an angular velocity about any arbitrary particle of the body.

201. *Angular velocity independent of origin particle.*

Theorem: The angular velocity of a rigid body in motion in any manner is independent of the particle of reference chosen.

For, let O_0, O_1, be two particles of reference, $\mathbf{\Omega}_0$, $\mathbf{\Omega}_1$ the angular velocities of the body about these particles. If $\mathbf{r}_0$, $\mathbf{r}_1$ are their position vectors with regard to a fixed origin O, then if $\mathbf{r}$ is any particle,

$$\dot{\mathbf{r}}=\dot{\mathbf{r}}_0+\mathbf{\Omega}_0\wedge(\mathbf{r}-\mathbf{r}_0)=\dot{\mathbf{r}}_1+\mathbf{\Omega}_1\wedge(\mathbf{r}-\mathbf{r}_1).$$

Also, since O_1, O_0 are particles of the body,

$$\dot{\mathbf{r}}_1=\dot{\mathbf{r}}_0+\mathbf{\Omega}_0\wedge(\mathbf{r}_1-\mathbf{r}_0).$$

Hence
$$\mathbf{\Omega}_0\wedge(\mathbf{r}-\mathbf{r}_1)=\mathbf{\Omega}_1\wedge(\mathbf{r}-\mathbf{r}_1).$$

This holds for all $\mathbf{r}$. Hence $\mathbf{\Omega}_1=\mathbf{\Omega}_0$.

We may accordingly speak of the *angular velocity* of the rigid body, without specifying the origin of reference. The momentary *rate of translation* of the rigid body ($\dot{\mathbf{r}}_0$ or $\dot{\mathbf{r}}_1$ in the above) depends on the origin of reference chosen.

202. *Reduction of the motion of a rigid body to that of a screw.* Let the motion of a rigid body be specified by the velocity $\mathbf{u}_0$ of a given particle, of position vector $\mathbf{r}_0$ with regard to a fixed origin O, and the angular velocity $\mathbf{\Omega}$. The velocity of any particle $\mathbf{r}$ is then given by

$$\frac{d\mathbf{r}}{dt}=\mathbf{u}_0+\mathbf{\Omega}\wedge(\mathbf{r}-\mathbf{r}_0).$$

If we refer the motion to some other particle $\mathbf{r}_1$, of velocity $\mathbf{u}_1$, then

$$\frac{d\mathbf{r}}{dt}=\mathbf{u}_1+\mathbf{\Omega}\wedge(\mathbf{r}-\mathbf{r}_1),$$

where
$$\mathbf{u}_1=\mathbf{u}_0+\mathbf{\Omega}\wedge(\mathbf{r}_1-\mathbf{r}_0).$$

Let us attempt to choose $\mathbf{r}_1$ so that $\mathbf{u}_1$ is parallel to $\mathbf{\Omega}$. Then, putting $\mathbf{u}_1 = p\mathbf{\Omega}$, we have

$$p\mathbf{\Omega} = \mathbf{u}_0 + \mathbf{\Omega} \wedge (\mathbf{r}_1 - \mathbf{r}_0).$$

Multiplying vectorially by $\mathbf{\Omega}$ we have

$$-\mathbf{\Omega}[(\mathbf{r}_1 - \mathbf{r}_0).\mathbf{\Omega}] + (\mathbf{r}_1 - \mathbf{r}_0)\Omega^2 = \mathbf{\Omega} \wedge \mathbf{u}_0$$

whence
$$\mathbf{r}_1 - \mathbf{r}_0 = \frac{\mathbf{\Omega} \wedge \mathbf{u}_0}{\Omega^2} + \lambda\mathbf{\Omega}.$$

The value of p is found by multiplying the same equation scalarly by $\mathbf{\Omega}$, when we get

$$p = \frac{\mathbf{\Omega}.\mathbf{u}_0}{\Omega^2}.$$

The formula for $\mathbf{r}_1 - \mathbf{r}_0$ shows that the locus of the particle $\mathbf{r}_1$ is a straight line. The motion therefore consists of a rate of displacement $p\mathbf{\Omega}$ along this straight line, and a rate of rotation or angular velocity $\mathbf{\Omega}$ about any point in this straight line. The motion is therefore one along a screw.

It is clear that, just as in the case of the small displacements of a rigid body, there is a complete analogy between the angular velocity $\mathbf{\Omega}$ of a rigid body and the vector sum $\mathbf{R}$ of a system of line vectors, and between the rate of displacement $\mathbf{u}_0$ of a given point $\mathbf{r}_0$ and the moment $\mathbf{G}$ of the system of line vectors about $\mathbf{r}_0$. Theorems concerning systems of line vectors have counterparts, concerning the motion of a rigid body.

203. *Theorem of relative angular velocities.* Let a rigid body A possess an angular velocity $\mathbf{\Omega}$ in a frame of reference F_1. Let the frame of reference F_1 possess an angular velocity $\mathbf{\Omega}_1$ in a frame of reference F_2. We have the following theorem.

Theorem: The angular velocity of the rigid body A in the frame of reference F_2 is $\mathbf{\Omega} + \mathbf{\Omega}_1$.

This is the first formal occasion on which we have had to consider *motion* of the frame of reference implied in any mention of a vector. In saying that a rigid body in motion has an angular velocity $\mathbf{\Omega}$, we mean that $\mathbf{\Omega}$ is reckoned relative to a tacitly implied frame of reference, itself a rigid body with possibly its own angular velocity relative to some second frame of reference. In the present theorem these frames of reference become explicit.

Without loss of generality we can consider the various bodies as in motion about a fixed point O, which corresponds to a particle possessed by them in common.

The velocity $d\mathbf{r}/dt$ of the particle $\mathbf{r}$ of the rigid body A, in the frame F_2, can be considered as the vector sum of the apparent velocity $\partial\mathbf{r}/\partial t$ of the particle $\mathbf{r}$ in the frame F_1 and the velocity $d\mathbf{r}'/dt$ in the frame F_2

of the particle $\mathbf{r}'$ of the frame F_1 momentarily coinciding with $\mathbf{r}$. Thus

$$\frac{d\mathbf{r}}{dt}=\frac{\partial\mathbf{r}}{\partial t}+\frac{d\mathbf{r}'}{dt},$$

where $$\frac{\partial\mathbf{r}}{\partial t}=\mathbf{\Omega}\wedge\mathbf{r}, \qquad \frac{d\mathbf{r}'}{dt}=\mathbf{\Omega}_1\wedge\mathbf{r}'=\mathbf{\Omega}_1\wedge\mathbf{r}.$$

Hence $$\frac{d\mathbf{r}}{dt}=(\mathbf{\Omega}+\mathbf{\Omega}_1)\wedge\mathbf{r}.$$

This is true for all $\mathbf{r}$. Hence the rigid body A possesses an angular velocity $\mathbf{\Omega}+\mathbf{\Omega}_1$ in the frame F_2.

204. *Theorem of relative angular velocities continued.* Further insight into this theorem may be obtained by putting in evidence the rigid body constituting the frame of reference F_1. Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be three unit vectors constituting an orthogonal triad rigidly attached to F_1. Then since the particles of position vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are rigidly attached to F_1, we have

$$\frac{d\mathbf{i}}{dt}=\mathbf{\Omega}_1\wedge\mathbf{i}, \qquad \frac{d\mathbf{j}}{dt}=\mathbf{\Omega}_1\wedge\mathbf{j}, \qquad \frac{d\mathbf{k}}{dt}=\mathbf{\Omega}_1\wedge\mathbf{k}.$$

Now if $\mathbf{r}$ is the position vector of a particle of the rigid body A,

$$\mathbf{r}=(\mathbf{r}.\mathbf{i})\mathbf{i}+(\mathbf{r}.\mathbf{j})\mathbf{j}+(\mathbf{r}.\mathbf{k})\mathbf{k},$$

and so $$\frac{d\mathbf{r}}{dt}=\frac{d(\mathbf{r}.\mathbf{i})}{dt}\mathbf{i}+\frac{d(\mathbf{r}.\mathbf{j})}{dt}\mathbf{j}+\frac{d(\mathbf{r}.\mathbf{k})}{dt}\mathbf{k}$$

$$+(\mathbf{r}.\mathbf{i})\frac{d\mathbf{i}}{dt}+(\mathbf{r}.\mathbf{j})\frac{d\mathbf{j}}{dt}+(\mathbf{r}.\mathbf{k})\frac{d\mathbf{k}}{dt}.$$

But $$\sum_{\mathbf{i},\mathbf{j},\mathbf{k}}(\mathbf{r}.\mathbf{i})\frac{d\mathbf{i}}{dt}=\sum_{\mathbf{i},\mathbf{j},\mathbf{k}}(\mathbf{r}.\mathbf{i})(\mathbf{\Omega}_1\wedge\mathbf{i})=\mathbf{\Omega}_1\wedge\sum_{\mathbf{i},\mathbf{j},\mathbf{k}}(\mathbf{r}.\mathbf{i})\mathbf{i}=\mathbf{\Omega}_1\wedge\mathbf{r},$$

and the vector $$\sum_{\mathbf{i},\mathbf{j},\mathbf{k}}\frac{d(\mathbf{r}.\mathbf{i})}{dt}\mathbf{i},$$

being the vector sum of the rates of change of the components of $\mathbf{r}$ along $\mathbf{i}, \mathbf{j}, \mathbf{k}$ when $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are considered fixed, is simply $\partial\mathbf{r}/\partial t$, the *apparent* rate of change of $\mathbf{r}$ in the frame F_1. Thus

$$\sum_{\mathbf{i},\mathbf{j},\mathbf{k}}\frac{d(\mathbf{r}.\mathbf{i})}{dt}\mathbf{i}=\frac{\partial\mathbf{r}}{\partial t}=\mathbf{\Omega}\wedge\mathbf{r}.$$

Combining, we get

$$\frac{d\mathbf{r}}{dt}=\sum_{\mathbf{i},\mathbf{j},\mathbf{k}}(\mathbf{r}.\mathbf{i})\frac{d\mathbf{i}}{dt}+\sum_{\mathbf{i},\mathbf{j},\mathbf{k}}\frac{d}{dt}(\mathbf{r}.\mathbf{i})\mathbf{i}=(\mathbf{\Omega}+\mathbf{\Omega}_1)\wedge\mathbf{r},$$

as in § 203.

205. As a further verification, we note that

$$\sum_{\mathbf{i},\mathbf{j},\mathbf{k}} \frac{d(\mathbf{r}.\mathbf{i})}{dt}\mathbf{i} = \sum_{\mathbf{i},\mathbf{j},\mathbf{k}} \left[\frac{d\mathbf{r}}{dt}.\mathbf{i} + \mathbf{r}.\frac{d\mathbf{i}}{dt}\right]\mathbf{i}$$

$$= \sum_{\mathbf{i},\mathbf{j},\mathbf{k}} [(\mathbf{\Omega}+\mathbf{\Omega}_1)\wedge\mathbf{r}.\mathbf{i} + \mathbf{\Omega}_1\wedge\mathbf{i}.\mathbf{r}]\mathbf{i}$$

$$= \sum_{\mathbf{i},\mathbf{j},\mathbf{k}} (\mathbf{\Omega}\wedge\mathbf{r}.\mathbf{i})\mathbf{i}$$

$$= \mathbf{\Omega}\wedge\mathbf{r} = \frac{\partial\mathbf{r}}{\partial t}.$$

206. *The operator* $\partial/\partial t$. In applying the operator $\partial/\partial t$ to any expression containing **i, j, k** or containing any vector rigidly connected to F_1, the vectors **i, j, k** or the vector concerned attached to F_1 may be treated as *constants*, in spite of the fact that they are varying. For example

$$\frac{\partial\mathbf{i}}{\partial t} = 0,$$

as is seen by setting $\mathbf{r}=\mathbf{i}$ in § 205. Again, we may evaluate

$$\frac{\partial}{\partial t}\sum_{\mathbf{i},\mathbf{j},\mathbf{k}} (\mathbf{r}.\mathbf{i})\mathbf{i},$$

which is just $\partial\mathbf{r}/\partial t$, in the form

$$\sum_{\mathbf{i},\mathbf{j},\mathbf{k}} \left(\frac{\partial\mathbf{r}}{\partial t}.\mathbf{i}\right)\mathbf{i}.$$

For the latter is just $$\sum_{\mathbf{i},\mathbf{j},\mathbf{k}} (\mathbf{\Omega}\wedge\mathbf{r}.\mathbf{i})\mathbf{i},$$

which equals $$\mathbf{\Omega}\wedge\mathbf{r}.$$

207. *Rate of change of any vector in a moving frame of reference.* An analogous procedure may be applied to any vector whatever which is specified with respect to the moving frame F_1. Let **P** be a vector which is specified with respect to F_1 at every instant t. Let $d\mathbf{P}/dt$ denote the rate of change of **P** relative to the frame F_2, relative to which F_1 has an angular velocity $\mathbf{\Omega}_1$. Let $\partial\mathbf{P}/\partial t$ denote the *apparent* rate of change of **P** relative to the frame F_1, i.e. *treating the frame* F_1 *as if fixed.* Then we have the following theorem.

Theorem: The rate of change $d\mathbf{P}/dt$ of **P** in the frame F_2 is given by

$$\frac{d\mathbf{P}}{dt} = \frac{\partial\mathbf{P}}{\partial t} + \mathbf{\Omega}_1\wedge\mathbf{P}.$$

For, choose an orthogonal triad of unit vectors **i, j, k** rigidly attached to F_1. Then

$$\mathbf{P} = (\mathbf{P}.\mathbf{i})\mathbf{i} + (\mathbf{P}.\mathbf{j})\mathbf{j} + (\mathbf{P}.\mathbf{k})\mathbf{k}.$$

Hence $$\frac{d\mathbf{P}}{dt} = \sum_{\mathbf{i},\mathbf{j},\mathbf{k}} \frac{d(\mathbf{P}.\mathbf{i})}{dt}\mathbf{i} + \Sigma(\mathbf{P}.\mathbf{i})\frac{d\mathbf{i}}{dt}.$$

But
$$\frac{d\mathbf{i}}{dt}=\mathbf{\Omega}_1\wedge\mathbf{i}$$

whence
$$\sum_{\mathbf{i},\mathbf{j},\mathbf{k}}(\mathbf{P}.\mathbf{i})\frac{d\mathbf{i}}{dt}=\mathbf{\Omega}_1\wedge\sum_{\mathbf{i},\mathbf{j},\mathbf{k}}(\mathbf{P}.\mathbf{i})\mathbf{i}=\mathbf{\Omega}_1\wedge\mathbf{P},$$

and by the definition of $\partial\mathbf{P}/\partial t$,
$$\frac{\partial\mathbf{P}}{\partial t}=\sum_{\mathbf{i},\mathbf{j},\mathbf{k}}\frac{d(\mathbf{P}.\mathbf{i})}{dt}\mathbf{i}.$$

Hence
$$\frac{d\mathbf{P}}{dt}=\frac{\partial\mathbf{P}}{\partial t}+\mathbf{\Omega}_1\wedge\mathbf{P}.$$

We observe that the vector
$$\frac{d\mathbf{P}}{dt}-\mathbf{\Omega}_1\wedge\mathbf{P}$$

is essentially a vector independent of the choice of triad **i, j, k.** Hence $\partial\mathbf{P}/\partial t$ is a *vector* independent of the choice of triad **i, j, k.** This justifies the notation in so far as it contains no explicit reference to **i, j, k.** It follows further, since $\partial\mathbf{P}/\partial t$ is a vector, that
$$\frac{\partial\mathbf{P}}{\partial t}=\sum_{\mathbf{i},\mathbf{j},\mathbf{k}}\left(\frac{\partial\mathbf{P}}{\partial t}.\mathbf{i}\right)\mathbf{i}$$

which justifies the use of the symbol $\partial/\partial t$ as passing over **i, j, k** as if they were constants.

208. If we apply the theorem of § 207 to the vector $\mathbf{\Omega}_1$ itself we get
$$\frac{d\mathbf{\Omega}_1}{dt}=\frac{\partial\mathbf{\Omega}_1}{\partial t}.$$

Thus the rate of change of $\mathbf{\Omega}_1$ (the angular velocity of F_1 in F_2) *relative* to the frame F_1, is equal to the rate of change relative to the frame F_2.

209. *Loci of the instantaneous axis.* The physical meaning of the last result is of some interest. Suppose we have a rigid body with one particle O fixed, in motion with angular velocity $\mathbf{\Omega}$. The direction of $\mathbf{\Omega}$ through O is the instantaneous axis (§ 199) of the body, and particles on this axis are instantaneously at rest. Let P be any *point* on the instantaneous axis. Then the axis OP will possess a definite locus in space, which is the cone described by $\mathbf{\Omega}$. But the axis OP will also have a locus in the body, and this will also be a cone. The motion of the body can therefore be reconstructed by bringing the generators of the body locus in succession into coincidence with those of the space locus, and giving the body at each instant the appropriate rate of rotation $|\mathbf{\Omega}|$.

The meaning of the result
$$\frac{d\mathbf{\Omega}}{dt}=\frac{\partial\mathbf{\Omega}}{\partial t}$$

is now contained in the statement that the two loci *touch*. For if $\boldsymbol{\Omega}$, $\boldsymbol{\Omega}+\mathrm{d}\boldsymbol{\Omega}$ are neighbouring positions of $\boldsymbol{\Omega}$, the tangent plane to the *space locus*, having a normal perpendicular to $\boldsymbol{\Omega}$ and to $\boldsymbol{\Omega}+\mathrm{d}\boldsymbol{\Omega}$, has for its tangent plane a plane normal to

$$\boldsymbol{\Omega}\wedge \mathrm{d}\boldsymbol{\Omega}$$

i.e. to
$$\left(\boldsymbol{\Omega}\wedge\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{dt}}\right)\mathrm{dt},$$

which is in the direction of the unit vector

$$\left(\boldsymbol{\Omega}\wedge\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{dt}}\right)\bigg/\left|\boldsymbol{\Omega}\wedge\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{dt}}\right|.$$

Similarly, in the frame moving with the body, neighbouring positions of the generators of the body locus are given by $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}+\dfrac{\partial\boldsymbol{\Omega}}{\partial t}\mathrm{dt}$, and the tangent plane to this locus has for its normal

$$\left(\boldsymbol{\Omega}\wedge\frac{\partial\boldsymbol{\Omega}}{\partial t}\right)\bigg/\left|\boldsymbol{\Omega}\wedge\frac{\partial\boldsymbol{\Omega}}{\partial t}\right|.$$

Since $\mathrm{d}\boldsymbol{\Omega}/\mathrm{dt}=\partial\boldsymbol{\Omega}/\partial t$, these vectors coincide, and accordingly at the instant to which $\boldsymbol{\Omega}$ relates, the space locus and body locus have coincident tangent planes.

It follows that as the motion of the body successively brings the generators of space locus and body locus into coincidence, it also brings their tangent planes into coincidence. Since the particles of the body in OP are instantaneously at rest, the body locus cannot be *slipping* over the space locus. Accordingly the motion of the body can be reproduced by *rolling* the body locus over the space locus with the appropriate instantaneous angular velocity. The body locus is called the polhode cone, the space locus the herpolhode cone.

210. *Body completely free.* When the body is completely free (without some particle being fixed) the axis of the instantaneous screw has similarly a locus in space and a locus in the body. These loci will now be in general ruled surfaces. It is readily shown that the motion consists of a rolling of the two ruled surfaces over one another, about the generators, with sliding along the generators.

For if a ruled surface is generated by the motion of a line of line co-ordinates $(\mathbf{i}, \mathbf{a})$, and if $\mathbf{n}$ is a unit vector perpendicular to the tangent plane, the condition of tangency is that $\mathbf{n}$ must be perpendicular to every chord joining points on the neighbouring lines

$$\mathbf{r}=\mathbf{i}\wedge\mathbf{a}+\lambda\mathbf{i}, \qquad \mathbf{r}'=(\mathbf{i}+\mathrm{d}\mathbf{i})\wedge(\mathbf{a}+\mathrm{d}\mathbf{a})+\lambda'(\mathbf{i}+\mathrm{d}\mathbf{i}).$$

Hence
$$\mathbf{n}.(\mathbf{r}'-\mathbf{r})=0$$

or
$$\mathbf{n}.[\mathrm{d}\mathbf{i}\wedge\mathbf{a}+\mathbf{i}\wedge\mathrm{d}\mathbf{a}+(\lambda'-\lambda)\mathbf{i}+\lambda'\mathrm{d}\mathbf{i}]=0,$$

for all λ, λ'. Hence

$$\mathbf{n}.\mathbf{i}=0, \qquad \mathbf{n}.\frac{d\mathbf{i}}{dt}=0,$$

and hence $\mathbf{n}$ is parallel to $\mathbf{i}\wedge d\mathbf{i}/dt$. The condition of tangency is then satisfied identically. In our case, the co-ordinate $\mathbf{i}$ is parallel to $\boldsymbol{\Omega}$, the angular velocity, and for the space locus the normal is accordingly parallel to

$$\frac{\boldsymbol{\Omega}}{|\boldsymbol{\Omega}|}\wedge\frac{d}{dt}\left(\frac{\boldsymbol{\Omega}}{|\boldsymbol{\Omega}|}\right)$$

which is equal to
$$\frac{1}{|\boldsymbol{\Omega}|^2}\left[\boldsymbol{\Omega}\wedge\frac{d\boldsymbol{\Omega}}{dt}\right].$$

We have a similar parallelism for the body locus, with $\partial\boldsymbol{\Omega}/\partial t$ replacing $d\boldsymbol{\Omega}/dt$, and hence the two tangent planes are coincident. But, now, mutual *sliding* along the common generator is permissible, because the body will in general possess a rate of translation in the direction of the axis of the instantaneous screw.

211. *Rate of change of a tensor in a moving frame of reference.* Let $\mathbf{T}$ be a tensor of the second rank, given with reference to a frame F_1, and let $\boldsymbol{\Omega}_1$ be the angular velocity of F_1 with respect to another frame F_2. Then, just as we considered the relation between $d\mathbf{P}/dt$ and $\partial\mathbf{P}/\partial t$, so we may consider the relation between $d\mathbf{T}/dt$ and $\partial\mathbf{T}/\partial t$, where these symbols denote respectively the rate of change of $\mathbf{T}$ relative to F_2 and its apparent rate of change relative to F_1. The relation is given by the following theorem.

Theorem: In the notation of cross-products of vectors and tensors, the rate of change $d\mathbf{T}/dt$ of $\mathbf{T}$ in F_2 is connected with its apparent rate of change $\partial\mathbf{T}/\partial t$ in F_1 by the relation

$$\frac{d\mathbf{T}}{dt}=\frac{\partial\mathbf{T}}{\partial t}+\boldsymbol{\Omega}_1\wedge\mathbf{T}-\mathbf{T}\wedge\boldsymbol{\Omega}_1.$$

For, let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be an orthogonal triad of unit vectors rigidly connected to F_1. Then

$$\frac{d\mathbf{i}}{dt}=\boldsymbol{\Omega}_1\wedge\mathbf{i}, \qquad \frac{d\mathbf{j}}{dt}=\boldsymbol{\Omega}_1\wedge\mathbf{j}, \qquad \frac{d\mathbf{k}}{dt}=\boldsymbol{\Omega}_1\wedge\mathbf{k}.$$

Now
$$\mathbf{T}=(\mathbf{T}:\mathbf{ii})\mathbf{ii}+(\mathbf{T}:\mathbf{ij})\mathbf{ji}+\ldots.$$

Hence
$$\frac{d\mathbf{T}}{dt}=\Sigma\frac{d(\mathbf{T}:\mathbf{ii})}{dt}\mathbf{ii}+\Sigma(\mathbf{T}:\mathbf{ii})\frac{d\mathbf{ii}}{dt}.$$

But
$$\frac{\partial\mathbf{T}}{\partial t}=\Sigma\frac{d(\mathbf{T}:\mathbf{ii})}{dt}\mathbf{ii},$$

since $\partial\mathbf{T}/\partial t$ is constructed from the rates of change of its components in the triad **i, j, k.** And

$$\frac{d(\mathbf{ii})}{dt}=\frac{d\mathbf{i}}{dt}\mathbf{i}+\mathbf{i}\frac{d\mathbf{i}}{dt}$$

$$=(\mathbf{\Omega}_1\wedge\mathbf{i})\mathbf{i}+\mathbf{i}(\mathbf{\Omega}_1\wedge\mathbf{i})$$

$$=(\mathbf{\Omega}_1\wedge\mathbf{i})\mathbf{i}-\mathbf{i}(\mathbf{i}\wedge\mathbf{\Omega}_1)$$

$$=\mathbf{\Omega}_1\wedge(\mathbf{ii})-(\mathbf{ii})\wedge\mathbf{\Omega}_1$$

by the theorem of § 66. Hence

$$\Sigma(\mathbf{T}:\mathbf{ii})\frac{d(\mathbf{ii})}{dt}=\mathbf{\Omega}_1\wedge\mathbf{T}-\mathbf{T}\wedge\mathbf{\Omega}_1.$$

Hence altogether

$$\frac{d\mathbf{T}}{dt}=\frac{\partial\mathbf{T}}{\partial t}+\mathbf{\Omega}_1\wedge\mathbf{T}-\mathbf{T}\wedge\mathbf{\Omega}_1.$$

212. The principal application of this is to the *inertia tensor* of a rigid body, which we define later. We mention however now that the inertia tensor is a self-conjugate tensor. Now *when* **T** *is self-conjugate* ($\mathbf{T}=\overline{\mathbf{T}}$), we have (§ 67)

$$-\mathbf{T}\wedge\mathbf{\Omega}_1=\overline{\mathbf{\Omega}_1\wedge\overline{\mathbf{T}}}=\overline{\mathbf{\Omega}_1\wedge\mathbf{T}}$$

and so
$$\frac{d\mathbf{T}}{dt}=\frac{\partial\mathbf{T}}{\partial t}+\mathbf{\Omega}_1\wedge\mathbf{T}+\overline{\mathbf{\Omega}_1\wedge\mathbf{T}}.$$

Thus $d\mathbf{T}/dt$ is also seen to be self-conjugate.

213. The following alternative mode of proof is alluring but deceptive. Suppose **T** is expressed as a dyadic

$$\mathbf{T}=\mathbf{AX}+\mathbf{BY}+\mathbf{CZ},$$

where **A, B, C** and **X, Y, Z** are vectors. To these vectors we may apply the theorem of § 207, obtaining

$$\frac{d\mathbf{T}}{dt}=\frac{d\mathbf{A}}{dt}\mathbf{X}+\frac{d\mathbf{B}}{dt}\mathbf{Y}+\frac{d\mathbf{C}}{dt}\mathbf{Z}+\mathbf{A}\frac{d\mathbf{X}}{dt}+\mathbf{B}\frac{d\mathbf{Y}}{dt}+\mathbf{C}\frac{d\mathbf{Z}}{dt}$$

$$=\Sigma\left(\frac{\partial\mathbf{A}}{\partial t}+\mathbf{\Omega}_1\wedge\mathbf{A}\right)\mathbf{X}+\Sigma\mathbf{A}\left(\frac{\partial\mathbf{X}}{\partial t}+\mathbf{\Omega}_1\wedge\mathbf{X}\right)$$

$$=\frac{\partial}{\partial t}(\Sigma\mathbf{AX})+\mathbf{\Omega}_1\wedge(\Sigma\mathbf{AX})-(\Sigma\mathbf{AX})\wedge\mathbf{\Omega}_1$$

$$=\frac{\partial\mathbf{T}}{\partial t}+\mathbf{\Omega}_1\wedge\mathbf{T}-\mathbf{T}\wedge\mathbf{\Omega}_1.$$

This proof, however, only becomes significant when a meaning is given to $\partial\mathbf{T}/\partial t$ beforehand. It is not sufficient to define this simply as $\partial(\Sigma\mathbf{AX})/\partial t$,

because then we could not be sure that the tensor so defined is independent of the particular way in which **T** is expressed as a dyadic. The above analysis does in fact show that $\partial(\Sigma\mathbf{AX})/\partial t$ is in fact dependent only on $d\mathbf{T}/dt$, **T** and $\mathbf{\Omega}_1$, and so is independent of the particular way in which **T** has been expressed as a dyadic. But we are not then sure that this entity which we call $\partial\mathbf{T}/\partial t$ is the tensor formed by the rates of change of the separate components. On the whole this second 'proof,' though terse, is specious, and avoids grappling with the real difficulties, and the longer proof of § 211 must be considered necessary; though the analysis of the present section is an instructive verification.

214. The above are the fundamental theorems relating to the motion of a rigid body, or of a rigid frame of reference. Many important results can be immediately derived from them. To some of these we now proceed.

215. *Accelerations in plane polar co-ordinates.* This is usually considered as a problem of particle dynamics, but it is instructive to derive the acceleration components of a particle specified by plane polar co-ordinates from the formula for rigid body motion.

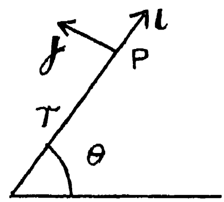

Fig. 48

Let O (Fig. 48) be an origin of polar co-ordinates, P a particle (r, θ). Take a unit vector **i** along OP, a unit vector **j** perpendicular to OP in the direction of θ increasing. Take a unit vector **k** forming with **i**, **j** a positive orthogonal triad. Then the triad **i**, **j**, **k** constitutes a rigid frame of reference in motion about the **k**-axis with angular speed $\dot\theta$, and hence possesses an angular velocity $\mathbf{\Omega}$ given by

$$\mathbf{\Omega}=\mathbf{k}\dot\theta.$$

Accordingly,
$$\frac{d\mathbf{i}}{dt}=\mathbf{\Omega}\wedge\mathbf{i}=\dot\theta(\mathbf{k}\wedge\mathbf{i})=\dot\theta\mathbf{j},$$

$$\frac{d\mathbf{j}}{dt}=\mathbf{\Omega}\wedge\mathbf{j}=\dot\theta(\mathbf{k}\wedge\mathbf{j})=-\dot\theta\mathbf{i}.$$

If **P** denotes the position vector of the particle P with respect to the fixed point O, then

$$\mathbf{P}=r\mathbf{i}.$$

Hence
$$\frac{d\mathbf{P}}{dt}=\dot r\mathbf{i}+r\frac{d\mathbf{i}}{dt}=\dot r\mathbf{i}+r\dot\theta\mathbf{j},$$

and
$$\frac{d^2\mathbf{P}}{dt^2}=\ddot r\mathbf{i}+\dot r\frac{d\mathbf{i}}{dt}+(\dot r\dot\theta+r\ddot\theta)\mathbf{j}+r\dot\theta\frac{d\mathbf{j}}{dt}$$

$$=(\ddot r-r\dot\theta^2)\mathbf{i}+\frac{1}{r}\frac{d}{dt}(r^2\dot\theta)\mathbf{j}.$$

These formulæ for $d\mathbf{P}/dt$ and $d^2\mathbf{P}/dt^2$ give the components of velocity and acceleration along $\mathbf{i}$ and $\mathbf{j}$.

216. *Accelerations in spherical polar co-ordinates.* Let O (Fig. 49) be an origin of spherical polar co-ordinates, P a particle (r, θ, φ). Let Oz be the axis from which θ is measured. Let $\mathbf{i}$ be a unit vector in the direction OP, $\mathbf{j}$ a unit vector perpendicular to OP in the plane zOP with sense in the direction of θ increasing. Let $\mathbf{k}=\mathbf{i}\wedge\mathbf{j}$. The plane zOP has the angular speed $\dot\varphi$ about Oz, and so the plane defined by $\mathbf{i}$ and $\mathbf{j}$ has the angular velocity $\dot\varphi\mathbf{z}$, where $\mathbf{z}$ is a unit vector along Oz. Relative to the rigid body moving with this angular velocity, the rigid body defined by the triad $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ has the angular speed $\dot\theta$ about an axis through P parallel to $\mathbf{k}$. Hence by the theorem of relative angular velocities, the angular velocity of the triad $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ relative to the frame Oxyz is given by

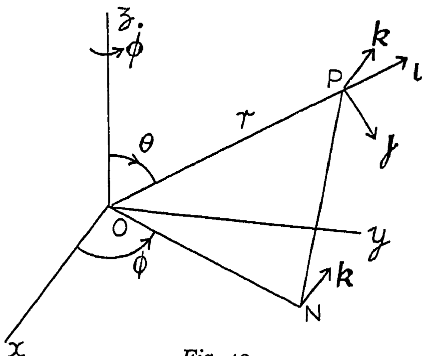

Fig. 49

$$\boldsymbol{\Omega}=\dot\varphi\mathbf{z}+\dot\theta\mathbf{k}.$$

But
$$\mathbf{z}=\mathbf{i}\cos\theta-\mathbf{j}\sin\theta.$$

Hence
$$\boldsymbol{\Omega}=\dot\varphi\cos\theta\mathbf{i}-\dot\varphi\sin\theta\mathbf{j}+\dot\theta\mathbf{k}.$$

It follows that, considering $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as position vectors with respect to O,

$$\frac{d\mathbf{i}}{dt}=\boldsymbol{\Omega}\wedge\mathbf{i}=\dot\theta\mathbf{j}+\dot\varphi\sin\theta\mathbf{k},$$

$$\frac{d\mathbf{j}}{dt}=\boldsymbol{\Omega}\wedge\mathbf{j}=-\dot\theta\mathbf{i}+\dot\varphi\cos\theta\mathbf{k},$$

$$\frac{d\mathbf{k}}{dt}=\boldsymbol{\Omega}\wedge\mathbf{k}=-\dot\varphi\cos\theta\mathbf{j}-\dot\varphi\sin\theta\mathbf{i}.$$

The position vector of P with respect to O is given by

$$\mathbf{P}=r\mathbf{i}.$$

Hence, by direct differentiation,

$$\frac{d\mathbf{P}}{dt}=\dot r\mathbf{i}+r\dot\theta\mathbf{j}+r\dot\varphi\sin\theta\mathbf{k},$$

which gives the component velocities of $\mathbf{P}$ parallel to $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. Again, using the formulæ for $d\mathbf{i}/dt$, etc.,

$$\frac{d^2\mathbf{P}}{dt^2}=\ddot{r}\mathbf{i}+\dot{r}(\dot{\theta}\mathbf{j}+\dot{\varphi}\sin\theta\mathbf{k})+\frac{d(r\dot{\theta})}{dt}\mathbf{j}+r\dot{\theta}(-\dot{\theta}\mathbf{i}+\dot{\varphi}\cos\theta\mathbf{k})$$

$$+\frac{d(r\dot{\varphi}\sin\theta)}{dt}\mathbf{k}+r\dot{\varphi}\sin\theta(-\dot{\varphi}\cos\theta\mathbf{j}-\dot{\varphi}\sin\theta\mathbf{i})$$

$$=\mathbf{i}(\ddot{r}-r\dot{\theta}^2-r\dot{\varphi}^2\sin^2\theta)+\mathbf{j}\left(\dot{r}\dot{\theta}+\frac{d}{dt}(r\dot{\theta})-r\dot{\varphi}^2\sin\theta\cos\theta\right)$$

$$+\mathbf{k}\left(\dot{r}\dot{\varphi}\sin\theta+r\dot{\theta}\dot{\varphi}\cos\theta+\frac{d}{dt}(r\dot{\varphi}\sin\theta)\right)$$

$$=\mathbf{i}(\ddot{r}-r\dot{\theta}^2-r\dot{\varphi}^2\sin^2\theta)+\mathbf{j}\left(\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})-r\dot{\varphi}^2\sin\theta\cos\theta\right)$$

$$+\mathbf{k}\frac{1}{r\sin\theta}\frac{d}{dt}(r^2\sin^2\theta\dot{\varphi}).$$

This gives the component accelerations of the particle P in the directions of r increasing ($\mathbf{i}$), θ increasing ($\mathbf{j}$) and φ increasing ($\mathbf{k}$).

217. *Accelerations of a particle with respect to rotating axes in two dimensions.* Let O (Fig. 50) be a fixed origin; $O\xi$, $O\eta$ a pair of perpendicular axes of co-ordinates rotating instantaneously with angular speed ω relative to a fixed frame of reference xOy in their plane. Let a particle P have co-ordinates ξ, η with respect to the rotating axes, and take unit vectors $\mathbf{i}$, $\mathbf{j}$ along $O\xi$, $O\eta$. Put $\mathbf{k}=\mathbf{i}\wedge\mathbf{j}$. Then the triad $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is a rigid frame of reference in motion with an angular velocity $\mathbf{\Omega}$ about O given by

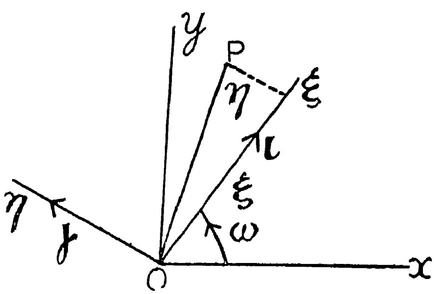

Fig. 50

$$\mathbf{\Omega}=\omega\mathbf{k}.$$

Hence $$\frac{d\mathbf{i}}{dt}=\mathbf{\Omega}\wedge\mathbf{i}=\omega\mathbf{j},\qquad \frac{d\mathbf{j}}{dt}=\mathbf{\Omega}\wedge\mathbf{j}=-\omega\mathbf{i}.$$

Then, since the position vector $\mathbf{P}$ of the particle P is given by

$$\mathbf{P}=\xi\mathbf{i}+\eta\mathbf{j},$$

we have $$\frac{d\mathbf{P}}{dt}=\dot{\xi}\mathbf{i}+\xi\omega\mathbf{j}+\dot{\eta}\mathbf{j}-\eta\omega\mathbf{i}$$

$$=(\dot{\xi}-\eta\omega)\mathbf{i}+(\dot{\eta}+\xi\omega)\mathbf{j}.$$

Again, $$\frac{d^2\mathbf{P}}{dt^2}=(\ddot{\xi}-\dot{\eta}\omega-\eta\dot{\omega})\mathbf{i}+(\dot{\xi}-\eta\omega)\omega\mathbf{j}$$

$$+(\ddot{\eta}+\dot{\xi}\omega+\xi\dot{\omega})\mathbf{j}+(\dot{\eta}+\xi\omega)(-\omega\mathbf{i})$$

$$=(\ddot{\xi}-2\dot{\eta}\omega-\xi\dot{\omega}^2-\eta\omega)\mathbf{i}+(\ddot{\;}+2\dot{\xi}\omega-\eta\omega^2+\xi\dot{\omega})\mathbf{j}.$$

These formulæ give the components of velocity and acceleration of P along the rotating axes.

218. Notice that the terms in the acceleration depending on the apparent velocity relative to the moving frame ($\dot{\xi}$, $\dot{\eta}$), namely

$$-2\dot{\eta}\omega\mathbf{i}+2\dot{\xi}\omega\mathbf{j},$$

may be written

$$2\omega\mathbf{k}\wedge(\dot{\xi}\mathbf{i}+\dot{\eta}\mathbf{j})$$

and thus correspond to a component of acceleration *perpendicular* to the apparent velocity ($\dot{\xi}\mathbf{i}+\dot{\eta}\mathbf{j}$), in a sense making a positive triad with the direction of rotation and the apparent velocity. This is a particular case of a more general theorem which we prove in the next section.

219. Notice that the above method of proof uses essentially *rotating vectors*. We may arrive at the same results by using the theorem of § 207 giving rates of change in terms of apparent rates of change relative to a *rotating frame*. For, let **P** be the position vector of a particle moving in a frame which is itself moving with angular velocity $\mathbf{\Omega}$ relative to a fixed frame. Then, applying the formula

$$\frac{d\mathbf{P}}{dt}=\frac{\partial\mathbf{P}}{\partial t}+\mathbf{\Omega}\wedge\mathbf{P}$$

with **P** replaced by d**P**/dt, we get

$$\frac{d^2\mathbf{P}}{dt^2}=\frac{\partial}{\partial t}\left(\frac{\partial\mathbf{P}}{\partial t}+\mathbf{\Omega}\wedge\mathbf{P}\right)+\mathbf{\Omega}\wedge\left(\frac{\partial\mathbf{P}}{\partial t}+\mathbf{\Omega}\wedge\mathbf{P}\right)$$

$$=\frac{\partial^2\mathbf{P}}{\partial t^2}+2\mathbf{\Omega}\wedge\frac{\partial\mathbf{P}}{\partial t}+\frac{\partial\mathbf{\Omega}}{\partial t}\wedge\mathbf{P}+\mathbf{\Omega}\wedge(\mathbf{\Omega}\wedge\mathbf{P}).$$

If now $\quad \mathbf{P}=\xi\mathbf{i}+\eta\mathbf{j}+\zeta\mathbf{k}, \qquad \mathbf{\Omega}=\omega_1\mathbf{i}+\omega_2\mathbf{j}+\omega_3\mathbf{k},$

then $\quad \dfrac{\partial\mathbf{P}}{\partial t}=\dot{\xi}\mathbf{i}+\dot{\eta}\mathbf{j}+\dot{\zeta}\mathbf{k}, \qquad \dfrac{\partial\mathbf{\Omega}}{\partial t}=\dot{\omega}_1\mathbf{i}+\dot{\omega}_2\mathbf{j}+\dot{\omega}_3\mathbf{k}$

and $\quad \dfrac{\partial^2\mathbf{P}}{\partial t^2}=\ddot{\xi}\mathbf{i}+\ddot{\eta}\mathbf{j}+\ddot{\zeta}\mathbf{k}.$

In applications it frequently occurs that $\mathbf{\Omega}$ is constant and $|\mathbf{\Omega}|$ is small. In this case approximately

$$\frac{d^2\mathbf{P}}{dt^2}=\frac{\partial^2\mathbf{P}}{\partial t^2}+2\mathbf{\Omega}\wedge\frac{\partial\mathbf{P}}{\partial t}.$$

220. *Components of angular velocity with reference to axes from which Eulerian angles are measured.* It is sometimes convenient to define the position of a moving set of axes by means of the angles they make with a fixed set of axes. It is then necessary to express the angular velocity of the moving frame in terms of the rates of change of the angles specifying the positions of the moving axes. We proceed to determine this angular velocity by means of the theorem of relative angular velocities (§ 203).

Let Oxyz be a fixed positive orthogonal triad (Fig. 51). Let a plane

zOz′ make an angle φ with the plane zOx, and let Oz′ make an angle θ with Oz ; let Ox′ be a perpendicular to Oz′ in the plane zOz′ making an angle $\frac{1}{2}\pi+\theta$ with Oz. Let Oy′ make with Oz′ and Ox′ a positive triad.

Again, let Ox″, Oy″ be the positions to which Ox′ and Oy′ are displaced when the triad Ox′y′z′ is rotated about Oz′ through an angle ψ in the positive sense.

If the rates of change of the angles θ, φ, ψ are specified, the triads Ox′y′z′ and Ox″y″z″ (where Oz″ is along Oz′) will possess definite angular velocities with respect to the triad Oxyz. Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{x}'$, ... denote unit vectors along the corresponding axes. Then the motion of the triad Ox′y′z′ consists of the angular velocity $\dot{\varphi}\mathbf{z}$ of the plane zOz′ relative to Oxyz, together with the angular velocity $\dot{\theta}\mathbf{y}'$ in this plane. Hence the angular velocity $\boldsymbol{\Omega}'$ of Ox′y′z′ relative to Oxyz is given by

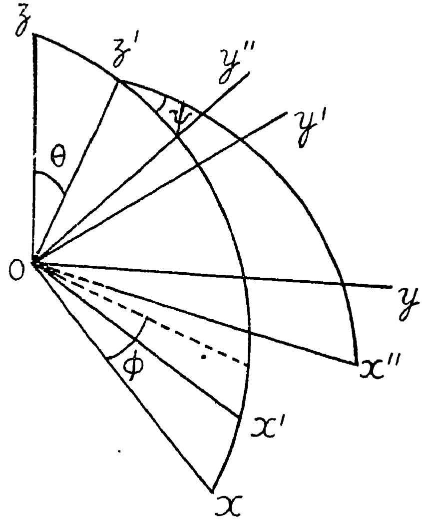

Fig. 51

$$\boldsymbol{\Omega}'=\dot{\varphi}\mathbf{z}+\dot{\theta}\mathbf{y}'.$$

The angular velocity of Ox″y″z″ is the angular velocity $\boldsymbol{\Omega}'$ of Ox′y′z′ together with the angular velocity $\dot{\psi}\mathbf{z}'$ relative to Ox′y′z′. Hence the angular velocity $\boldsymbol{\Omega}''$ of Ox″y″z″ relative to Oxyz is given by

$$\boldsymbol{\Omega}''=\dot{\varphi}\mathbf{z}+\dot{\theta}\mathbf{y}'+\dot{\psi}\mathbf{z}'.$$

The angular velocities $\boldsymbol{\Omega}'$ and $\boldsymbol{\Omega}''$ can now be expressed in terms of the vectors $\mathbf{x}'$, $\mathbf{y}'$, $\mathbf{z}'$ or $\mathbf{x}''$, $\mathbf{y}''$, $\mathbf{z}''$ respectively.

Thus we have

$$\mathbf{z}=\mathbf{z}'\cos\theta-\mathbf{x}'\sin\theta,$$

whence
$$\boldsymbol{\Omega}'=-\dot{\varphi}\sin\theta\mathbf{x}'+\dot{\theta}\mathbf{y}'+\dot{\varphi}\cos\theta\mathbf{z}'.$$

The coefficients of $\mathbf{x}'$, $\mathbf{y}'$, $\mathbf{z}'$ are called *the spins of the axes* Ox′y′z′ *about themselves*.

Again, we have

$$\mathbf{x}'=\mathbf{x}''\cos\psi-\mathbf{y}''\sin\psi,$$
$$\mathbf{y}'=\mathbf{x}''\sin\psi+\mathbf{y}''\cos\psi,$$
$$\mathbf{z}''=\mathbf{z}'.$$

Hence

$$\begin{aligned}\boldsymbol{\Omega}''&=\dot{\varphi}[\cos\theta\mathbf{z}''-\sin\theta(\cos\psi\mathbf{x}''-\sin\psi\mathbf{y}'')]+\dot{\theta}[\sin\psi\mathbf{x}''+\cos\psi\mathbf{y}'']+\dot{\psi}\mathbf{z}'',\\&=(\dot{\theta}\sin\psi-\dot{\varphi}\sin\theta\cos\psi)\mathbf{x}''+(\dot{\theta}\cos\psi+\dot{\varphi}\sin\theta\sin\psi)\mathbf{y}''\\&\quad+(\dot{\varphi}\cos\theta+\dot{\psi})\mathbf{z}''.\end{aligned}$$

The coefficients of $\mathbf{x}''$, $\mathbf{y}''$, $\mathbf{z}''$ give the spins of the axes Ox″y″z″ about themselves.

Example. Show that the spins of Ox″y″z″ about Oxyz are given by $\mathbf{\Omega}''=(-\dot\theta\sin\varphi+\dot\psi\sin\theta\cos\varphi)\mathbf{x}+(\dot\theta\cos\varphi+\dot\psi\sin\theta\sin\varphi)\mathbf{y}+(\dot\varphi+\dot\psi\cos\theta)\mathbf{z}$.

The essence of the foregoing derivations of these well-known formulæ s that we write down the *given* angular velocities, in terms of $\dot\theta$, $\dot\varphi$, $\dot\psi$, *directly*; we thus avoid all awkward mental steps involved in *resolving* angular velocities, and resolve only unit vectors.

It may be mentioned that the free use of moving vectors often obviates the use of special systems of co-ordinates (such as the Eulerian set θ, φ, ψ used above) in the solution of dynamical problems. Thus in the present work, the Eulerian formulæ just found will prove to be of minor importance.

221. *The curvature and torsion of a twisted curve.* Let C be a given twisted curve. Choose a sense of direction along C, and let s be the arc-length along C measured from some zero to a variable point P. Let $\mathbf{T}$ be a unit vector along the tangent at P to C. Then $d\mathbf{P}=\mathbf{T}ds$, whence

$$\mathbf{T}=\frac{d\mathbf{P}}{ds}.$$

Now let $\mathbf{N}$ be a unit vector in the direction of $d\mathbf{T}/ds$. Then, since $\mathbf{T}^2=1$, we have

$$\mathbf{T}.\frac{d\mathbf{T}}{ds}=0$$

or
$$\mathbf{T}.\mathbf{N}=0.$$

Thus $\mathbf{N}$ is perpendicular to $\mathbf{T}$. Construct the unit vector $\mathbf{B}$ making a positive orthogonal triad with $\mathbf{T}$ and $\mathbf{N}$, so that

$$\mathbf{B}=\mathbf{T}\wedge\mathbf{N}.$$

The direction of $\mathbf{N}$ is said to be that of the *principal normal* to C at P; the direction of $\mathbf{B}$ is said to be that of the *binormal*.

As P moves along C, the orthogonal triad $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ will possess a certain angular velocity $\mathbf{\Omega}$. If we choose to measure the time by the variable s, i.e. if P moves with unit speed along C, the value of $\mathbf{\Omega}$ must satisfy

$$\frac{d\mathbf{T}}{ds}=\mathbf{\Omega}\wedge\mathbf{T}.$$

Multiply each side of this equality vectorially by $\mathbf{N}$. Then, since $\mathbf{N}$ and $d\mathbf{T}/ds$ are parallel, we have

$$(\mathbf{\Omega}\wedge\mathbf{T})\wedge\mathbf{N}=0,$$

which yields
$$\mathbf{T}(\mathbf{\Omega}.\mathbf{N})=0.$$

But $\mathbf{T}$, being a unit vector, is non-zero. Hence $\mathbf{\Omega}.\mathbf{N}=0$. Hence $\mathbf{\Omega}$ has no component along $\mathbf{N}$. Put then

$$\mathbf{\Omega}=\frac{\mathbf{T}}{\sigma}+\frac{\mathbf{B}}{\rho},$$

where σ and ρ are scalars. The number $1/\rho$ is the rate of turning of the triad **T, N, B** about the binormal, **B**, and $1/\sigma$ is the rate of turning of the triad about the tangent **T**.

222. We now have

$$\frac{d\mathbf{T}}{ds}=\mathbf{\Omega}\wedge\mathbf{T}=\frac{\mathbf{B}\wedge\mathbf{T}}{\rho}=\frac{\mathbf{N}}{\rho}$$

$$\frac{d\mathbf{B}}{ds}=\mathbf{\Omega}\wedge\mathbf{B}=\frac{\mathbf{T}\wedge\mathbf{B}}{\sigma}=-\frac{\mathbf{N}}{\sigma}$$

and
$$\frac{d\mathbf{N}}{ds}=\mathbf{\Omega}\wedge\mathbf{N}=\frac{\mathbf{T}\wedge\mathbf{N}}{\sigma}+\frac{\mathbf{B}\wedge\mathbf{N}}{\rho}=\frac{\mathbf{B}}{\sigma}-\frac{\mathbf{T}}{\rho}.$$

These are known as Frenet's formulæ. We proceed to find geometrical meanings for ρ and σ.

223. Since **N** is drawn parallel to $d\mathbf{T}/ds$, from the first of Frenet's formulæ it follows that ρ is essentially positive. Further, since **N** is a unit vector,

$$\frac{1}{\rho}=\left|\frac{d\mathbf{T}}{ds}\right|\sim\frac{|\delta\mathbf{T}|}{\delta s}.$$

But since **T**, $\mathbf{T}+\delta\mathbf{T}$ are unit vectors along two neighbouring tangents, $|\delta\mathbf{T}|$ is the angle ε between these tangents, and thus

$$\frac{1}{\rho}\sim\frac{\varepsilon}{\delta s}.$$

Thus
$$\rho=\lim\frac{\delta s}{\varepsilon},$$

and hence, according to the usual definition of curvature, ρ is the *radius of curvature* at P. The number $1/\rho$ is called the *curvature*.

Again, we have from the second of Frenet's formulæ,

$$\delta\mathbf{B}\sim-\frac{\mathbf{N}\delta s}{\sigma},$$

so that
$$\frac{1}{|\sigma|}=\frac{|\delta\mathbf{B}|}{\delta s}.$$

But $|\delta\mathbf{B}|$ is the angle η between neighbouring binormals, and so, with a suitable sign convention,

$$\frac{1}{\sigma}\sim\frac{\eta}{\delta s}.$$

Hence
$$\sigma=\lim\frac{\delta s}{\eta},$$

and thus σ is the *radius of torsion* at P of the curve C. The convention fixing the sign of η must be such that

$$\mathbf{\Omega}.\mathbf{T}=\frac{1}{\sigma}.$$

Hence σ is positive or negative according as the triad rotates in the positive or negative direction about the tangent **T**. If we reverse the direction of motion of P along the curve C, we reverse the signs of $\mathbf{\Omega}$ and of **T**, and the sign of σ is unaltered. Thus the sign of σ is a characteristic of the curve itself. The number $1/\sigma$ is called the torsion.

224. By means of Frenet's formulæ the vector distance of a point P of C from a neighbouring point P_0 can be expanded in powers of s, the length of the arc P_0P. For we have

$$\mathbf{P}=\mathbf{P}_0+s\left(\frac{d\mathbf{P}}{ds}\right)_0+\frac{s^2}{2!}\left(\frac{d^2\mathbf{P}}{ds^2}\right)_0+\frac{s^3}{3!}\left(\frac{d^3\mathbf{P}}{ds^3}\right)_0+\dots$$

or

$$\mathbf{P}-\mathbf{P}_0=s\mathbf{T}_0+\frac{s^2}{2!}\left(\frac{d\mathbf{T}}{ds}\right)_0+\frac{s^3}{3!}\left(\frac{d^2\mathbf{T}}{ds^2}\right)_0+\dots$$

$$=s\mathbf{T}_0+\frac{s^2}{2!}\frac{\mathbf{N}_0}{\rho_0}+\frac{s^3}{3!}\frac{d}{ds}\left(\frac{\mathbf{N}}{\rho}\right)_0+\dots.$$

$$=s\mathbf{T}_0+\frac{s^2}{2!}\frac{\mathbf{N}_0}{\rho_0}+\frac{s^3}{3!}\left[\frac{1}{\rho_0}\left(\frac{\mathbf{B}_0}{\sigma_0}-\frac{\mathbf{T}_0}{\rho_0}\right)+\mathbf{N}_0\frac{d}{ds}\left(\frac{1}{\rho}\right)_0\right]+\dots.$$

Collecting terms according to the vectors concerned we have

$$\mathbf{P}-\mathbf{P}_0=\mathbf{T}_0\left(s-\frac{s^3}{6\rho_0{}^2}+\dots\right)+\mathbf{N}_0\left(\frac{s^2}{2\rho_0}+\frac{s^3}{6}\frac{d}{ds}\left(\frac{1}{\rho}\right)_0\right)+\mathbf{B}_0\left(\frac{s^3}{6\rho_0\sigma_0}+\dots\right).$$

This formula shows that $\mathbf{B}_0$ is the direction in which $\mathbf{P}-\mathbf{P}_0$ has its smallest component for small s, namely a component $O(s^3)$. The plane normal to $\mathbf{B}_0$, i.e. the plane of $\mathbf{T}_0$ and $\mathbf{N}_0$ is accordingly called the *osculating plane* of the curve C at P_0. Further, the plane curve which is the projection of C on the osculating plane at P_0 clearly has $\mathbf{T}_0$ and $\mathbf{N}_0$ for its tangent and normal at P_0, and ρ_0 for its radius of curvature there.

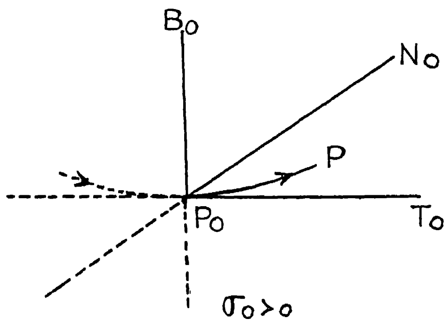

Fig. 52

Again, for small $s>0$, the $\mathbf{B}_0$ component of $\mathbf{P}-\mathbf{P}_0$ is positive or negative according as $\sigma_0 \gtrless 0$. The curve C thus *rises* through the plane of $\mathbf{T}_0$ and $\mathbf{N}_0$ (the osculating plane) from the negative to the positive side in the direction of the vector $\mathbf{B}_0$, which it must be remembered makes a positive triad with $\mathbf{T}_0$ and $\mathbf{N}_0$, provided $\sigma_0>0$; and conversely if $\sigma_0<0$ (Fig. 52).

225. *Values of curvature and torsion.* Differentiating the formula

$$\frac{d\mathbf{T}}{ds}=\frac{\mathbf{N}}{\rho}$$

with respect to s, we have

$$\frac{d^2\mathbf{T}}{ds^2}=\frac{1}{\rho}\left(\frac{\mathbf{B}}{\sigma}-\frac{\mathbf{T}}{\rho}\right)+\mathbf{N}\frac{d}{ds}\left(\frac{1}{\rho}\right).$$

Multiply these two together vectorially. We get

$$\frac{d\mathbf{T}}{ds}\wedge\frac{d^2\mathbf{T}}{ds^2}=\frac{1}{\rho^2}\left(\frac{\mathbf{T}}{\sigma}+\frac{\mathbf{B}}{\rho}\right).$$

Taking the modulus of the first formula of this section,

$$\frac{1}{\rho}=\left|\frac{d\mathbf{T}}{ds}\right|=\left|\frac{d^2\mathbf{P}}{ds^2}\right|,$$

and multiplying the third scalarly by $\mathbf{T}$,

$$\frac{1}{\sigma\rho^2}=\mathbf{T}.\frac{d\mathbf{T}}{ds}\wedge\frac{d^2\mathbf{T}}{ds^2},$$

or
$$\frac{1}{\sigma}=\frac{\frac{d\mathbf{P}}{ds}.\frac{d^2\mathbf{P}}{ds^2}\wedge\frac{d^3\mathbf{P}}{ds^3}}{\left(\frac{d^2\mathbf{P}}{ds^2}\right)^2}=\rho^2\left(\frac{d\mathbf{P}}{ds}.\frac{d^2\mathbf{P}}{ds^2}\wedge\frac{d^3\mathbf{P}}{ds^3}\right).$$

Again
$$\mathbf{T}\wedge\frac{d\mathbf{T}}{ds}=\frac{\mathbf{B}}{\rho},$$

whence
$$\frac{1}{\rho}=\left|\frac{d\mathbf{P}}{ds}\wedge\frac{d^2\mathbf{P}}{ds^2}\right|.$$

These formulæ determine ρ and σ when $\mathbf{P}$ is given as a function of the arc s.

226. P *given in terms of a parameter.* If $\mathbf{P}$ is given as a function not of the arc s but of some parameter λ, the above formulæ need modification. Using primes to denote differential coefficients with respect to the parameter λ, we have

$$\mathbf{P}'=\frac{d\mathbf{P}}{ds}s'=\mathbf{T}s',$$

$$\mathbf{P}''=\frac{d\mathbf{T}}{ds}s'^2+\mathbf{T}s''.$$

Hence
$$\mathbf{P}'\wedge\mathbf{P}''=s'^3\left(\mathbf{T}\wedge\frac{d\mathbf{T}}{ds}\right).$$

Also
$$\mathbf{P}'''=\frac{d^2\mathbf{T}}{ds^2}s'^3+\frac{3d\mathbf{T}}{ds}s's''+\mathbf{T}s''',$$

whence
$$\mathbf{P}'\wedge\mathbf{P}''.\mathbf{P}'''=s'^6\left(\mathbf{T}\wedge\frac{d\mathbf{T}}{ds}.\frac{d^2\mathbf{T}}{ds^2}\right).$$

It follows from § 225 that

$$\frac{1}{\rho}=\frac{|\mathbf{P}'\wedge\mathbf{P}''|}{s'^3}$$

and

$$\frac{1}{\sigma}=\frac{\rho^2}{s'^6}\mathbf{P}'\wedge\mathbf{P}''.\mathbf{P}'''.$$

227. *Osculating sphere to a twisted curve.* The foregoing formulæ suffice to determine many of the properties of a twisted curve by vector methods. As an example, we propose to determine the osculating sphere to a twisted curve.

We have seen that a tangent line, meeting the curve in a point P, is a line such that if P′ is a neighbouring point on the curve, N the foot of the perpendicular from P′ to the tangent, then P′N is of order PP'^2, i.e. $P'N/PP'^2$ has a finite limit as $P'\to P$. Similarly an osculating plane, meeting the curve in a point P, is a plane such that if N is the foot of the perpendicular from P′, then P′N is of the order PP'^3. We may define similarly higher orders of contact.

A sphere may be described to satisfy four conditions; a plane, three. We may therefore expect it to be possible to construct a sphere having an order of contact with a twisted curve one higher than that of an osculating plane. Such a sphere is called an osculating sphere. Let Q be the centre of such a sphere, and let P′ be a neighbouring point on the curve. Then the length QP′ must differ from the length QP by a length of the highest possible order in PP′.

The position vector of Q may be specified as

$$\mathbf{Q}=\mathbf{P}+\xi\mathbf{T}+\eta\mathbf{N}+\zeta\mathbf{B},$$

where ξ, η, ζ are functions of the position of P. Then

$$|\mathbf{P}'-\mathbf{Q}|=|(\mathbf{P}'-\mathbf{P})+(\mathbf{P}-\mathbf{Q})|$$

$$=\left|-(\xi\mathbf{T}+\eta\mathbf{N}+\zeta\mathbf{B})+\mathbf{T}\left(s-\frac{s^3}{6\rho^2}\right)+\mathbf{N}\left(\frac{s^2}{2\rho}+\frac{s^3}{6}\frac{d}{ds}\left(\frac{1}{\rho}\right)\right)+\mathbf{B}\frac{s^3}{6\rho\sigma}+\dots\right|$$

The values of ξ, η, ζ must be accordingly chosen so that

$$\left[\xi-\left(s-\frac{s^3}{6\rho^2}+\dots\right)\right]^2+\left[\eta-\left(\frac{s^2}{2\rho}-\frac{s^3\rho'}{6\rho^2}+\dots\right)\right]^2+\left[\zeta-\frac{s^3}{6\rho\sigma}+\dots\right]^2$$

has an expansion in powers of s of the form

$$\xi^2+\eta^2+\zeta^2+As^n,$$

where n is as large an integer as possible. This allows us to choose ξ, η, ζ so that the coefficients of s, s^2, s^3 are all zero. We find:

Coefficient of s: $\xi=0$;

Coefficient of s^2: $1-\frac{\eta}{\rho}=0$;

Coefficient of s^3: $$\frac{\eta\rho'}{3\rho^2}-\frac{\zeta}{3\rho\sigma}=0.$$

Hence $$\xi=0, \qquad \eta=\rho, \qquad \zeta=\sigma\rho'.$$

The centre of the osculating sphere is thus

$$\mathbf{Q}=\mathbf{P}+\rho\mathbf{N}+\sigma\rho'\mathbf{B},$$

and its radius is $$[\rho^2+(\sigma\rho')^2]^{\frac{1}{2}}.$$

228. As a further example we will determine the sphere touching four neighbouring osculating planes to a given twisted curve (more strictly, the limit of such a sphere). If R is the radius of the sphere, the distance of its centre from the osculating plane at P must be a vector **RB**, and the position **Q** of its centre may accordingly be written

$$\mathbf{Q}=\mathbf{P}+\xi\mathbf{T}+\eta\mathbf{N}+\mathrm{R}\mathbf{B}.$$

The sphere corresponding to a neighbouring position P′ of P must then possess the same centre Q and the same radius R. Consequently we obtain conditions to determine ξ and η by differentiating the foregoing relation keeping **Q** and **R** constant. We get

$$0=\frac{d\mathbf{P}}{ds}+\frac{d\xi}{ds}\mathbf{T}+\xi\frac{d\mathbf{T}}{ds}+\frac{d\eta}{ds}\mathbf{N}+\eta\frac{d\mathbf{N}}{ds}+\mathrm{R}\frac{d\mathbf{B}}{ds},$$

or, using Frenet's formula,

$$0=\mathbf{T}\left(1+\frac{d\xi}{ds}\right)+\xi\frac{\mathbf{N}}{\rho}+\frac{d\eta}{ds}\mathbf{N}+\eta\left(\frac{\mathbf{B}}{\sigma}-\frac{\mathbf{T}}{\rho}\right)+\mathrm{R}\left(-\frac{\mathbf{N}}{\sigma}\right).$$

The right-hand side is a linear function of the three linearly independent vectors **T, N, B,** and the coefficient of each must thus be zero.

Hence
$$\mathbf{T}:\quad 1+\frac{d\xi}{ds}-\frac{\eta}{\rho}=0$$

$$\mathbf{N}:\quad \frac{\xi}{\rho}+\frac{d\eta}{ds}-\frac{\mathrm{R}}{\sigma}=0$$

$$\mathbf{B}:\quad \frac{\eta}{\sigma}=0.$$

These give in turn

$$\eta=0, \qquad \xi=\mathrm{R}\frac{\rho}{\sigma}, \qquad \frac{d}{ds}\left(\mathrm{R}\frac{\rho}{\sigma}\right)=-1.$$

But in the differentiation with respect to s, R is to be treated as constant. Hence

$$\frac{1}{\mathrm{R}}=-\frac{d}{ds}\left(\frac{\rho}{\sigma}\right).$$

An alternative derivation of the above result is of some interest, and may be considered as possessing greater rigour. Fix a point P_0 on the given twisted curve, and choose a point Q_0, a function of P_0. Let P be a current point on the curve. Put

$$\mathbf{Q}_0-\mathbf{P}_0=\xi_0\mathbf{T}_0+\eta_0\mathbf{N}_0+R_0\mathbf{B}_0,$$
$$\mathbf{Q}_0-\mathbf{P}=\xi\mathbf{T}+\eta\mathbf{N}+R\mathbf{B},$$

where in the first equation $\mathbf{T}_0$, $\mathbf{N}_0$, $\mathbf{B}_0$ refer to P_0 and in the second equation $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ refer to P. Then (ξ, η, R) are the co-ordinates of the fixed point Q_0 with regard to the variable triad defined by the principal directions at P. If s_0, s denote the arcs to P_0 and P, ξ, η, R are functions of s and s_0 reducing to ξ_0, η_0, R_0 when s reduces to s_0. We now seek to choose the function $R=R(s, s_0)$ so that it is as 'stationary' as possible at $s=s_0$, i.e. so that as many as possible of the derivatives $(dR/ds)_{s=s_0}$, $(d^2R/ds^2)_{s=s_0}$, ... vanish. Since

$$R=(\mathbf{Q}_0-\mathbf{P}).\mathbf{B},$$

this requires that the successive derivatives

$$\frac{d}{ds}[(\mathbf{Q}_0-\mathbf{P}).\mathbf{B}], \qquad \frac{d^2}{ds^2}[(\mathbf{Q}_0-\mathbf{P}).\mathbf{B}], \ldots$$

shall vanish as far as possible. Differentiating out (keeping $\mathbf{Q}_0$ constant), and omitting the suffix o after differentiation, we have

$$(\mathbf{Q}-\mathbf{P}).\frac{d\mathbf{B}}{ds}-\frac{d\mathbf{P}}{ds}.\mathbf{B}=0$$

$$(\mathbf{Q}-\mathbf{P}).\frac{d^2\mathbf{B}}{ds^2}-2\frac{d\mathbf{P}}{ds}.\frac{d\mathbf{B}}{ds}-\frac{d^2\mathbf{P}}{ds^2}.\mathbf{B}=0,$$

and thirdly $$(\mathbf{Q}-\mathbf{P}).\frac{d^3\mathbf{B}}{ds^3}-3\frac{d\mathbf{P}}{ds}.\frac{d^2\mathbf{B}}{ds^2}-3\frac{d^2\mathbf{P}}{ds^2}.\frac{d\mathbf{B}}{ds}-\frac{d^3\mathbf{P}}{ds^3}.\mathbf{B}=0.$$

These three equations determine the three components of $\mathbf{Q}-\mathbf{P}$ along the three vectors $d\mathbf{B}/ds$, $d^2\mathbf{B}/ds^2$, $d^3\mathbf{B}/ds^3$. Substitution of the expressions for the derivatives of $\mathbf{P}$ and $\mathbf{B}$ as functions of $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ in these equations determines ξ, η and R. The results will be found to coincide with the earlier solution. It is remarkable that the earlier, more purely vector solution determines ξ, η, R by a single differentiation, whilst the second method requires three successive differentiations.

The student should attempt to work out by vector methods the standard examples given in treatises on elementary differential geometry.

As a still further example, we explore by vector methods the properties of the helix. We choose this because the definition of the helix is essentially a kinematic one, describing the motion of a particle in time.

229. *Helix. Definition.* Let a point N describe a circle of radius a with uniform angular speed ω. Let P be a point in the normal at N to

the plane of the circle, and let it move along the moving normal with constant velocity u. Then P is said to describe a helix.

The normal to the plane through the centre O of the given circle is called the axis of the helix, and the ratio of the velocity component of P parallel to the axis to the speed of N, namely the ratio $u/a\omega$ is called the pitch of the helix and is denoted by $\tan\alpha$; α is called the angle of the helix.

Position vector of P. Let $\mathbf{z}$ be a unit vector along the axis, $\mathbf{i}$ a unit vector along ON. Then if $\mathbf{r}$ is the position vector of P,

$$\frac{d\mathbf{r}}{dt}=a\frac{d\mathbf{i}}{dt}+u\mathbf{z}, \qquad (\mathbf{z}.\mathbf{i}=0)$$

where, by the fundamental angular velocity formula,

$$\frac{d\mathbf{i}}{dt}=\omega\mathbf{z}\wedge\mathbf{i}.$$

Hence $$\mathbf{r}=a\mathbf{i}+ut\mathbf{z},$$

with a suitable choice of origins of t and $\mathbf{r}$, t being the time. Thus

$$\mathbf{r}=a\mathbf{i}+a\omega\tan\alpha\; t\mathbf{z}.$$

Arc length. We have

$$\frac{d\mathbf{r}}{ds}=a\frac{d\mathbf{i}}{ds}+a\omega\tan\alpha\frac{dt}{ds}\mathbf{z}=[a\omega(\mathbf{z}\wedge\mathbf{i})+a\omega\tan\alpha\mathbf{z}]\frac{dt}{ds}.$$

But $$|d\mathbf{r}/ds|=1.$$

Hence $$\frac{ds}{dt}=a\omega(1+\tan^2\alpha)^{\frac{1}{2}}=a\omega\sec\alpha,$$

whence $$s=a\omega\sec\alpha\; t.$$

Hence the equation of the helix may be written

$$\mathbf{r}=a\mathbf{i}+s\sin\alpha\;\mathbf{z}, \qquad (1)$$

where $$\frac{d\mathbf{i}}{ds}=\frac{dt}{ds}\omega(\mathbf{z}\wedge\mathbf{i})=\frac{\cos\alpha}{a}(\mathbf{z}\wedge\mathbf{i}). \qquad (2)$$

Equations (1) and (2) describe the helix fully in terms of the arc s as parameter. The various properties of the helix now follow from Frenet's formulæ. For the unit vector along the tangent we have

$$\mathbf{T}=\frac{d\mathbf{r}}{ds}=\cos\alpha(\mathbf{z}\wedge\mathbf{i})+\sin\alpha\;\mathbf{z}$$

and, for the unit vector along the normal,

$$\frac{\mathbf{N}}{\rho}=\frac{d\mathbf{T}}{ds}=\frac{\cos^2\alpha}{a}\mathbf{z}\wedge(\mathbf{z}\wedge\mathbf{i})=-\frac{\cos^2\alpha}{a}\mathbf{i}$$

whence, since $\rho>0$, $\quad \mathbf{N}=-\mathbf{i}, \qquad \rho=a\sec^2\alpha.$

For the binormal

$$\mathbf{B}=\mathbf{T}\wedge\mathbf{N}=\cos\alpha(\mathbf{z}\wedge\mathbf{i})\wedge(-\mathbf{i})-\sin\alpha(\mathbf{z}\wedge\mathbf{i})$$
$$=\cos\alpha\,\mathbf{z}-\sin\alpha\,(\mathbf{z}\wedge\mathbf{i}).$$

For the torsion

$$-\frac{\mathbf{N}}{\sigma}=\frac{d\mathbf{B}}{ds}=-\frac{\sin\alpha\cos\alpha}{a}\mathbf{z}\wedge(\mathbf{z}\wedge\mathbf{i})=\frac{\sin\alpha\cos\alpha}{a}\mathbf{i}$$

whence $\sigma=+a\sec\alpha\operatorname{cosec}\alpha.$

It follows that for given $a>0$ we can construct two essentially distinct helices with torsions of opposite signs, corresponding to $\alpha>0$ and $\alpha<0$.

The student should note the ease with which vector methods fix the signs of quantities and the directions of lines, without appeals to diagrams.

230. *The acceleration of a particle in motion along a twisted curve.* If t denotes the time, $\mathbf{P}$ the vector position of the particle, we have

$$\frac{d\mathbf{P}}{dt}=\frac{d\mathbf{P}}{ds}\,\frac{ds}{dt}=v\mathbf{T},$$

where v is the speed of the particle, $\mathbf{T}$ a unit vector along the tangent. Hence

$$\frac{d^2\mathbf{P}}{dt^2}=\frac{dv}{dt}\mathbf{T}+v\frac{d\mathbf{T}}{ds}\,\frac{ds}{dt}$$
$$=\frac{dv}{dt}\mathbf{T}+\frac{v^2}{\rho}\mathbf{N},$$

by the first of Frenet's formulæ.

It follows that the acceleration of a particle in motion in three dimensions in any manner may be resolved into a component dv/dt along the tangent to the path and a component v^2/ρ along the principal normal, $1/\rho$ being the curvature of the path. The component of acceleration along the binormal is zero, and the acceleration lies wholly in the osculating plane of the path.

Example. Evaluate $d^3\mathbf{P}/dt^3$.

231. *Angular velocity of a moving triad in terms of the motion of one of its members.* If a unit vector $\mathbf{i}$ is a function of t, the representation of $\mathbf{i}$ with respect to an origin O will move in a definite way. We cannot, however, speak unambiguously about its angular velocity, for an origin O and a radius vector $\mathbf{i}$ do not specify a rigid body. The motion of the representative point $\mathbf{i}$ can in fact be specified by any one of a class of angular velocities about appropriate axes. Amongst these different axes, one is of pre-eminent interest, namely that which is perpendicular to $\mathbf{i}$.

Let us therefore attempt to find an angular velocity $\boldsymbol{\Omega}$ with the two properties: (i) that $\boldsymbol{\Omega}$ is perpendicular to $\mathbf{i}$; (ii) that if $\mathbf{i}$ is supposed embedded in a rigid body moving with angular velocity $\boldsymbol{\Omega}$, then the motion of the rigid body reproduces the actual motion of $\mathbf{i}$.

These conditions require

$$\mathbf{\Omega}.\mathbf{i}=0, \qquad \frac{d\mathbf{i}}{dt}=\mathbf{\Omega}\wedge\mathbf{i}.$$

Hence
$$\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}=\mathbf{i}\wedge(\mathbf{\Omega}\wedge\mathbf{i})=\mathbf{\Omega}.$$

This determines $\mathbf{\Omega}$ in terms of $\mathbf{i}$ and its prescribed motion.

The motion of $\mathbf{i}$ is now also reproduced by the motion of any rigid body possessed of an angular velocity $\mathbf{\Omega}'$ defined by

$$\mathbf{\Omega}'=n\mathbf{i}+\mathbf{i}\wedge\frac{d\mathbf{i}}{dt},$$

where n is an arbitrary scalar. This is the most general form of angular velocity of a rigid body of which $\mathbf{i}$ can form a part. The scalar n is called the *spin* of the rigid body about the axis $\mathbf{i}$. This formula is very useful in dealing with the motion of a *top*, namely a solid of revolution whose axis is specified by a unit vector $\mathbf{i}$.

The motion of $\mathbf{i}$ being given, of all the lines perpendicular to $\mathbf{i}$ there is one which is parallel to $d\mathbf{i}/dt$. If we construct a unit vector $\mathbf{j}$ parallel to $d\mathbf{i}/dt$, then the triad $\mathbf{i}$, $\mathbf{j}$, $\mathbf{i}\wedge\mathbf{j}$ defines a positive unit orthogonal triad, and so specifies a rigid body; for, since $\mathbf{i}^2=1$, we have $\mathbf{i}.\frac{d\mathbf{i}}{dt}=0$, or $\mathbf{i}.\mathbf{j}=0$. The question arises: what is the angular velocity $\mathbf{\Omega}$ of this rigid body? i.e. what is the value of the spin n of this body about $\mathbf{i}$?

Since $\mathbf{j}$ is parallel to $d\mathbf{i}/dt$, we have

$$\mathbf{j}\wedge\frac{d\mathbf{i}}{dt}=0$$

whence, differentiating,
$$\frac{d\mathbf{j}}{dt}\wedge\frac{d\mathbf{i}}{dt}+\mathbf{j}\wedge\frac{d^2\mathbf{i}}{dt^2}=0.$$

But
$$\frac{d\mathbf{i}}{dt}=\mathbf{\Omega}\wedge\mathbf{i}, \qquad \frac{d\mathbf{j}}{dt}=\mathbf{\Omega}\wedge\mathbf{j},$$

and so
$$\frac{d\mathbf{j}}{dt}\wedge\frac{d\mathbf{i}}{dt}=(\mathbf{\Omega}\wedge\mathbf{j})\wedge\frac{d\mathbf{i}}{dt}=-\mathbf{\Omega}\left(\mathbf{j}.\frac{d\mathbf{i}}{dt}\right).$$

Hence
$$\mathbf{\Omega}=\frac{\mathbf{j}\wedge\dfrac{d^2\mathbf{i}}{dt^2}}{\mathbf{j}.\dfrac{d\mathbf{i}}{dt}},$$

or, since $\mathbf{j}$ is parallel to $\frac{d\mathbf{i}}{dt}$,

$$\mathbf{\Omega}=\frac{\dfrac{d\mathbf{i}}{dt}\wedge\dfrac{d^2\mathbf{i}}{dt^2}}{\left(\dfrac{d\mathbf{i}}{dt}\right)^2}.$$

This is the expression of $\mathbf{\Omega}$ in terms of the specified motion of $\mathbf{i}$.

The spin n of the triad about **i** is then given by

$$n=\boldsymbol{\Omega}.\mathbf{i}=\left(\mathbf{i}.\frac{d\mathbf{i}}{dt}\wedge\frac{d^2\mathbf{i}}{dt^2}\right)\Big/\left(\frac{d\mathbf{i}}{dt}\right)^2.$$

Example. Show that the foregoing formula for $\boldsymbol{\Omega}$ is equivalent to

$$\boldsymbol{\Omega}=\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}+\mathbf{i}\frac{\mathbf{i}.\frac{d\mathbf{i}}{dt}\wedge\frac{d^2\mathbf{i}}{dt^2}}{\left(\frac{d\mathbf{i}}{dt}\right)^2}.$$

The relation of this formula to the analysis of the motion of the triad formed by the tangent, normal and binormal to a twisted curve, should be investigated by the reader.

Example. A moving plane has a unit vector **i** for its normal, **i** being a given function of t. Show that the plane may be considered to be turning instantaneously about an axis in itself in the direction of the unit vector

$$\left(\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}\right)\Big/\left|\frac{d\mathbf{i}}{dt}\right|.$$

Example. Two unit vectors **i, j** (not necessarily perpendicular) define the direction of the normal to a variable plane, **i** and **j** being given functions of t. Show that the angular velocity of this plane, apart from an arbitrary spin about its normal, may be written

$$\left(\mathbf{i}\frac{d\mathbf{j}}{dt}-\mathbf{j}\frac{d\mathbf{i}}{dt}\right).(\mathbf{i}\wedge\mathbf{j})/|\mathbf{i}\wedge\mathbf{j}|^2.$$

(*Note.* The expression $\left(\mathbf{i}\frac{d\mathbf{j}}{dt}-\mathbf{j}\frac{d\mathbf{i}}{dt}\right)$ is a dyadic.)

232. *Hooke's joint.* This is a mechanism for transmitting a rotation about a given axis to a second given axis which intersects the first at an angle.

Let PQ, RS (Fig. 53) be the given axes, intersecting in O. AB is a rod perpendicular to PQ, rigidly attached to PQ, and CD is another rod rigidly attached to RS. The rods AB, CD are rigidly attached to one another, at their point of intersection O, so as to be perpendicular. The rod AB is capable of turning about itself as axis, and similarly CD.

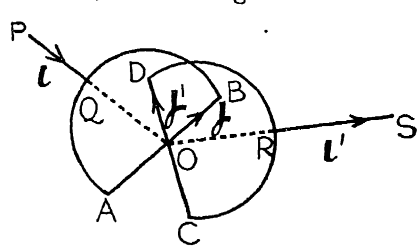

Fig. 53

Let **i, i′** be unit vectors along the given fixed directions of the axes PQ, RS, and let **j, j′** be unit vectors along AB and CD. Let ω be the spin of the axis PQ, ω′ that of RS, and let $\boldsymbol{\Omega}$ be the angular velocity of

the rigid body formed by AB and CD. We note that there are three rigid bodies concerned: the points A, B are members of two of them, the points C, D of a different two of them. Then the motions of the points of position vectors $\mathbf{j}$ and $\mathbf{j}'$ with respect to O are given by

$$\frac{d\mathbf{j}}{dt}=\mathbf{\Omega}\wedge\mathbf{j}=\omega\mathbf{i}\wedge\mathbf{j}, \qquad \frac{d\mathbf{j}'}{dt}=\mathbf{\Omega}\wedge\mathbf{j}'=\omega'\mathbf{i}'\wedge\mathbf{j}'.$$

Hence $$\mathbf{\Omega}-\omega\mathbf{i}=\lambda\mathbf{j}, \qquad \mathbf{\Omega}-\omega'\mathbf{i}'=\lambda'\mathbf{j}'.$$

Hence, subtracting, $$\omega'\mathbf{i}'-\omega\mathbf{i}=\lambda\mathbf{j}-\lambda'\mathbf{j}'.$$

Multiplying scalarly in turn by $\mathbf{j}$ and $\mathbf{j}'$ we have

$$\omega'(\mathbf{i}'.\mathbf{j})=\lambda, \qquad -\omega(\mathbf{i}.\mathbf{j}')=-\lambda',$$

since we have $$\mathbf{i}.\mathbf{j}=0, \quad \mathbf{i}'.\mathbf{j}'=0, \quad \mathbf{j}.\mathbf{j}'=0.$$

Hence $$\mathbf{\Omega}=\omega\mathbf{i}+\omega'(\mathbf{i}'.\mathbf{j})\mathbf{j}=\omega'\mathbf{i}'+\omega(\mathbf{i}.\mathbf{j}')\mathbf{j}'.$$

This gives us the desired relation between ω and ω'. For multiplying scalarly in turn by $\mathbf{i}$ and $\mathbf{i}'$ we have

$$\omega[1-(\mathbf{i}.\mathbf{j}')^2]=\omega'(\mathbf{i}.\mathbf{i}'),$$
$$\omega'[1-(\mathbf{i}'.\mathbf{j})^2]=\omega(\mathbf{i}.\mathbf{i}'),$$

whence $$\frac{\omega'}{\omega}=\frac{|\mathbf{i}\wedge\mathbf{j}'|^2}{\mathbf{i}.\mathbf{i}'}=\frac{\mathbf{i}.\mathbf{i}'}{|\mathbf{i}'\wedge\mathbf{j}|^2}.$$

The value of $\mathbf{i}.\mathbf{i}'$ is fixed. As $\mathbf{j}'$ moves, it describes a plane perpendicular to $\mathbf{i}'$. This plane intersects the plane perpendicular to $\mathbf{i}$ in a straight line. Thus twice during a complete revolution, $\mathbf{j}'$ is perpendicular to $\mathbf{i}$, and here $|\mathbf{i}\wedge\mathbf{j}'|$ takes its maximum value, unity. Hence the maximum value of ω'/ω is $1/(\mathbf{i}.\mathbf{i}')$. By a similar argument, the minimum value of ω'/ω is $\mathbf{i}.\mathbf{i}'$. (It is assumed for simplicity that $\mathbf{i}.\mathbf{i}'>0$.) Thus ω'/ω oscillates between $\mathbf{i}.\mathbf{i}'$ and $1/\mathbf{i}.\mathbf{i}'$, and actually attains these limits. If $\mathbf{i}$ and $\mathbf{i}'$ are inclined at a small angle, $\mathbf{i}.\mathbf{i}'$ differs from unity by a small quantity of the second order, and if ω is constant, so also approximately is ω'. (It is readily verified that $(\mathbf{i}\wedge\mathbf{j}').(\mathbf{i}'\wedge\mathbf{j})=\mathbf{i}.\mathbf{i}'$, which guarantees the consistency of the two formulæ for ω'/ω.)

It may be noted that since $\mathbf{j}.\mathbf{j}'=0$, we must have

$$(\mathbf{\Omega}-\omega\mathbf{i}).(\mathbf{\Omega}-\omega'\mathbf{i}')=0.$$

But $$\mathbf{\Omega}.\mathbf{i}=\omega, \qquad \mathbf{\Omega}.\mathbf{i}'=\omega'.$$

Hence $$\mathbf{\Omega}^2=\omega^2+\omega'^2-\omega\omega'(\mathbf{i}.\mathbf{i}').$$

An alternative expression for ω'/ω can be obtained by noting that since $\mathbf{j}.\mathbf{j}'=0$, we must have

$$\frac{d\mathbf{j}}{dt}.\mathbf{j}'+\mathbf{j}.\frac{d\mathbf{j}'}{dt}=0,$$

or $$\omega(\mathbf{i}\wedge\mathbf{j}).\mathbf{j}'+\omega'(\mathbf{i}'\wedge\mathbf{j}').\mathbf{j}=0$$

or $$\frac{\omega'}{\omega}=\frac{\mathbf{j}\wedge\mathbf{j}'.\mathbf{i}}{\mathbf{j}\wedge\mathbf{j}'.\mathbf{i}'}.$$

When $\mathbf{j}'$ is perpendicular to $\mathbf{i}$, $\mathbf{j}\wedge\mathbf{j}'$ is along $\mathbf{i}$, and the value of ω'/ω is $1/(\mathbf{i}.\mathbf{i}')$; when $\mathbf{j}$ is perpendicular to $\mathbf{i}'$, $\mathbf{j}\wedge\mathbf{j}'$ is along $\mathbf{i}'$, and the value of ω'/ω is $\mathbf{i}.\mathbf{i}'$. That these are extreme values of ω'/ω can be seen by noticing that at a maximum or minimum of $(\mathbf{j}\wedge\mathbf{j}').\mathbf{i}$ we must have

$$0=\frac{d}{dt}[\mathbf{j}\wedge\mathbf{j}'.\mathbf{i}]=[\omega(\mathbf{i}\wedge\mathbf{j})\wedge\mathbf{j}'+\omega'\mathbf{j}\wedge(\mathbf{i}'\wedge\mathbf{j}')].\mathbf{i}=-\omega'(\mathbf{i}.\mathbf{j}')(\mathbf{i}'.\mathbf{j}).$$

Hence in these positions either $\mathbf{j}$ is perpendicular to $\mathbf{i}'$ or $\mathbf{j}'$ is perpendicular to $\mathbf{i}$. By the symmetry of the result, these give also the stationary values of $\mathbf{j}\wedge\mathbf{j}'.\mathbf{i}'$, and it is easily seen that a maximum of $\mathbf{j}\wedge\mathbf{j}'.\mathbf{i}$ coincides with a minimum of $\mathbf{j}\wedge\mathbf{j}'.\mathbf{i}'$.

233. *Examples on the motion of a rigid body.* Many of the following examples are taken from a standard textbook (Lamb's *Higher Mechanics*). The solutions are worked out in detail here in the hope of persuading the reader that vector methods are usually just as powerful in dealing with examples as in proving standard theorems.

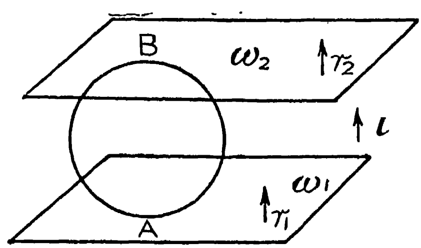

Fig. 54

Example (1). A rough sphere is pressed between two parallel plane boards rotating with angular velocities ω_1, ω_2 about non-coincident axes normal to themselves. Prove that the path of the centre of the sphere is a circle, and determine its centre and the angular velocity round it.

Take a fixed origin O in the plane of one of the boards. Let $\mathbf{r}_1$ be the position vector of the intersection with this board of the axis of rotation of the board. Let $\mathbf{i}$ be a unit vector normal to the boards. Take an origin O′ in the plane of the second board at the projection of O on this board, and let $\mathbf{r}_2$ be the position vector, with respect to O′, of the intersection of this board with its axis of rotation. Let A, B (Fig. 54) be the points of contact of the rolling sphere, $\mathbf{r}$ their position vector with respect to O or O′ respectively. Let $\mathbf{\Omega}$ be the angular velocity of the moving sphere.

Then the condition of rolling at A is that the velocity of A considered as a particle of the sphere is equal to the velocity of A considered as a particle of the board. Considered as a particle of the sphere, the velocity of A is

$$\frac{d\mathbf{r}}{dt}+\mathbf{\Omega}\wedge(-a\mathbf{i}),$$

where a is the radius of the sphere ; considered as a particle of the plane, the velocity of A is

$$\omega_1\mathbf{i}\wedge(\mathbf{r}-\mathbf{r}_1).$$

Hence $$\frac{d\mathbf{r}}{dt}-a\boldsymbol{\Omega}\wedge\mathbf{i}=\omega_1\mathbf{i}\wedge(\mathbf{r}-\mathbf{r}_1).$$

Conditions at B give similarly

$$\frac{d\mathbf{r}}{dt}+a\boldsymbol{\Omega}\wedge\mathbf{i}=\omega_2\mathbf{i}\wedge(\mathbf{r}-\mathbf{r}_2).$$

Adding, $$\frac{d\mathbf{r}}{dt}=\tfrac{1}{2}\mathbf{i}\wedge[\omega_1(\mathbf{r}-\mathbf{r}_1)+\omega_2(\mathbf{r}-\mathbf{r}_2)]$$

$$=\tfrac{1}{2}(\omega_1+\omega_2)\mathbf{i}\wedge\left(\mathbf{r}-\frac{\omega_1\mathbf{r}_1+\omega_2\mathbf{r}_2}{\omega_1+\omega_2}\right).$$

This formula asserts that the centre of the sphere is in motion with an angular velocity $\frac{1}{2}(\omega_1+\omega_2)\mathbf{i}$ about the point

$$\frac{\omega_1\mathbf{r}_1+\omega_2\mathbf{r}_2}{\omega_1+\omega_2};$$

this point divides the line joining $\mathbf{r}_1$ to $\mathbf{r}_2$ in the ratio $\omega_2:\omega_1$. Since this point is fixed, and since $\frac{1}{2}(\omega_1+\omega_2)\mathbf{i}$ is a fixed vector normal to the planes, the path must be a circle.

To obtain the angular velocity $\boldsymbol{\Omega}$ of the sphere itself, we subtract the equations expressing the conditions of rolling contact, obtaining

$$a\boldsymbol{\Omega}\wedge\mathbf{i}=\tfrac{1}{2}\mathbf{i}\wedge[\omega_2(\mathbf{r}-\mathbf{r}_2)-\omega_1(\mathbf{r}-\mathbf{r}_1)].$$

Multiplying vectorially by $\mathbf{i}$, we have

$$a[-\boldsymbol{\Omega}+(\boldsymbol{\Omega}.\mathbf{i})\mathbf{i}]=\tfrac{1}{2}(\omega_2-\omega_1)\mathbf{r}-\tfrac{1}{2}(\omega_2\mathbf{r}_2-\omega_1\mathbf{r}_1).$$

This evaluates $\boldsymbol{\Omega}$ as far as the conditions of rolling contact fix it, for $\boldsymbol{\Omega}.\mathbf{i}$, the spin of the sphere about the diameter AB, is clearly arbitrary.

Example (2). A rough sphere, of diameter $(b-a)$, is pressed between two concentric spheres of radii a, b $(b>a)$ which are compelled to rotate about their centres with given angular velocities $\boldsymbol{\Omega}_1$, $\boldsymbol{\Omega}_2$. Show that the path of the first sphere is a circle; and ascertain the motion.

Let $\boldsymbol{\Omega}$ be the angular velocity of the moving rough sphere, $\mathbf{i}$ a unit vector along the common radii to the points of contact. Then the velocity $\mathbf{v}$ of the centre of the free sphere is given by

$$\mathbf{v}=\tfrac{1}{2}(a+b)\frac{d\mathbf{i}}{dt}.$$

The conditions of rolling contact are

$$\mathbf{v}+\boldsymbol{\Omega}\wedge\tfrac{1}{2}(b-a)\mathbf{i}=\boldsymbol{\Omega}_2\wedge b\mathbf{i},$$
$$\mathbf{v}+\boldsymbol{\Omega}\wedge[\tfrac{1}{2}(b-a)(-\mathbf{i})]=\boldsymbol{\Omega}_1\wedge a\mathbf{i}.$$

Adding $$\mathbf{v}=\tfrac{1}{2}(b\boldsymbol{\Omega}_2+a\boldsymbol{\Omega}_1)\wedge\mathbf{i}$$

or $$\mathbf{v}=\frac{b\boldsymbol{\Omega}_2+a\boldsymbol{\Omega}_1}{a+b}\wedge\tfrac{1}{2}(a+b)\mathbf{i}.$$

Now $\frac{1}{2}(a+b)\mathbf{i}$ is the position vector of the centre of the free sphere. Hence the last result asserts that the centre of the free sphere is being compelled to move with uniform angular velocity $\mathbf{\Omega}'$ given by

$$\mathbf{\Omega}' = \frac{a\mathbf{\Omega}_1 + b\mathbf{\Omega}_2}{a+b}.$$

Its path is therefore in a plane normal to $\mathbf{\Omega}'$ and consists of a circle described with uniform speed. The value of Ω is readily determined as in Example (1).

Example (3). A rough right-circular cone of semi-vertical angle α is rolling on a rough horizontal table, the contact generator revolving round the vertical through the vertex with angular velocity ω_1. Find the angular velocity of the cone, and discuss the motion of the cone relative to the moving vertical plane containing the contact generator.

Let $\mathbf{i}$ be a unit vector along the contact generator. This line of particles, considered as forming part of the cone, is momentarily at rest. Hence it is the instantaneous axis of the cone, and the angular velocity of the cone is accordingly of the form $\omega\mathbf{i}$.

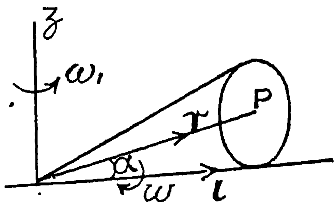

Fig. 55

Let $\mathbf{r}$ be the position vector of a point P (Fig. 55) in the axis of the cone with respect to the vertex O of the cone. The line of particles OP, considered as belonging to the rigid body constituted by the cone, has the angular velocity $\omega\mathbf{i}$ about O; but considered as a member of the rigid body defined by the vertical through O and the revolving contact generator, OP has the angular velocity $\omega_1\mathbf{z}$, where $\mathbf{z}$ is a unit vector vertically upwards. Hence, equating the velocities thus calculated for any point on OP, we get

$$\omega\mathbf{i}\wedge\mathbf{r} = \omega_1\mathbf{z}\wedge\mathbf{r}.$$

Hence $\omega\mathbf{i} - \omega_1\mathbf{z}$ is parallel to $\mathbf{r}$. But $\mathbf{r}$ is parallel to

$$\mathbf{i}\cos\alpha + \mathbf{z}\sin\alpha.$$

Hence $$\frac{\omega}{\cos\alpha} = \frac{-\omega_1}{\sin\alpha}$$

or $$\omega = -\omega_1\cot\alpha.$$

The angular velocity of the rotating vertical plane through the contact generator being $\omega_1\mathbf{z}$, the angular velocity of the cone *relative to* this plane is

$$-\omega_1\cot\alpha\,\mathbf{i} - \omega_1\mathbf{z}.$$

Hence the time for the complete revolution of a generator of the rolling cone from contact to contact is

$$\frac{2\pi}{\omega_1|\mathbf{i}\cot\alpha + \mathbf{z}|} = \frac{2\pi\sin\alpha}{\omega_1}.$$

This may be checked by noticing that if l is the slant length of the cone, the circumference of the base is $2\pi l \sin\alpha$, and hence the time taken to describe, with angular speed ω_1, an arc of length $2\pi l \sin\alpha$ extending round an azimuthal circle of radius l is $(2\pi \sin\alpha)/\omega_1$. An observer attached to the revolving vertical plane and provided with a revolution-counter would measure the modulus not of the angular velocity ω_1 of the cone but of the *relative* angular velocity $\omega\mathbf{i}-\omega_1\mathbf{z}$, of modulus ω_1 cosec α.

Example (4). The plane $Ax+By+Cz=1$ is fixed in space, but the rectangular axes to which it is referred are rotating with an angular velocity (p, q, r). Prove that

$$\frac{dA}{dt}=Br-Cq, \text{ etc.}$$

The triplet of numbers (A, B, C) form a vector $\mathbf{N}$ normal to the plane and therefore fixed in space ; for, if $(x, y, z)=\mathbf{R}$, the given equation is $\mathbf{N}.\mathbf{R}=1$, and hence by the quotient theorem $\mathbf{N}$ is a vector. Accordingly, $d\mathbf{N}/dt=0$. But

$$\frac{d\mathbf{N}}{dt}=\frac{\partial\mathbf{N}}{\partial t}+\mathbf{\Omega}\wedge\mathbf{N},$$

where $\mathbf{\Omega}=(p, q, r)$. Hence

$$\frac{\partial\mathbf{N}}{\partial t}=-\mathbf{\Omega}\wedge\mathbf{N}.$$

But the components of $\partial\mathbf{N}/\partial t$ are the apparent rates of change of A, B, C in the moving frame defined by the co-ordinate system. Hence the required result.

An alternative procedure is to note that the given plane $\mathbf{N}.\mathbf{R}=1$ has an angular velocity $-\mathbf{\Omega}$ with respect to the axes. Hence

$$\frac{\partial\mathbf{N}}{\partial t}.\mathbf{R}+\mathbf{N}.\frac{\partial\mathbf{R}}{\partial t}=0.$$

But

$$\frac{\partial\mathbf{R}}{\partial t}=-\mathbf{\Omega}\wedge\mathbf{R}.$$

Hence

$$\left(\frac{\partial\mathbf{N}}{\partial t}-\mathbf{N}\wedge\mathbf{\Omega}\right).\mathbf{R}=0.$$

This is true for all $\mathbf{R}$. Hence

$$\partial\mathbf{N}/\partial t=\mathbf{N}\wedge\mathbf{\Omega}=-\mathbf{\Omega}\wedge\mathbf{N}.$$

Example (5). A quadric whose equation relative to fixed axes is a any instant

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$$

is of invariable form, but is rotating round the origin with angular velocity (p, q, r). Prove that

$$\frac{da}{dt}=2(gq-hr), \qquad \frac{df}{dt}=(b-c)p+gr-hq,$$

etc.

The equation of the quadric at any instant is $\mathbf{T}:\mathbf{rr}=1$, where $\mathbf{T}$ is the self-conjugate tensor

$$\begin{matrix} a & h & g \\ h & b & f \\ g & f & c. \end{matrix}$$

In the rotating frame defined by the rigid body associated with the invariable quadric, $\mathbf{T}$ has constant components. Thus $\partial\mathbf{T}/\partial t=0$. Hence, applying the theorem of § 211,

$$\frac{d\mathbf{T}}{dt}=\mathbf{\Omega}\wedge\mathbf{T}-\mathbf{T}\wedge\mathbf{\Omega}=\mathbf{\Omega}\wedge\mathbf{T}+\overline{\mathbf{\Omega}\wedge\overline{\mathbf{T}}}=\mathbf{\Omega}\wedge\mathbf{T}+\overline{\mathbf{\Omega}\wedge\mathbf{T}}.$$

The (1, 1) component of $d\mathbf{T}/dt$ is da/dt ; that of $\mathbf{\Omega}\wedge\mathbf{T}$ is by its definition

$$A_{123}\Omega_2 T_{31}+A_{132}\Omega_3 T_{21}$$

or

$$qg-rh.$$

That of $\overline{\mathbf{\Omega}\wedge\mathbf{T}}$ is the same, and so we have the first result stated. The (2, 3) component of $d\mathbf{T}/dt$ is df/dt ; that of $\mathbf{\Omega}\wedge\mathbf{T}$ is

$$A_{231}\Omega_3 T_{13}+A_{213}\Omega_1 T_{33}$$

or

$$rg-pc\ ;$$

that of $\overline{\mathbf{\Omega}\wedge\mathbf{T}}$ is the (3, 2) component of $\mathbf{\Omega}\wedge\mathbf{T}$ which is

$$A_{312}\Omega_1 T_{22}+A_{321}\Omega_2 T_{12}$$

or

$$pb-qh.$$

Hence the second result stated.

From the foregoing we can deduce the conditions for the quadric to be a surface of revolution. For if the quadric is one of revolution, there exists an angular velocity (p, q, r) which leaves the equation of the quadric unaltered. For such values of p, q, r we have accordingly

$$\begin{aligned} gq-hr&=0, &\quad (b-c)p+gr-hq&=0,\\ hr-fp&=0, &\quad (c-a)q+hp-fr&=0,\\ fp-gq&=0, &\quad (a-b)r+fq-gp&=0. \end{aligned}$$

Hence

$$fp=gq=hr,$$

or

$$p:q:r=\frac{1}{f}:\frac{1}{g}:\frac{1}{h},$$

provided none of f, g, h are zero. The second set of equations then gives

$$\frac{b-c}{f}+\frac{g}{h}-\frac{h}{g}=0,$$

etc., or

$$b-\frac{fh}{g}=c-\frac{gf}{h}=a-\frac{hg}{f}.$$

If, on the other hand, one of f, g, h is zero, say $g=0$, then $fp=hr=0$. If both p and r are zero, the second group of equations gives $hq=0$, $(c-a)q=0$, $fq=0$, requiring $f=h=0$, $c=a$, and the quadric reduces to $ax^2+by^2+az^2=1$. If $p=0$, $r\neq 0$, $q\neq 0$, then $h=0$, $g=0$, and the second group of equations gives $(c-a)q=fr$, $(a-b)r=-fq$, whence $(a-b)(a-c)=f^2$. The latter set of conditions includes the case $f=g=h=0$.

Example (6). A solid is rolling in contact with a fixed plane with angular velocity $\mathbf{\Omega}$. Find the acceleration of the particle of contact in terms of the principal radii of curvature of the surface of the solid and the components of $\mathbf{\Omega}$ along the lines of curvature.

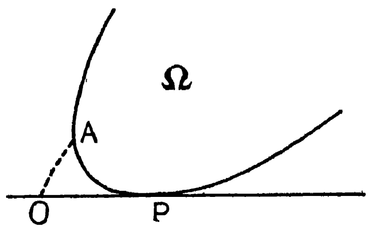

Fig. 56

Let A (Fig. 56) be the particle of the solid in contact with the plane at a point O of the plane at some instant $t=0$. Let P be the point of contact at time t. Let $OA=\mathbf{r}$. Then, since the particle P of the body is instantaneously at rest,

$$\frac{d\mathbf{r}}{dt}+\mathbf{\Omega}\wedge AP=0.$$

(In writing down this relation we have used the facts that $d\mathbf{r}/dt$ is the velocity of the particle A relative to O and $\mathbf{\Omega}\wedge AP$ is the velocity of the particle P relative to the particle A.) Differentiating this relation (which holds good for all t) with respect to the time t, we have

$$\frac{d^2\mathbf{r}}{dt^2}+\frac{d\mathbf{\Omega}}{dt}\wedge AP+\mathbf{\Omega}\wedge\frac{dAP}{dt}=0.$$

When $t=0$, $AP=0$. Hence

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_0=-\mathbf{\Omega}\wedge\left(\frac{dAP}{dt}\right)_0.$$

Take now a frame of reference fixed in the body. Then in the standard notation,

$$\frac{dAP}{dt}=\frac{\partial AP}{\partial t}+\mathbf{\Omega}\wedge AP,$$

so that

$$\left(\frac{dAP}{dt}\right)_0=\left(\frac{\partial AP}{\partial t}\right)_0.$$

Referred to a frame of reference fixed in the body, the plane of contact has an angular velocity $-\mathbf{\Omega}$. Take a unit vector $\mathbf{n}$ normal to the tangent plane at P, and let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be a triad of unit vectors along the principal lines of curvature and along the normal at A.

Referred to A as origin, P will have co-ordinates (ξ', η', ζ'), so that

$$\mathrm{AP} = \xi'\mathbf{i} + \eta'\mathbf{j} + \zeta'\mathbf{k},$$

and the approximate equation of the surface is

$$2\zeta = \frac{\xi^2}{\rho_1} + \frac{\eta^2}{\rho_2}$$

where ρ_1, ρ_2 are the principal radii of curvature at A. The tangent plane at P has for its equation

$$\zeta + \zeta' = \frac{\xi\xi'}{\rho_1} + \frac{\eta\eta'}{\rho_2},$$

whence the normal $\mathbf{n}$ to this plane is the vector

$$\mathbf{n} = \frac{1}{\theta}\left[-\frac{\xi'}{\rho_1}\mathbf{i} - \frac{\eta'}{\rho_2}\mathbf{j} + \mathbf{k}\right]$$

where $$\theta^2 = 1 + (\xi'/\rho_1)^2 + (\eta'/\rho_2)^2.$$

This vector $\mathbf{n}$ can be considered to have an angular velocity $-\mathbf{\Omega}$ referred to the frame of reference fixed in the body, so that

$$\frac{\partial}{\partial t}\left[\frac{1}{\theta}\left\{-\frac{\xi'}{\rho_1}\mathbf{i} - \frac{\eta'}{\rho_2}\mathbf{j} + \mathbf{k}\right\}\right] = -\mathbf{\Omega}\wedge\frac{1}{\theta}\left\{-\frac{\xi'}{\rho_1}\mathbf{i} - \frac{\eta'}{\rho_2}\mathbf{j} + \mathbf{k}\right\}.$$

Substituting $$\mathbf{\Omega} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k},$$

putting $t=0$ and noting that at $t=0$, $\theta=1$ and $\partial\theta/\partial t=0$, we have, omitting primes,

$$-\frac{\dot{\xi}_0}{\rho_1}\mathbf{i} - \frac{\dot{\eta}_0}{\rho_2}\mathbf{j} = -(\omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k})\wedge\left[-\frac{\xi}{\rho_1}\mathbf{i} - \frac{\eta}{\rho_2}\mathbf{j} + \mathbf{k}\right]_{t=0}$$
$$= \omega_1\mathbf{j} - \omega_2\mathbf{i}.$$

Hence $$\dot{\xi}_0 = \rho_1\omega_2, \qquad \dot{\eta}_0 = -\omega_1\rho_2.$$

Hence $$\left[\frac{\partial \mathrm{AP}}{\partial t}\right]_{t=0} = \dot{\xi}_0\mathbf{i} + \dot{\eta}_0\mathbf{j} = \rho_1\omega_2\mathbf{i} - \rho_2\omega_1\mathbf{j}.$$

Finally $$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_0 = -\mathbf{\Omega}\wedge\left[\frac{\partial \mathrm{AP}}{\partial t}\right]_0$$
$$= -(\omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k})\wedge(\rho_1\omega_2\mathbf{i} - \rho_2\omega_1\mathbf{j})$$
$$= -\rho_2\omega_1\omega_3\mathbf{i} - \rho_1\omega_2\omega_3\mathbf{j} + (\rho_1\omega_2{}^2 + \rho_2\omega_1{}^2)\mathbf{k}.$$

In this formula the coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ give the components of acceleration of the particle of contact along the directions of the lines of curvature and along the normal.

234. *Motion in a plane curve.* This is readily dealt with in an elemen way, but it is desirable to exhibit its relation to our general analysis of motion in a twisted curve. A plane curve may be regarded as a twisted curve for which the torsion $1/\sigma$ is zero. The binormal is then a constant vector. It is more instructive, however, to derive the fundamental formulæ *ab initio.*

Let **T** be a unit vector in the direction of the tangent to a plane curve, with some convention as to sense. The normal **N** is defined as a unit vector parallel to $d\mathbf{T}/ds$, where s is the arc. Take **z**, a unit vector normal to the plane of the curve. It would be possible to choose the sense of **z** so that **T, N, z** form a positive triad, in which case the curvature (to be defined later) is always positive. It is more convenient, however, to select a definite sense for **z** (e.g. the upward sense when the curve lies in a horizontal plane) and then to define the sense of **N** so that **T, N, z** form a positive triad. The curvature may be then either positive or negative.

Adopting this procedure, we now define the curvature $1/\rho$ at a point P of the curve as such that the triad **T, N, z** has the angular velocity $\mathbf{z}/\rho$ at this point when P moves with unit speed along the curve. We then have

$$\frac{d\mathbf{T}}{ds}=\frac{\mathbf{z}}{\rho}\wedge\mathbf{T}=\frac{\mathbf{N}}{\rho},\qquad \frac{d\mathbf{N}}{ds}=\frac{\mathbf{z}}{\rho}\wedge\mathbf{N}=-\frac{\mathbf{T}}{\rho}.$$

It follows that $\rho \gtrless 0$ according as the curve near P is on the same side of the tangent as **N** or on the opposite side (see diagram, Fig. 57).

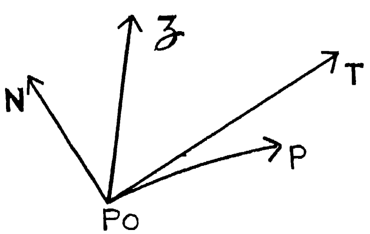

Fig. 57*a*

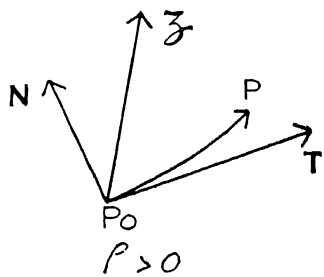

Fig. 57*b*

The form of the curve near P is given by

$$\begin{aligned}\mathbf{P}-\mathbf{P}_0&=s\left(\frac{d\mathbf{P}}{ds}\right)_0+\frac{s^2}{2!}\left(\frac{d^2\mathbf{P}}{ds^2}\right)_0+\frac{s^3}{3!}\left(\frac{d^3\mathbf{P}}{ds^3}\right)_0+\ldots\\ &=s\mathbf{T}_0+\frac{s^2}{2}\,\frac{\mathbf{N}_0}{\rho}+\frac{s^3}{6}\,\frac{d}{ds}\left(\frac{\mathbf{N}}{\rho}\right)_0\\ &=s\mathbf{T}_0+\frac{s^2}{2}\,\frac{\mathbf{N}_0}{\rho}+\frac{s^3}{6}\left[-\frac{\mathbf{T}}{\rho^2}+\mathbf{N}\frac{d}{ds}\left(\frac{1}{\rho}\right)\right]_0\\ &=\mathbf{T}_0\left(s-\frac{s^3}{6\rho^2}\right)+\mathbf{N}_0\left(\frac{s^2}{2\rho}+\frac{s^3}{6}\,\frac{d}{ds}\left(\frac{1}{\rho}\right)_0\right)+\ldots.\end{aligned}$$

The numerical value of the curvature is given by

$$\frac{1}{\rho}=\pm\left|\frac{d^2\mathbf{P}}{ds^2}\right|,$$

the upper sign being taken if $d\mathbf{T}/ds$ is in the same sense as $\mathbf{N}$, the lower if in the opposite sense.

If $\mathbf{P}$ is given, not as a function of s but in terms of a parameter, and if primes denote differentiation with respect to this parameter, then

$$\mathbf{P}'=\mathbf{T}s'$$

$$\mathbf{P}''=\frac{d\mathbf{T}}{ds}s'^2+\mathbf{T}s''$$

and
$$\mathbf{P}'\wedge\mathbf{P}''=s'^3\mathbf{T}\wedge d\mathbf{T}/ds=(s'^3/\rho)\mathbf{z}.$$

The acceleration of a particle following a plane curve is now obtained simply from

$$\frac{d\mathbf{P}}{dt}=\frac{d\mathbf{P}}{ds}\frac{ds}{dt}=v\mathbf{T},$$

$$\frac{d^2\mathbf{P}}{dt^2}=\frac{dv}{dt}\mathbf{T}+v\frac{d\mathbf{T}}{ds}\frac{ds}{dt}=\frac{dv}{dt}\mathbf{T}+\frac{v^2}{\rho}\mathbf{N},$$

where v is the speed.

The centre of curvature has a position vector $\rho\mathbf{N}$ with respect to P, where ρ has its proper sign.

235. *Kinematics of the motion of a lamina in a plane.* Let a lamina be in motion in its own plane in such a way as to pass continuously through a 'linear series' of configurations. Without loss of generality it may be made to traverse the same series of configurations at constant angular speed ω. Let $\mathbf{u}$ be the velocity of a particle O of the lamina whose position vector is $\mathbf{r}_0$ with respect to some fixed origin in the plane. Let $\mathbf{r}$ be the position vector of an arbitrary particle P of the lamina with respect to the same fixed origin. Let $\mathbf{z}$ be a unit vector normal to the plane of the lamina.

Instantaneous centre. Then since the lamina has angular velocity $\omega\mathbf{z}$, we have

$$\frac{d\mathbf{r}}{dt}=\mathbf{u}+\omega\mathbf{z}\wedge(\mathbf{r}-\mathbf{r}_0),$$

where
$$\frac{d\mathbf{r}_0}{dt}=\mathbf{u}.$$

There will be a particle $\mathbf{r}=\mathbf{r}_1$ of the lamina which will be momentarily at rest provided a solution $\mathbf{r}_1$ can be found of the equation

$$0=\mathbf{u}+\omega\mathbf{z}\wedge(\mathbf{r}_1-\mathbf{r}_0).$$

To solve this equation, multiply vectorially by $\mathbf{z}$. We find

$$\mathbf{r}_1-\mathbf{r}_0=\frac{\mathbf{z}\wedge\mathbf{u}}{\omega}.$$

We then have
$$\frac{d\mathbf{r}}{dt}=\omega\mathbf{z}\wedge(\mathbf{r}-\mathbf{r}_1).$$

Hence every particle of the lamina has the same velocity as if the lamina were instantaneously rotating round the particle $\mathbf{r}=\mathbf{r}_1$ with angular velocity $\omega\mathbf{z}$. For this reason, the point $\mathbf{r}=\mathbf{r}_1$ is called the *instantaneous centre*, I.

Accelerations. The acceleration of the particle $\mathbf{r}$ of the lamina is given by

$$\frac{d^2\mathbf{r}}{dt^2}=\frac{d\mathbf{u}}{dt}+\omega\mathbf{z}\wedge\left[\frac{d\mathbf{r}}{dt}-\frac{d\mathbf{r}_0}{dt}\right].$$

Now
$$\frac{d(\mathbf{r}-\mathbf{r}_0)}{dt}=\omega\mathbf{z}\wedge(\mathbf{r}-\mathbf{r}_0).$$

Hence
$$\frac{d^2\mathbf{r}}{dt^2}=\frac{d\mathbf{u}}{dt}-\omega^2(\mathbf{r}-\mathbf{r}_0).$$

There will be a particle $\mathbf{r}=\mathbf{r}_2$ of the lamina which will have zero acceleration provided a solution $\mathbf{r}_2$ can be found of the equation

$$0=\frac{d\mathbf{u}}{dt}-\omega^2(\mathbf{r}_2-\mathbf{r}_0).$$

This particle $\mathbf{r}_2$ clearly exists, namely

$$\mathbf{r}_2=\mathbf{r}_0+\frac{1}{\omega^2}\frac{d\mathbf{u}}{dt},$$

so that
$$\mathbf{r}_2-\mathbf{r}_1=\frac{1}{\omega^2}\frac{d\mathbf{u}}{dt}-\frac{\mathbf{z}\wedge\mathbf{u}}{\omega}.$$

The acceleration of the arbitrary particle $\mathbf{r}$ can now be written

$$\frac{d^2\mathbf{r}}{dt^2}=-\omega^2(\mathbf{r}-\mathbf{r}_2).$$

Thus the accelerations of all the particles of the lamina are directed towards the point J given by $\mathbf{r}=\mathbf{r}_2$, and are proportional to their distances from J. The point J is called the *centre of accelerations*.

We have seen, from the formula $d\mathbf{r}/dt=\omega\mathbf{z}\wedge(\mathbf{r}-\mathbf{r}_1)$ that the velocity of any particle P is perpendicular to the line joining it to the instantaneous centre, i.e. to PI. Hence when P is such that PJ is perpendicular to PI, the velocity and acceleration of P lie in the same straight line, and, at such a position for P, the acceleration being along the path, the curvature

of the path is zero, and at the point concerned, the locus of P in space has a *point of inflexion.* It follows that the locus of particles passing through points of inflexion in their space loci is the circle on IJ as diameter. This circle is called the *circle of inflexions.* In the diagram (Fig. 58), N is the position of a particle P passing through a point of inflexion; the angle INJ is a right angle.

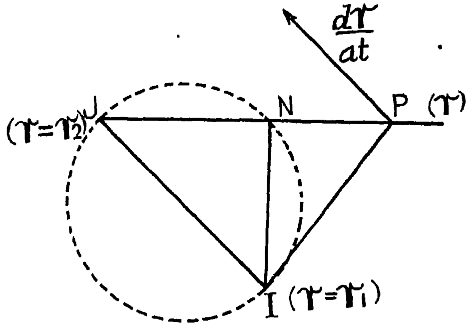

Fig. 58

Since the position of I is given by

$$\mathbf{r}_1=\mathbf{r}_0+\frac{\mathbf{z}\wedge\mathbf{u}}{\omega},$$

the velocity of the *point* I in space, i.e. the velocity of the instantaneous centre, is given by

$$\frac{d\mathbf{r}_1}{dt}=\frac{d}{dt}\left[\mathbf{r}_0+\frac{\mathbf{z}\wedge\mathbf{u}}{\omega}\right]=\mathbf{u}+\frac{1}{\omega}\mathbf{z}\wedge\frac{d\mathbf{u}}{dt}.$$

If we choose for the reference particle $\mathbf{r}_0$ the particle of the lamina in the instantaneous position of I, then $\mathbf{u}=0$, and moreover

$$\frac{d\mathbf{u}}{dt}=-\omega^2(\mathbf{r}_1-\mathbf{r}_2).$$

Hence, for this choice, we have

$$\frac{d\mathbf{r}_1}{dt}=-\omega\mathbf{z}\wedge(\mathbf{r}_1-\mathbf{r}_2).$$

It follows that the *point* I is instantaneously rotating round J with angular velocity $-\omega\mathbf{z}$. This gives a very convenient way of determining J in cases where the velocity of I is known by inspection.

The curvature of the path of any particle P of the lamina is now readily obtained. For if, as usual, $\mathbf{T}$ denotes a unit vector along the tangent to its path, $\mathbf{N}$ a unit vector along the normal, then

$$\mathbf{T}=\frac{\mathbf{z}\wedge(\mathbf{r}-\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|},\qquad \mathbf{N}=\mathbf{z}\wedge\mathbf{T}=-\frac{\mathbf{r}-\mathbf{r}_1}{|\mathbf{r}-\mathbf{r}_1|},$$

and the acceleration of P, which we have already seen to be $-\omega^2(\mathbf{r}-\mathbf{r}_2)$, is also equal to

$$\frac{dv}{dt}\mathbf{T}+\frac{v^2}{\rho}\mathbf{N}.$$

Hence $$-\omega^2(\mathbf{r}-\mathbf{r}_2)=\frac{dv}{dt}\mathbf{z}\wedge\frac{\mathbf{r}-\mathbf{r}_1}{|\mathbf{r}-\mathbf{r}_1|}+\frac{\omega^2(\mathbf{r}-\mathbf{r}_1)^2}{\rho}\left[-\frac{\mathbf{r}-\mathbf{r}_1}{|\mathbf{r}-\mathbf{r}_1|}\right]$$

Multiplying scalarly by $\mathbf{r}-\mathbf{r}_1$ we have at once

$$\rho=\frac{|\mathbf{r}-\mathbf{r}_1|^3}{(\mathbf{r}-\mathbf{r}_1).(\mathbf{r}-\mathbf{r}_2)}=\frac{\mathrm{PI}^3}{\mathrm{PN.PJ}}.$$

This again shows that the curvature passes through zero and changes sign if P passes through N.

236. *Example* (1). Obtain the positions of the instantaneous centres and centres of acceleration (*a*) for a disc rolling along a straight line, (*b*) for a disc rolling along the inside of a fixed circle, (*c*) for a disc rolling on the outside of a fixed circle.

Example (2). A lamina is in motion with given uniform velocity $\mathbf{u}$ (a vector). A point P describes a given curve in the lamina with uniform speed $\dot{s}$ (a scalar) relative to the lamina. Find the curvature of the space path of P in terms of the curvature of the given curve on the lamina.

Let $\mathbf{R}$ be the position vector of the moving point relative to an origin fixed in space, $\mathbf{r}$ its position vector relative to an origin fixed in the lamina. Let $\mathbf{T}, \mathbf{N}, \mathbf{z}$ be the triad formed by the tangent, normal and vertical for the space locus, $\mathbf{t}, \mathbf{n}, \mathbf{z}$ the corresponding triad for the locus in the lamina. If dots denote differentiations with respect to the time, we have

$$\dot{\mathbf{R}}=\dot{s}\mathbf{t}+\mathbf{u},$$

whence, if $\dot{S}$ is the speed over the space locus,

$$\dot{S}=|\dot{s}\mathbf{t}+\mathbf{u}|.$$

Next, since $\dot{s}$ and $\mathbf{u}$ are constants,

$$\ddot{\mathbf{R}}=\dot{s}\dot{\mathbf{t}}=\dot{s}^2\frac{d\mathbf{t}}{ds}=\dot{s}^2\frac{\mathbf{n}}{\rho},$$

where $1/\rho$ is the curvature of the locus in the lamina. But, by the standard formula for the acceleration of a particle moving in a curve (§§ 230, 234)

$$\ddot{\mathbf{R}}=\ddot{S}\mathbf{T}+\frac{\dot{S}^2}{\rho'}\mathbf{N},$$

where $1/\rho'$ is the curvature of the space locus. Equating the two expressions and multiplying scalarly by $\mathbf{N}$,

$$\frac{\dot{S}^2}{\rho'}=\frac{\dot{s}^2\mathbf{n}.\mathbf{N}.}{\rho}$$

But $$\mathbf{T}=\frac{\dot{s}\mathbf{t}+\mathbf{u}}{|\dot{s}\mathbf{t}+\mathbf{u}|},\qquad \mathbf{N}=\mathbf{z}\wedge\mathbf{T}=\frac{\mathbf{z}\wedge(\dot{s}\mathbf{t}+\mathbf{u})}{|\dot{s}\mathbf{t}+\mathbf{u}|}.$$

Hence $$\frac{(\dot{s}\mathbf{t}+\mathbf{u})^2}{\rho'}=\frac{\dot{s}^2}{\rho}\frac{\mathbf{n}.[\mathbf{z}\wedge(\dot{s}\mathbf{t}+\mathbf{u})]}{|\dot{s}\mathbf{t}+\mathbf{u})}.$$

But $$\mathbf{n}\wedge\mathbf{z}.\mathbf{t}=1,\qquad \mathbf{n}.(\mathbf{z}\wedge\mathbf{u})=(\mathbf{n}\wedge\mathbf{z}).\mathbf{u}=\mathbf{t}.\mathbf{u}.$$

Hence $$\frac{1}{\rho'}=\frac{\dot{s}^2}{\rho}\,\frac{\dot{s}+\mathbf{t}.\mathbf{u}}{|\dot{s}\mathbf{t}+\mathbf{u}|^3}.$$

This evaluates ρ' in terms of the given quantities together with the angle θ between the direction of motion u and the tangent to the curve in the lamina. In terms of θ, the result is

$$\frac{1}{\rho'}=\frac{\dot{s}^2}{\rho}\,\frac{\dot{s}+|\mathbf{u}|\cos\theta}{[\dot{s}^2+\mathbf{u}^2+2\dot{s}|\mathbf{u}|\cos\theta]^{\frac{3}{2}}}.$$

Example (3). A lamina is in motion with uniform angular velocity about an axis normal to its plane as in § 235. Determine a particle K of the lamina such that the rate of change of its acceleration is momentarily zero, and discuss its properties.

Example (4). A rough disc of radius $\frac{1}{2}(b-a)$ fits closely between two concentric rings of radii a and b $(b>a)$; the rings are made to rotate with angular speeds ω_1, ω_2 respectively about the normal to their common plane through their centre. Show that the angular velocity of the disc is

$$(b\omega_2-a\omega_1)/(b-a)$$

and that the angular velocity of its centre about the common centre of the two concentric rings is

$$(b\omega_2+a\omega_1)/(b+a).$$

Example (5). A ring in the form of a circle of radius a is made to rotate with angular velocity ω_1 about an axis through a point O in its *circumference* normal to its plane. A second coplanar ring, in the form of a circle of radius $b(b<a)$, has its centre fixed to the moving centre A of the first circle, but is made to rotate *relative to it* with angular velocity ω_2. A rough circular disc of diameter $(a-b)$ fits between the rings. Determine its motion.

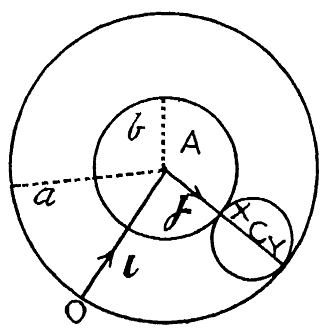

Fig. 59

Let C (Fig. 59) be the centre of the disc, X and Y the points of contact. Let $\mathbf{i}$ be a unit vector along OA, $\mathbf{j}$ a unit vector along AXCY. The rigid body constituted by the circle of radius *a* has an angular velocity $\omega_1\mathbf{z}$ about the fixed point O, $\mathbf{z}$ being a unit vector normal to the plane. The rigid body constituted by the circle of radius *b* has its particle A moving with velocity $a\,d\mathbf{i}/dt$, together with an angular velocity $(\omega_1+\omega_2)\mathbf{z}$ about A. The disc has the velocity

$$\frac{d}{dt}[\tfrac{1}{2}(a+b)\mathbf{j}+a\mathbf{i}]$$

of its centre C together with an angular velocity, say $\omega\mathbf{z}$ about C. The conditions of contact at Y and X give respectively

$$a\frac{d\mathbf{i}}{dt}+\tfrac{1}{2}(a+b)\frac{d\mathbf{j}}{dt}+\omega\mathbf{z}\wedge\tfrac{1}{2}(a-b)\mathbf{j}=\omega_1\mathbf{z}\wedge a(\mathbf{i}+\mathbf{j}),$$

$$a\frac{d\mathbf{i}}{dt}+\tfrac{1}{2}(a+b)\frac{d\mathbf{j}}{dt}-\omega\mathbf{z}\wedge\tfrac{1}{2}(a-b)\mathbf{j}=a\frac{d\mathbf{i}}{dt}+(\omega_1+\omega_2)\mathbf{z}\wedge b\mathbf{j}.$$

But $$\frac{d\mathbf{i}}{dt}=\omega_1\mathbf{z}\wedge\mathbf{i}.$$

The conditions accordingly reduce to

$$\tfrac{1}{2}(a+b)\frac{d\mathbf{j}}{dt}+\tfrac{1}{2}(a-b)\,\omega\mathbf{z}\wedge\mathbf{j}=a\,\omega_1\mathbf{z}\wedge\mathbf{j}$$

$$\tfrac{1}{2}(a+b)\frac{d\mathbf{j}}{dt}-\tfrac{1}{2}(a-b)\,\omega\mathbf{z}\wedge\mathbf{j}=b(\omega_1+\omega_2)\mathbf{z}\wedge\mathbf{j}.$$

These are independent of $\mathbf{i}$. Adding, we have

$$\frac{d\mathbf{j}}{dt}=\left(\omega_1+\frac{b}{a+b}\,\omega_2\right)\mathbf{z}\wedge\mathbf{j},$$

which determines the rate of rotation of $\mathbf{j}$ as

$$\omega_1+\frac{b}{a+b}\,\omega_2.$$

Subtracting the same two relations we have

$$(a-b)\,\omega\mathbf{z}\wedge\mathbf{j}=[(a-b)\,\omega_1-b\,\omega_2]\mathbf{z}\wedge\mathbf{j},$$

whence $$\omega=\omega_1-\frac{b}{a-b}\,\omega_2.$$

The motion of C is now determined as that of the vector $a\mathbf{i}+\tfrac{1}{2}(a+b)\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are uniformly rotating vectors whose rotational speeds are now known. The velocity of C, namely

$$\omega_1\mathbf{z}\wedge a\mathbf{i}+\left(\omega_1+\frac{b}{a+b}\,\omega_2\right)\mathbf{z}\wedge\tfrac{1}{2}(a+b)\mathbf{j},$$

may be written in the form

$$\omega_1\mathbf{z}\wedge[a\mathbf{i}+\tfrac{1}{2}(a+b)\mathbf{j}]+\tfrac{1}{2}\,\omega_2\mathbf{z}\wedge b\mathbf{j},$$

which exhibits it as the sum of a vector velocity normal to OC, of amount $\omega_1|OC|$, together with a component varying in direction but of constant magnitude $\tfrac{1}{2}b\,\omega_2$.

237. *Isotropic tensors of the second rank.** An isotropic tensor is defined to be one whose components take the same numerical values in all triads of reference. Examples of such tensors are **U** and **A**, of ranks 2 and 3 respectively.

* H. Jeffreys, *Cartesian Tensors*, Chap. VII.

It is readily proved by the methods of this chapter that there is no isotropic tensor of rank 1, i.e. no isotropic vector, and that the only isotropic tensors of rank 2 are multiples of **U**.

Consider a tensor **T**, assumed to be isotropic. If the rigid body constituted by the frame of reference is moved in any way, then by definition the components of **T** with respect to a triad of reference moving with the rigid body are unaltered, and so

$$\frac{\partial \mathbf{T}}{\partial t}=0.$$

But **T** being a tensor, and so independent of the triad of reference employed, we must also have

$$\frac{d\mathbf{T}}{dt}=0.$$

These two conditions must be satisfied for all possible motions of the triad of reference, in particular for all angular velocities **Ω**.

Now take the case where **T** is a tensor of rank 1, i.e. a vector, say **P**. Then by § 207, we must have

$$0=0+\mathbf{\Omega}\wedge\mathbf{P},$$

for all **Ω**. Hence

$$\mathbf{P}=0.$$

Take next the case where **T** is of rank 2. Then by § 209

$$0=0+\mathbf{\Omega}\wedge\mathbf{T}-\mathbf{T}\wedge\mathbf{\Omega}$$

or

$$\mathbf{\Omega}\wedge\mathbf{T}=\mathbf{T}\wedge\mathbf{\Omega}$$

for all vectors **Ω**. By § 70, the only solution **T** of this equation satisfying it for all **Ω** is

$$\mathbf{T}=\lambda\mathbf{U}.$$

This is the desired result.

238. To examine isotropic tensors of rank 3 or higher, the suffix notation is required. It is convenient and instructive to give first the following alternative discussion of isotropic tensors of rank 2.

Let the members $l_{\alpha\beta}$ (not the components of a tensor) denote the set of direction cosines of the triad of unit vectors $\mathbf{i}'_\alpha$ with respect to the triad $\mathbf{i}_\beta$, as in § 38, so that

$$l_{\alpha\beta}=\mathbf{i}'_\alpha.\mathbf{i}_\beta.$$

Let us determine the values of $l_{\alpha\beta}$ when the $\mathbf{i}'_\alpha$ triad is the result of the infinitesimal displacement $\boldsymbol{\theta}$ of the given triad $\mathbf{i}_\alpha$ about the origin O, $\boldsymbol{\theta}$ being a vector. Then, if **P** is any vector rigidly attached to the given triad, $\mathbf{P}+\delta\mathbf{P}$ its value when the triad has undergone the small displacement $\boldsymbol{\theta}$, then

$$\delta\mathbf{P}=\boldsymbol{\theta}\wedge\mathbf{P}.$$

Hence

$$\mathbf{i}'_\alpha=\mathbf{i}_\alpha+\boldsymbol{\theta}\wedge\mathbf{i}_\alpha.$$

Hence $$l_{\alpha\beta}=(\mathbf{i}_\alpha+\boldsymbol{\theta}\wedge\mathbf{i}_\alpha).\mathbf{i}_\beta.$$

But $$\mathbf{i}_\alpha.\mathbf{i}_\beta=\delta_{\alpha\beta}.$$

Hence $$l_{\alpha\beta}=\delta_{\alpha\beta}+\mathbf{i}_\alpha\wedge\mathbf{i}_\beta.\boldsymbol{\theta}.$$

Now $$\mathbf{i}_\alpha\wedge\mathbf{i}_\beta=0$$
if $\alpha=\beta$. And
$$\mathbf{i}_\alpha\wedge\mathbf{i}_{\alpha'}=\mathbf{i}_{\alpha''}$$
if α, α', α'' are in cyclic order, whilst
$$\mathbf{i}_\alpha\wedge\mathbf{i}_{\alpha'}=-\mathbf{i}_{\alpha''}$$
if α, α', α'' are in non-cyclic order. If therefore, as in § 57, $\varepsilon_{\alpha\beta\gamma}$ denote the twenty-seven members such that $\varepsilon_{\alpha\beta\gamma}=0$ if any two of α, β, γ are equal, and such that $\varepsilon_{\alpha\beta\gamma}=\pm 1$ if all of α, β, γ are unequal, the upper or lower sign being taken according as the order of α, β, γ is cyclic or non-cyclic, we can write
$$\mathbf{i}_\alpha\wedge\mathbf{i}_\beta=\varepsilon_{\gamma\alpha\beta}\mathbf{i}_\gamma$$
where summation with respect to γ is implied. Hence
$$l_{\alpha\beta}=\delta_{\alpha\beta}+(\boldsymbol{\theta}.\mathbf{i}_\gamma)\varepsilon_{\gamma\alpha\beta}.$$
Now $\boldsymbol{\theta}.\mathbf{i}_\gamma$ is the γ-component of $\boldsymbol{\theta}$, which we can write as θ_γ. Thus
$$l_{\alpha\beta}=\delta_{\alpha\beta}+\theta_\gamma\varepsilon_{\gamma\alpha\beta}.$$
This formula, due to D. R. Hartree,* is a two-suffix form of the well-known relations giving the direction cosines of a slightly displaced triad with regard to the undisplaced triad. Written out, it gives the scheme

	$\beta=1$	$\beta=2$	$\beta=3$
$\alpha=1$	1	θ_3	$-\theta_2$
$\alpha=2$	$-\theta_3$	1	θ_1
$\alpha=3$	θ_2	$-\theta_1$	1

This set of numbers is easily obtained from a diagram, but the present method gives the algebraic signs unambiguously, without appeal to geometrical intuition.

Now let $\mathbf{T}$ be an isotropic tensor of rank 2, $\mathbf{T}'$ its description in the displaced triad ; then by the rule of transformation
$$T'_{\alpha\beta}=l_{\alpha\mu}l_{\beta\nu}T_{\mu\nu}.$$
But since $\mathbf{T}$ is isotropic,
$$T'_{\alpha\beta}=T_{\alpha\beta}.$$
Also
$$\begin{aligned}l_{\alpha\mu}l_{\beta\nu}&=(\delta_{\alpha\mu}+\theta_\gamma\varepsilon_{\gamma\alpha\mu})(\delta_{\beta\nu}+\theta_{\gamma'}\varepsilon_{\gamma'\beta\nu})\\&=\delta_{\alpha\mu}\delta_{\beta\nu}+\theta_\gamma(\varepsilon_{\gamma\alpha\mu}\delta_{\beta\nu}+\varepsilon_{\gamma\beta\nu}\delta_{\alpha\mu}),\end{aligned}$$

* Privately communicated.

where we have altered a dummy suffix. Combining these relations, we get

$$T_{\alpha\beta}=T_{\alpha\beta}+\theta_\gamma(\varepsilon_{\gamma\alpha\mu}T_{\mu\beta}+\varepsilon_{\gamma\beta\nu}T_{\alpha\nu}).$$

Since this must hold for all θ_γ, we must have

$$\varepsilon_{\gamma\alpha\mu}T_{\mu\beta}+\varepsilon_{\gamma\beta\nu}T_{\alpha\nu}=0.$$

Multiply by $\varepsilon_{\gamma\alpha\sigma}$ and carry out the implied summations. Then since

$$\varepsilon_{\gamma\alpha\mu}\varepsilon_{\gamma\alpha\sigma}=2\delta_{\mu\sigma},$$

$$\varepsilon_{\gamma\beta\nu}\varepsilon_{\nu\alpha\sigma}=\delta_{\beta\alpha}\delta_{\nu\sigma}-\delta_{\beta\sigma}\delta_{\nu\alpha},$$

we have $$2T_{\sigma\beta}+T_{\beta\sigma}-\delta_{\beta\sigma}T_{\alpha\alpha}=0.$$

Interchanging the suffixes σ and β we get

$$2T_{\beta\sigma}+T_{\sigma\beta}-\delta_{\beta\sigma}T_{\alpha\alpha}=0.$$

Subtracting, $$T_{\sigma\beta}-T_{\beta\sigma}=0,$$

so that **T** is self-conjugate. Hence

$$T_{\beta\sigma}=\tfrac{1}{3}T_{\alpha\alpha}\delta_{\beta\sigma},$$

so that **T** is a multiple of **U**.

239. We can now apply the same method to determine the isotropic tensors of rank 3. If $T_{\alpha\beta\gamma}$ is such a tensor, we have in the notation of the preceding section

$$T'_{\alpha\beta\gamma}=l_{\alpha\mu}l_{\beta\nu}l_{\gamma\sigma}T_{\mu\nu\sigma}=T_{\alpha\beta\gamma},$$

whence on substituting the expressions found above for the l's we get

$$\theta_\rho[\varepsilon_{\rho\alpha\mu}T_{\mu\beta\gamma}+\varepsilon_{\rho\beta\nu}T_{\alpha\nu\gamma}+\varepsilon_{\rho\gamma\sigma}T_{\alpha\beta\sigma}]=0.$$

This must be true for all θ_ρ, consequently (changing the surviving dummy suffix in each term to μ) we have

$$\varepsilon_{\rho\alpha\mu}T_{\mu\beta\gamma}+\varepsilon_{\rho\beta\mu}T_{\alpha\mu\gamma}+\varepsilon_{\rho\gamma\mu}T_{\alpha\beta\mu}=0.$$

This relation contains four independent suffixes, ρ, α, β, γ, and involves summation with respect to μ.

Taking $\rho=1$, $\alpha=1$, $\beta=1$, $\gamma=2$ we get

$$\sum_\mu \varepsilon_{12\mu}T_{11\mu}=0$$

or $$T_{113}=0.$$

Hence a component of **T** for which two suffixes are equal is zero.

Taking $\rho=3$, $\alpha=1$, $\beta=1$, $\gamma=3$, we get

$$\sum_\mu(\varepsilon_{31\mu}T_{\mu 13}+\varepsilon_{31\mu}T_{1\mu 3}+\varepsilon_{33\mu}T_{11\mu})=0,$$

or $$T_{213}+T_{123}=0.$$

Hence the interchange of a pair of adjacent suffixes changes the sign of the component. By a similar process,

$$T_{123}+T_{132}=0$$

whence $$T_{132}=-T_{123}=T_{213},$$

and, by repetition of the argument

$$T_{213}=T_{321}.$$

Hence components of **T** with three unequal suffixes take only two values which are equal and opposite. Hence $T_{\alpha\beta\gamma}=\lambda\varepsilon_{\alpha\beta\gamma}$, so that $\mathbf{T}=\lambda\mathbf{A}$.

It should be possible to derive this result synthetically, without recourse to the substitution of special values for the suffixes, by operating suitably on the relation concerned. The search for this operator may be left to the reader.

The determination of isotropic tensors of higher ranks may be carried out similarly.

CHAPTER XI

PARTICLE DYNAMICS

240. *Kinematics and dynamics.* Kinematics is the study of types of motion. Dynamics is the study of the motions which will actually occur in nature under given circumstances. To discuss these, at least on the level on which this book is written, an appeal to the results of observation is necessary. We arrange these appeals in the following way.

241. *Concept of the particle.* A *particle* is a quantity of matter whose spatio-temporal behaviour is sufficiently fully described by a position vector $\mathbf{r}$ varying with the time. It is often convenient to represent a particle to the imagination as a *small* quantity of matter, that is to say small in its linear dimensions ; so small, in fact, that its configuration at any instant t may be identified with the position of a geometrical point. But just as there is no logical necessity to associate with a geometrical point the notion of being 'without parts or magnitude,' so there is no logical necessity to consider a particle as small.

242. *Definition of zero force.* If a free particle is moving so that the rate of change of its position vector $\mathbf{r}$ with regard to the time t is constant, i.e. $d\mathbf{r}/dt=$const. vector, the particle is said to be under the action of no forces. If the particle is constrained and $d\mathbf{r}/dt=$const., the particle is said to be under the action of forces of resultant zero. These statements define zero force.

243. *Mass.* If the velocity $d\mathbf{r}/dt$ is not constant, there will be at any instant t a definite acceleration $d^2\mathbf{r}/dt^2$. Let us suppose that, in a portion of the universe which may legitimately be considered as isolated, we have two particles P_0 and P_1 which are connected to one another in some way or which influence one another's motion. The connexion may be of the nature of an elastic string or spiral spring, or the influence may be what we ordinarily call gravitation.

We introduce now the undefined concept of an *unaccelerated frame of reference.* It is difficult in an elementary treatment to say exactly what is meant by an unaccelerated frame of reference : it is any one of a class moving with uniform relative velocities with respect to one another, but further than this an unaccelerated frame is just one for which the following empirical propositions are true.

Let $\ddot{\mathbf{r}}_0$, $\ddot{}_1$ be the accelerations of the particles P_0 and P_1 in an unaccelerated frame of reference, at any instant. Then it is consistent with

the results of observation to say that $\ddot{\mathbf{r}}_0$, $\ddot{\mathbf{r}}_1$ are vectors lying in the line P_0P_1 joining the particles, and are opposite in sense, and that the ratio $|\ddot{\mathbf{r}}_1|/|\ddot{\mathbf{r}}_0|$ is the same for different distances apart of the particles, different modes of connexion between the particles and for different relative velocities of the particles provided these relative velocities are not too large.*

Choose a number m_0 and associate it with P_0. Associate with P_1 a number m_1 determined by the relation

$$m_1 = m_0\left|\frac{\ddot{\mathbf{r}}_0}{\ddot{\mathbf{r}}_1}\right|.$$

Then
$$m_1\ddot{\mathbf{r}}_1 = -m_0\ddot{\mathbf{r}}_0.$$

The vector $m_1\ddot{\mathbf{r}}_1$ is said to be the force $\mathbf{F}_{01}$ exerted by P_0 on P_1; (or, alternatively, by the elastic string, etc., on P_1); and the vector $m_0\ddot{\mathbf{r}}_0$ is said to be the force $\mathbf{F}_{10}$ exerted by P_1 on P_0 (etc.). Then

$$\mathbf{F}_{01} = m_1\ddot{\mathbf{r}}_1, \qquad \mathbf{F}_{10} = m_0\ddot{\mathbf{r}}_0,$$

and
$$\mathbf{F}_{01} + \mathbf{F}_{10} = 0.$$

Now let a new particle P_2 be substituted for P_1. By a similar procedure, it is possible to determine a number m_2 associated with P_2.

Next let the two particles P_1 and P_2 (otherwise isolated), influence one another, or be connected in any way. Then it is consistent with observation to say that if $\ddot{\mathbf{R}}_1$, $\ddot{\mathbf{R}}_2$ are their accelerations at any instant in an unaccelerated frame, then

$$m_1\ddot{\mathbf{R}}_1 = -m_2\ddot{\mathbf{R}}_2.$$

It follows that the number m_1 is characteristic of the particle P_1, independent of the comparison particle used. (If we had first compared P_2 and P_0, and then P_1 and P_2, we should have reached the same number m_2 for P_2.) This number m_1 is called the *mass* of the particle P_1. Clearly every particle P has a characteristic mass m. The mass of any one particle can be fixed arbitrarily; the masses of the rest are then determinate. In general we now have that the acceleration $\ddot{\mathbf{r}}$ of a particle P 'under the influence' of, or acted on by, a force $\mathbf{F}$, is given by

$$\mathbf{F} = m\ddot{\mathbf{r}}.$$

This is called the equation of motion of P.

244. *Equality of action and reaction.* When more than two particles are moving in one another's presence, or influencing one another, or connected with one another in certain ways, we assume as a rational generalization of the above that if $\ddot{\mathbf{r}}_n$ is the acceleration of a particle P_n of mass m_n, and if we define the force $\mathbf{F}_n$ acting on P_n by the vector equation

$$\mathbf{F}_n = m_n\ddot{\mathbf{r}}_n,$$

* New effects come into play when the velocities of the particles become appreciable compared with the velocity of light.

then $\mathbf{F}_n$ is the vector sum of a number of line vectors $\mathbf{F}_{1n}, \mathbf{F}_{2n}, \ldots, \mathbf{F}_{sn}, \ldots$ passing through P_n and such that

$$\mathbf{F}_{sn} \equiv -\mathbf{F}_{ns}.$$

This statement implies that the force acting on the sth particle can be dissected into forces $\mathbf{F}_{1s}, \mathbf{F}_{2s}, \ldots, \mathbf{F}_{ns}, \ldots$ acting along the lines joining the particles 1 and s, 2 and s, etc. The constituents $\mathbf{F}_{sn}$ are not necessarily uniquely defined by these relations, but the dissection is supposed to be possible and to satisfy these conditions subject to any other empirical rules determining the **F**'s.

245. The above contains the substance of Newton's three laws of motion. The first law is just a statement of the situation in which a particle can be recognized as moving under zero force. The second law introduces the two concepts of *force* and of *quantity of motion*, here to be defined as $m\dot{\mathbf{r}}$, but states explicitly no *a priori* way of measuring either force or mass, that is, no means of recognizing when forces or masses are equal. The third law, whilst presenting situations in which equal and opposite forces can be recognized, still affords no means of measuring forces ; but permits the recognition in nature of just one force equal in magnitude to a given force, and so permits the investigation of the action of *equal* forces on *different* particles. In conjunction with the second law, this then permits the attribution of unique mass numbers to the members of a pair of particles, given one particle as standard. It asserts tacitly the conservation of the ratio of the mass numbers for a given pair of particles exerting on different occasions different (though still equal and opposite) forces on one another. This then leads to the possibility of attaching *measures* to forces in general.

The above dynamical definition of force is linked with the statical definition of force given in Chapter VII by choosing the same mass number m_0 for the standard particle.

The mass numbers we have defined through ratios of accelerations are called 'inertial' masses ; 'inertial' because they afford a measure of the *acceleration response* of the particle concerned to a given force. We proceed to connect the 'inertial' mass of a particle with its 'gravitational' mass.

246. *Gravitational mass.* Let m_2 be the inertial mass of a particle P_2 in the presence of a particle P_1 of inertial mass m_1. Let f_{12} be the scalar acceleration of P_2 when at a distance r_{12} from P_1, relative to an unaccelerated frame of reference. When P_1 and P_2 may be considered as isolated from the rest of the universe, we have the experimental result that

$$f_{12} \propto 1/r_{12}^2,$$

for the two given particles P_1 and P_2. Put then

$$f_{12} = \lambda_{12}/r_{12}^2.$$

Then the scalar value of the force exerted by P_1 on P_2, say F_{12}, by the above definition of force is given by

$$F_{12}=m_2\frac{\lambda_{12}}{r_{12}^2}.$$

Now let P_3 be any third particle capable of being observed when isolated in the presence of P_1 above. By an extension of the notation,

$$F_{13}=m_3\frac{\lambda_{13}}{r_{13}^2}.$$

It is now an experimental result that for different particles P_2, P_3, ... in the presence of P_1, the coefficients of proportionality λ_{12}, λ_{13} determining the accelerations are all equal. Thus

$$\lambda_{12}=\lambda_{13}=\ldots=\lambda_1 .$$

The common value of these λ's is denoted by M_1, and is called the *gravitational* mass of P_1 ; for it depends only on P_1.

Now consider the motions of P_1 itself in the presence respectively of P_2, P_3, ... The scalar force F_{21} exerted by P_2 on P_1 is of the same absolute magnitude as the force F_{12} exerted by P_1 on P_2, i.e.

$$F_{21}=F_{12}.$$

Hence, using the same notation

$$\frac{\lambda_{12}m_2}{r_{12}^2}=\frac{\lambda_{21}m_1}{r_{21}^2}$$

when $r_{21}=r_{12}$. But $\lambda_{12}=M_1$, $\lambda_{21}=M_2$. Hence

$$\frac{M_1}{m_1}=\frac{M_2}{m_2}.$$

That is, gravitational mass is proportional to inertial mass.

This proportionality between gravitational and inertial mass is seen to be an immediate consequence of the experimental law that the acceleration in a gravitational field is independent of the particle being accelerated, together with the law of equality of action and reaction. In the present context, the latter is a *definition* enabling us to define inertial mass. (Compare the treatment in Mach's *Science of Mechanics*.)

247. *Momentum.* If m is the mass of a particle P, $\dot{\mathbf{r}}$ its velocity at time t, then the *line vector* $m\dot{\mathbf{r}}$ through P is called the *momentum* of P. The moment of this line vector about any point O is said to be the *moment of momentum*, or the *angular momentum*, of P about O. The *momentum of a system* of particles is the *system of line vectors* constituted by the momenta of the separate particles. In this chapter and in Chapter XII we continue to confine attention to the dynamics of a single particle.

248. *Equation of motion.* The relation

$$\mathbf{F}=m\ddot{\mathbf{r}}$$

which we have obtained as a result of ideal experiments with particles is called the *equation of motion* of the particle P, of mass m and position vector $\mathbf{r}$, under the force $\mathbf{F}$. Since m is constant, it may be written in the form

$$\mathbf{F}=\frac{d^2}{dt^2}(m\mathbf{r})=\frac{d}{dt}(m\dot{\mathbf{r}}),$$

or, in words, 'force equals rate of change of momentum.' By a rational generalization of the preceding laws, in this form it may be taken as describing the motion of a particle of varying mass m under the force $\mathbf{F}$. The notion of a varying mass is, however, not easy to define unless the mass can at any instant be isolated and experimented on as constant; nor is the notion of 'force' in this case free from difficulty. Cases are, however, best discussed as encountered.

249. *Equation of rate of change of the angular momentum of a particle.* If we multiply the equation of motion

$$\mathbf{F}=m\ddot{\mathbf{r}}$$

vectorially by $\mathbf{r}$, the result may be written in the form

$$\mathbf{r}\wedge\mathbf{F}=\frac{d}{dt}(\mathbf{r}\wedge m\dot{\mathbf{r}}).$$

But $\mathbf{r}\wedge m\dot{\mathbf{r}}$ is the angular momentum of the particle about the origin O, say $\mathbf{H}$; and $\mathbf{r}\wedge\mathbf{F}$ is the moment about the origin of the applied force $\mathbf{F}$, say $\mathbf{G}$. Thus we have

$$\mathbf{G}=\frac{d\mathbf{H}}{dt},$$

or, in words, 'moment of applied force equals rate of change of angular momentum.'

When the line of action of $\mathbf{F}$ always passes through O, $\mathbf{r}$ and $\mathbf{F}$ are in the same line, and $\mathbf{G}=\mathbf{r}\wedge\mathbf{F}=0$. Hence integrating,

$$\mathbf{H}=\text{const.},$$

or

$$\mathbf{r}\wedge m\dot{\mathbf{r}}=\text{const.}$$

250. *Areal velocity.* The vector $\frac{1}{2}\mathbf{r}\wedge\dot{\mathbf{r}}$ is a measure of the areal velocity of the particle about the origin. For the area of an elementary triangle subtended at O by the path during an interval dt is represented by the vector $\frac{1}{2}\mathbf{r}\wedge d\mathbf{r}$, or $\frac{1}{2}(\mathbf{r}\wedge\dot{\mathbf{r}})dt$. Accordingly, when the applied force passes through a fixed point, the areal velocity about this point is constant.

Further, if we call the constant areal velocity $\frac{1}{2}\mathbf{A}$, then multiplying the integral

$$\mathbf{r}\wedge\dot{\mathbf{r}}=\mathbf{A}$$

scalarly by $\mathbf{r}$, we have

$$\mathbf{r}.\mathbf{A}=0$$

Hence $\mathbf{r}$ is constantly perpendicular to the fixed vector $\mathbf{A}$, and so the path of the particle P lies in a plane perpendicular to $\mathbf{A}$. Likewise $\dot{\mathbf{r}}.\mathbf{A}=0$, or the velocity of P is also perpendicular to $\mathbf{A}$.

A force which passes through a fixed point is called a *central* force. Thus any trajectory under a central force lies in a plane through the centre of force, and is described with constant areal velocity about the centre of force.

Conversely, if the areal velocity $\frac{1}{2}(\mathbf{r}\wedge\dot{\mathbf{r}})$ is constant, then differentiating

$$\mathbf{r}\wedge\ddot{\mathbf{r}}=0$$

whence

$$\mathbf{r}\wedge\mathbf{F}=0.$$

Hence $\mathbf{F}$ is parallel to $\mathbf{r}$, and so passes through a fixed point, namely the origin about which the areal velocity is measured.

251. *Equation of rate of change of the energy of a particle.*

The work of a force $\mathbf{F}$ when its point of action undergoes a displacement $d\mathbf{r}$ is $\mathbf{F}.d\mathbf{r}$. Accordingly the rate of performance of work is $\mathbf{F}.\dot{\mathbf{r}}$. Multiplying the equation of motion scalarly by $\dot{\mathbf{r}}$ we get

$$\mathbf{F}.\dot{\mathbf{r}}=m\dot{\mathbf{r}}.\ddot{\mathbf{r}}=\frac{d}{dt}(\tfrac{1}{2}m\dot{\mathbf{r}}^2).$$

If W denotes the total work performed by the force, then $\mathbf{F}.\dot{\mathbf{r}}=dW/dt$, and so

$$W=\tfrac{1}{2}m\dot{\mathbf{r}}^2+\text{const.}$$

The scalar E given by

$$E=\tfrac{1}{2}m\dot{\mathbf{r}}^2$$

is called the *kinetic energy* of the particle. Thus, in any interval of time, the work performed by the resultant force on a particle is equal to the increase of its kinetic energy. The relation between W and E is called the *energy integral* of the motion.

252. *Motion in a conservative field of force.* Suppose now that the force $\mathbf{F}$ acting on the particle at P is a function of the vector $\mathbf{r}$. Then the work done by the force as the particle passes from P_1 to P_2 is

$$W=\int_{\mathbf{r}_1}^{\mathbf{r}_2}\mathbf{F}.d\mathbf{r}.$$

Now suppose that $\mathbf{F}$ is the negative gradient of a scalar function V of $\mathbf{r}$. Then the relation

$$\mathbf{F}=-dV/d\mathbf{r}$$

gives

$$W=-\int_{\mathbf{r}_1}^{\mathbf{r}_2}\frac{dV}{d\mathbf{r}}.d\mathbf{r}=-[V(\mathbf{r}_2)-V(\mathbf{r}_1)],$$

and so the work performed by the force depends only on the initial and final positions, and is independent of the intervening path traversed.

In such a case, the field of force $\mathbf{F}$ is said to be *conservative*, and V is said to be its potential. If a particle of mass m is in motion in this field of force, we have

$$-\frac{dV}{d\mathbf{r}}=m\ddot{\mathbf{r}}, \qquad -\frac{dV}{d\mathbf{r}}.\dot{\mathbf{r}}=m\dot{\mathbf{r}}.\ddot{\mathbf{r}},$$

whence, integrating,

$$\tfrac{1}{2}m\dot{\mathbf{r}}^2+V=\text{const.}$$

This is the form which the energy integral takes for motion in a conservative field of force. V is now said to be the *potential energy* of the particle. The sum of the kinetic and potential energies of a particle moving in a conservative field of force is accordingly constant.

CHAPTER XII

TYPES OF PARTICLE MOTION

253. *Resisted motion under gravity. Resistance proportional to velocity.* Let a particle be projected from the origin with velocity $\mathbf{V}_0$, under the action of gravity and of a resistance proportional to the velocity. If $\mathbf{r}$ is the position vector at time t, $\mathbf{V}$ the velocity, $-k\mathbf{V}$ the deceleration due to resistance, and if $\mathbf{z}$ is a unit vector vertically upwards, the equation of motion gives

$$\frac{d\mathbf{V}}{dt} = -g\mathbf{z} - k\mathbf{V}.$$

If we put $d\mathbf{V}/dt = \mathbf{f}$, differentiation of the last equation gives

$$\frac{d\mathbf{f}}{dt} = -k\mathbf{f}$$

or

$$\frac{d}{dt}(\mathbf{f}e^{kt}) = 0.$$

Hence

$$\mathbf{f} = \mathbf{f}_0 c^{-kt},$$

where $\mathbf{f}_0$ is a constant vector. Hence the deceleration is always parallel to a fixed vector and decreases exponentially with the time. As $t \to \infty$ $\mathbf{f} \to 0$, i.e. $d\mathbf{V}/dt \to 0$ and so $\mathbf{V} \to -g\mathbf{z}/k$. This is accordingly the limiting velocity.

The equation of motion may be integrated once as it stands, since $\mathbf{V} = d\mathbf{r}/dt$, giving

$$\mathbf{V} = -gt\mathbf{z} - k\mathbf{r} + \mathbf{V}_0.$$

If $\mathbf{x}$ is a unit horizontal vector, the horizontal displacement is given by

$$\mathbf{r}.\mathbf{x} = \mathbf{x}.\left(\frac{\mathbf{V}_0 - \mathbf{V}}{k}\right),$$

whence, letting $t \to \infty$ and $\mathbf{V} \to -g\mathbf{z}/k$, we have

$$\lim_{t \to \infty} \mathbf{r}.\mathbf{x} = \mathbf{x}.\mathbf{V}_0/k.$$

There is accordingly a limiting horizontal range.

The maximum height occurs when $\mathbf{V}.\mathbf{z} = 0$. Hence maximum height is attained when

$$\mathbf{r}.\mathbf{z} = \frac{\mathbf{V}_0.\mathbf{z} - gt}{k},$$

but this does not of itself determine the maximum height, since here t, the time of flight, is unknown. To determine this we need a second integral.

The integral of the equation of motion already obtained may be written, after multiplication by e^{kt}, in the form

$$\frac{d}{dt}(\mathbf{r}e^{kt}) = -gte^{kt}\mathbf{z} + \mathbf{V}_0 e^{kt}.$$

This integrates as it stands in the form

$$\mathbf{r}e^{kt} = \frac{\mathbf{V}_0}{k}(e^{kt} - 1) + \frac{g}{k^2}(e^{kt} - kte^{kt} - 1)\mathbf{z}.$$

where we have added constants to make $\mathbf{r} = 0$ at $t = 0$. This gives

$$\mathbf{r} = \frac{\mathbf{V}_0}{k}(1 - e^{-kt}) + \frac{g}{k^2}(1 - kt - e^{-kt})\mathbf{z}.$$

Hence at any time t,

$$\mathbf{V} = \frac{d\mathbf{r}}{dt} = \mathbf{V}_0 e^{-kt} - \frac{g}{k}(1 - e^{-kt})\mathbf{z}.$$

This gives **V** explicitly as a function of t. The time of flight to maximum height, t_0, is a root of the equation obtained by putting $\mathbf{V}.\mathbf{z} = 0$, namely

$$e^{-kt_0}(\mathbf{V}_0.\mathbf{z}) = \frac{g}{k}(1 - e^{-kt_0}).$$

Hence

$$[k(\mathbf{V}_0.\mathbf{z}) + g] = ge^{kt_0},$$

or $$t_0 = \frac{1}{k}\log\left[1 + \frac{k}{g}(\mathbf{V}_0.\mathbf{z})\right].$$

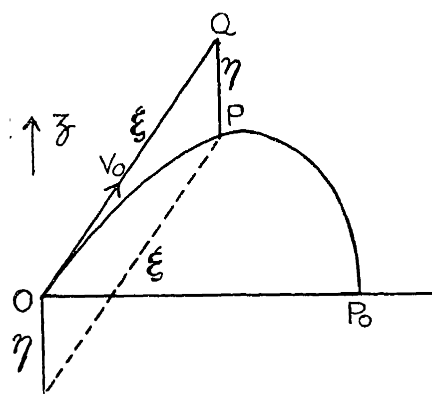

Fig. 60

From the equation giving **r** explicitly as a function of t, it appears that the natural co-ordinates to use for defining the position of the moving particle P are $\xi = OQ$ and $\eta = QP$ (Fig. 60), where Q is the point on the initial tangent vertically above P. The equation then gives

$$\mathbf{r} = \frac{\mathbf{V}_0}{|\mathbf{V}_0|}\xi(t) - \eta(t)\mathbf{z},$$

where $$\xi(t) = \frac{1 - e^{-kt}}{k}|\mathbf{V}_0|$$

$$\eta(t) = g\frac{-1 + kt + e^{-kt}}{k^2}.$$

It will be observed that $\xi(t)$ and $\eta(t)$ are functions of t only, and are independent of the *direction* of projection ; the drop from the tangent, $\eta(t)$ is in fact independent of the magnitude of $|\mathbf{V}_0|$. We notice that

$$\xi(t) < |\mathbf{V}_0|t, \qquad \eta(t) < \tfrac{1}{2}gt^2$$

It follows that for given $|\mathbf{V}_0|$ and constant t, if the angle of projection (i.e. the direction of $\mathbf{V}_0$) is varied, the locus of P is a circle centred at a point C distant $\eta(t)$ from O and vertically below O, and of radius $\xi(t)$.

Let P, P_0 (Fig. 61) be two points reached in the same time of flight t, P_0 being on the horizontal through O. Let α be the angle of projection

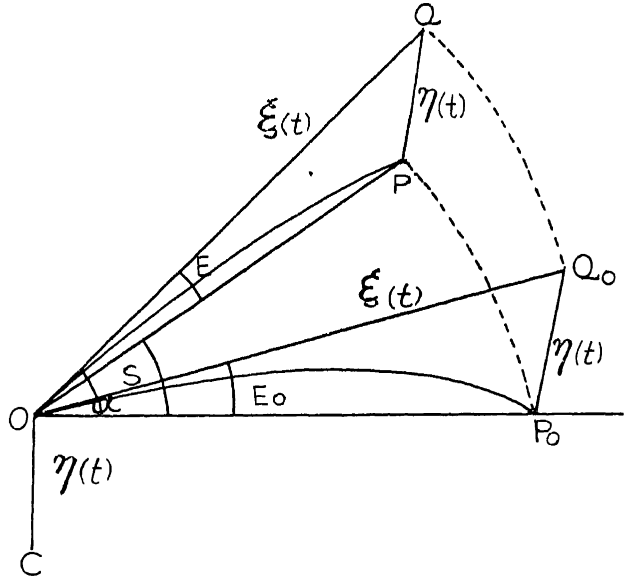

Fig. 61

for P, i.e. the angle QOP_0, S the 'angle of sight' of P, i.e. the angle POP_0, E the 'tangent elevation,' for P, i.e. the angle QOP. For P_0, the value of S is zero, and the angle of projection is the tangent elevation E_0. We now establish an interesting relation between E, E_0 and S.

We have

$$\sin E_0 = \frac{Q_0P_0}{OQ_0} = \frac{\eta(t)}{\xi(t)},$$

and
$$\frac{\sin E}{\sin(\frac{1}{2}\pi - \alpha)} = \frac{QP}{OP} = \frac{QP}{OQ\cos\alpha\sec S} = \frac{\eta(t)}{\xi(t)\cos\alpha}\cos S.$$

Hence
$$\sin E = \sin E_0 \cos S.$$

This formula enables E to be calculated for any angle of sight S in terms of its value for the same time of flight to the horizontal. The gunners'

'cosine rule' is an approximate formula of this type, save that it specifies constant range instead of constant time of flight. The interest of the type of formula we have obtained lies in the absence of any explicit mention of resistance (k) or muzzle velocity ($|\mathbf{V}_0|$), so that it is the type of formula to be expected to hold good approximately for other laws of resistance.

Example. Prove that with the same notation, for constant time of flight,

$$\mathrm{OP} = \mathrm{OP}_0\left[\frac{\cos \mathrm{E}}{\cos \mathrm{E}_0} - \sin \mathrm{S} \tan \mathrm{E}_0\right].$$

Another interesting property which is also a consequence of the possibility of expressing the motion by means of two functions $\xi(t)$ and

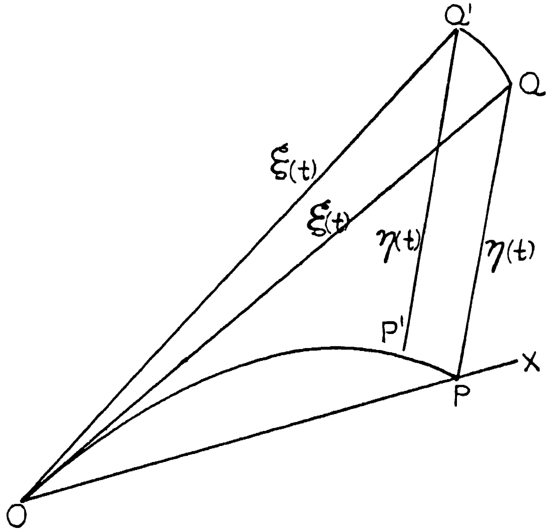

Fig. 62

$\eta(t)$ is that if P is the point of fall on a given inclined plane for the maximum value of the range OP on that plane (for given $|\mathbf{V}_0|$, then the tangents to the trajectory at O and P are perpendicular.

For, let the tangent at O meet the vertical through P in Q (Fig. 62). Then, if t is the time of flight to P, $\mathrm{OQ} = \xi(t)$ and $\mathrm{QP} = \eta(t)$. Take a point Q′, near Q, such that OQ′ = OQ, and take P′, vertically below Q′, and such that Q′P′ = QP. Then if the particle is projected from O along OQ′ with the given velocity of projection $|\mathbf{V}_0|$, then it will reach P′ in the same time of flight t. Since OP is a maximum, this neighbouring trajectory must have to the first order the same range OP along OPX as the trajectory for projection along OQ. Hence the particle projected along OQ′ and passing through P′ must pass approximately through P, so that PP′ is parallel to the tangent at P to the original trajectory. But

since $Q'P'=QP$, $P'P$ is parallel to $Q'Q$, which is perpendicular to OQ since $OQ=OQ'=\xi(t)$. Hence, when OP is a maximum for the inclined plane OX, the tangents at P and O are perpendicular.

The actual angle of descent of a trajectory to the ground is readily found. If $\mathbf{x}$ is as before a unit horizontal vector, we have at any point

$$\mathbf{V}.\mathbf{x}=(\mathbf{V}_0.\mathbf{x})e^{-kt}$$

and
$$\mathbf{V}.\mathbf{z}=(\mathbf{V}_0.\mathbf{z})e^{-kt}-\frac{g}{k}(1-e^{-kt}).$$

whilst at the point of fall we have

$$0=\mathbf{r}.\mathbf{z}=\frac{\mathbf{V}_0.\mathbf{z}}{k}(1-e^{-kt})+\frac{g}{k^2}(1-e^{-kt}-kt).$$

Eliminating g by means of the last equation, we find, if ω is the angle of descent

$$\tan\omega=-\frac{\mathbf{V}.\mathbf{z}}{\mathbf{V}.\mathbf{x}}=\frac{\mathbf{V}_0.\mathbf{z}}{\mathbf{V}_0.\mathbf{x}}\cdot\frac{e^{kt}-(1+kt)}{e^{-kt}-(1-kt)}=\tan\alpha\frac{e^{kt}-(1+kt)}{e^{-kt}-(1-kt)}.$$

Example. Prove that $\omega>\alpha$.

The above are merely to be taken as illustrations of the use of vector methods in solving problems of resisted motion under a linear law of resistance.

254. *Motion of a particle in a circle.* Let a particle P of mass m be in uniform motion with speed v in a circle of radius a. If $\mathbf{i}$ is a unit vector normal to the plane of the circle, in an appropriate sense, we have for the motion

$$\frac{d\mathbf{r}}{dt}=\frac{v}{a}\mathbf{i}\wedge\mathbf{r}, \qquad (\mathbf{i}.\mathbf{r}=0),$$

where $\mathbf{r}$ is the position vector of P with the centre of the circle as origin. Hence the acceleration of P is given by

$$\frac{d^2\mathbf{r}}{dt^2}=\frac{v}{a}\mathbf{i}\wedge\frac{d\mathbf{r}}{dt}=\frac{v^2}{a^2}\mathbf{i}\wedge(\mathbf{i}\wedge\mathbf{r})=-\frac{v^2}{a^2}\mathbf{r}.$$

Since $|\mathbf{r}|=a$, the acceleration is of absolute value v^2/a, and is directed inwards to the centre of the circle. This is, of course, merely a particular case of the result of § 230, that the acceleration of any particle in motion along a curve of curvature $1/\rho$ with speed v is dv/dt along the tangent and v^2/ρ along the principal normal.

Simple as this example is, the equivalence of the equation

$$d\mathbf{r}/dt=\omega(\mathbf{i}\wedge\mathbf{r}), \qquad (\mathbf{i}.\mathbf{r}=0),$$

to the equation
$$d^2\mathbf{r}/dt^2=-\omega^2\mathbf{r}$$

is the principle underlying the solutions of a host of problems by vector methods.

255. *Simple harmonic motion. The motion of a particle under a force to a fixed point proportional to its distance.* Let m be the mass of the particle, $-m\mu\mathbf{r}$ the external force. Then the equation of motion reduces to

$$\frac{d^2\mathbf{r}}{dt^2} = -\mu\mathbf{r}. \qquad (\mu > 0). \quad (1)$$

Multiplying scalarly by $d\mathbf{r}/dt$ and integrating we have

$$\tfrac{1}{2}\left(\frac{d\mathbf{r}}{dt}\right)^2 = -\tfrac{1}{2}\mu\mathbf{r}^2 + \text{const.},$$

which is the energy integral. Multiplying vectorially by $\mathbf{r}$ we have

$$\mathbf{r}\wedge\frac{d^2\mathbf{r}}{dt^2} = 0$$

or, integrating,
$$\mathbf{r}\wedge\frac{d\mathbf{r}}{dt} = \text{const.} = \mathbf{A}, \qquad (2)$$

say. Then $\mathbf{r}.\mathbf{A} = 0$, so that the vector $OP = \mathbf{r}$ always lies in the plane through O normal to $\mathbf{A}$, and the motion lies wholly in one plane. Similarly, $(d\mathbf{r}/dt).\mathbf{A} = 0$. The vector $\mathbf{A}$ is proportional to the angular momentum of P about O, namely $\mathbf{r}\wedge m\frac{d\mathbf{r}}{dt}$, which is accordingly constant. As we have already seen (§ 250), these are general properties of the motion of a particle under a force to a fixed centre.

The constant vector $\mathbf{A}$ is fixed in direction by the directions of $\mathbf{r}$ and $d\mathbf{r}/dt$ at some given initial instant $t = t_0$. We now require to determine the motion in the plane normal to $\mathbf{A}$.

Let $\mathbf{i}$ be a unit vector parallel to $\mathbf{A}$, so that

$$\mathbf{i}.\mathbf{r} = 0.$$

Consider the motion defined by

$$\frac{d\mathbf{r}}{dt} = \omega\mathbf{i}\wedge\mathbf{r},$$

where ω is some constant. This gives

$$\frac{d^2\mathbf{r}}{dt^2} = \omega\mathbf{i}\wedge(\omega\mathbf{i}\wedge\mathbf{r}) = -\omega^2\mathbf{r},$$

and accordingly will satisfy the equation of motion (1) provided that

$$\omega^2 = \mu, \qquad \omega = \pm\mu^{\frac{1}{2}}.$$

Now consider two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ of arbitrary constant modulus and arbitrary initial position, satisfying $\mathbf{i}.\mathbf{r}_1 = 0$, $\mathbf{i}.\mathbf{r}_2 = 0$, and rotating with constant angular speeds $+\mu^{\frac{1}{2}}$, $-\mu^{\frac{1}{2}}$ according to the equations

$$\left.\frac{d\mathbf{r}_1}{dt} = \mu^{\frac{1}{2}}\mathbf{i}\wedge\mathbf{r}_1, \qquad \frac{d\mathbf{r}_2}{dt} = -\mu^{\frac{1}{2}}\mathbf{i}\wedge\mathbf{r}_2.\right\} \quad (3)$$

Put
$$\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2.$$

Then $\mathbf{r}$ satisfies (1). Moreover this is the most general solution of (1). For the most general solution of (1) will originate when P is projected, at a given time $t=t_0$, through an arbitrary position vector $\mathbf{r}_0$ with an arbitrary velocity $(d\mathbf{r}/dt)_0$. Each of the latter vectors is equivalent to three scalar numbers, so that the general solution of (1) will contain the equivalent of six arbitrary constants. But the solution (3) contains the equivalent of six arbitrary constants, for the unit vector $\mathbf{i}$ is equivalent to the choice of two constants, and once $\mathbf{i}$ has been chosen the initial vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ normal to $\mathbf{i}$ are equivalent to the choice of four more constants, making six in all.

The most general solution $\mathbf{r}$ of (1) may thus be constructed as the vector sum of two vectors of constant modulus, rotating in opposite senses with angular velocities $\pm\mu^{\frac{1}{2}}\mathbf{i}$. The loci so obtained are particular cases of Lissajous's figures. The period is $2\pi/\mu^{\frac{1}{2}}$.

It will be noted that we have effected a kinematic description of the most general motion of P when subject to the second order differential equation (1) by means of two *first* integrals given by (3) together with the information contained in (2).

It is true that we have not expressed $\mathbf{r}$ as an explicit function of t. We have, however, obtained a picture of the *structure of the motion*, and this is actually more useful and more insight-giving than an explicit expression for $\mathbf{r}$ as a function of t.

The solution of a second order vector differential equation by means of uniformly rotating vectors is as fundamental in vector calculus as the solution of a scalar differential equation by means of exponentials. It is, however, in principle simpler. The integration of an equation of motion is effectively complete when we have obtained a description of the motion, and we secure this by one integration, not two. The vector method yields the moving picture of the motion one stage earlier than the complete integration of the corresponding three scalar equations and subsequent combination of the three solutions.

That the solution (3) is consistent with the integral (2) follows since

$$\mathbf{r}\wedge\frac{d\mathbf{r}}{dt}=(\mathbf{r}_1+\mathbf{r}_2)\wedge\left(\frac{d\mathbf{r}_1}{dt}+\frac{d\mathbf{r}_2}{dt}\right)=(\mathbf{r}_1+\mathbf{r}_2)\wedge[\mu^{\frac{1}{2}}(\mathbf{i}\wedge\mathbf{r}_1)-\mu^{\frac{1}{2}}(\mathbf{i}\wedge\mathbf{r}_2)]$$
$$=\mu^{\frac{1}{2}}\mathbf{i}(\mathbf{r}_1{}^2-\mathbf{r}_2{}^2),$$

and the right-hand side here is constant since $\mathbf{r}_1$ and $\mathbf{r}_2$ have constant moduli. It follows that the angular momentum about O is

$$m\mu^{\frac{1}{2}}(\mathbf{r}_1{}^2-\mathbf{r}_2{}^2).$$

As a corollary, the angular momentum vanishes if $|\mathbf{r}_1|=|\mathbf{r}_2|$. In that case, the direction of the velocity must pass through O, and the particle must describe a straight line through O. Thus the combination of two circular motions of equal radius and opposite angular velocities results in rectilinear motion through the origin.

The velocity of P is given by

$$\frac{d\mathbf{r}}{dt}=\mu^{\frac{1}{2}}\mathbf{i}\wedge(\mathbf{r}_1-\mathbf{r}_2).$$

Thus if $OM=\mathbf{r}_1$, $MP=\mathbf{r}_2$ (Fig. 63), so that P is the position of the particle, and if we produce PM to P′ so that MP′=PM, then P has always the velocity of P′, namely a rotation about O with angular velocity $\mu^{\frac{1}{2}}\mathbf{i}$ (see Fig. 63). When $|\mathbf{r}_1|=|\mathbf{r}_2|$, $d\mathbf{r}/dt$ must be constant in direction, as we have seen, and therefore $\mathbf{r}_1-\mathbf{r}_2$ must be constant in direction. This is obvious geometrically (see Fig. 64).

Motion governed by an equation of type (1) is called *simple harmonic motion*. It is clear that simple harmonic motion is essentially a *two-*

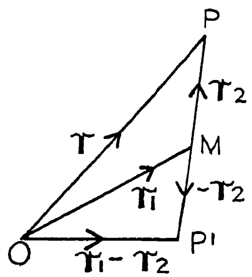

Fig. 63

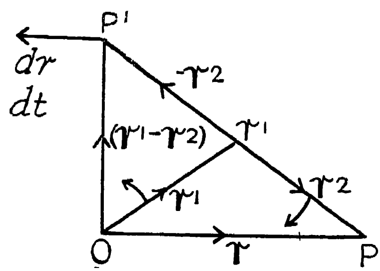

Fig. 64

dimensional motion, reducing as a particular case to rectilinear motion. The term simple harmonic motion is often confined to the case of rectilinear motion. The vector method, however, insists on the consideration of the two-dimensional case first.

Another particular case is when one of the rotating vectors has modulus zero. In that case the particle P describes a circle.

The most general form of the algebraic locus of P is obtained by writing down the Cartesian equations corresponding to the sum of two oppositely rotating vectors of the form

$$\mathbf{r}_1=a_1[\mathbf{j}\cos(\mu^{\frac{1}{2}}t+\varepsilon_1)+\mathbf{k}\sin(\mu^{\frac{1}{2}}t+\varepsilon_1)],$$

$$\mathbf{r}_2=a_2[\mathbf{j}\cos(-\mu^{\frac{1}{2}}t+\varepsilon_2)+\mathbf{k}\sin(-\mu^{\frac{1}{2}}t+\varepsilon_2)],$$

namely $$x=(\mathbf{r}_1+\mathbf{r}_2).\mathbf{j}, \qquad y=(\mathbf{r}_1+\mathbf{r}_2).\mathbf{k}$$

and eliminating t. The resulting equation is that of an ellipse, centre the origin. But for any required properties of the ellipse, it is best to return to the kinematic definition of the locus as the vector sum of two arbitrary, oppositely rotating vectors. For example, the extremities of the principal axes of the ellipse occur when

$$\mathbf{r}.\frac{d\mathbf{r}}{dt}=0,$$

i.e.
$$\mu^{\frac{1}{2}}(\mathbf{r}_1+\mathbf{r}_2).\mathbf{i}\wedge(\mathbf{r}_1-\mathbf{r}_2)=0$$
or
$$(\mathbf{r}_1\wedge\mathbf{r}_2).\mathbf{i}=0.$$
This requires $\mathbf{r}_1\wedge\mathbf{r}_2=0$, or $\mathbf{r}_1$ to be parallel to $\mathbf{r}_2$ (or antiparallel). These give the directions of the major and minor axes respectively.

256. *Repulsive force proportional to distance.* The equation of motion is in this case
$$\frac{d^2\mathbf{r}}{dt^2}=+\mu\mathbf{r}. \qquad (\mu>0).$$
As usual we have an energy integral,
$$\tfrac{1}{2}\left(\frac{d\mathbf{r}}{dt}\right)^2=\tfrac{1}{2}\mu\mathbf{r}^2+\text{const.},$$
and an angular momentum integral
$$\mathbf{r}\wedge\frac{d\mathbf{r}}{dt}=\text{const.}=\mathbf{A}.$$
As before, since $\mathbf{r}.\mathbf{A}=0$, $\mathbf{r}$ lies in a fixed plane.

To solve the equation of motion, let us seek a solution of the form
$$\frac{d\mathbf{r}}{dt}=\omega\mathbf{i}\wedge\mathbf{r}-k\mathbf{r},$$
where
$$\mathbf{i}.\mathbf{r}=0.$$
Then
$$\frac{d^2\mathbf{r}}{dt^2}=\omega\mathbf{i}\wedge(\omega\mathbf{i}\wedge\mathbf{r}-k\mathbf{r})-k(\omega\mathbf{i}\wedge\mathbf{r}-k\mathbf{r})$$
$$=(k^2-\omega^2)\mathbf{r}-2k\omega(\mathbf{i}\wedge\mathbf{r}).$$
This solution will accordingly satisfy the equation of motion only if
$$2k\omega=0, \qquad k^2-\omega^2=\mu.$$
Hence either $k=0$, or $\omega=0$. If $k=0$, we get $\omega^2=-\mu$, so that μ must be negative if ω is real, and we recover the case of attraction. If $\omega=0$, then $k=\pm\mu^{\frac{1}{2}}$. A solution is accordingly
$$\mathbf{r}=\mathbf{r}_1+\mathbf{r}_2,$$
where $\mathbf{r}_1$, $\mathbf{r}_2$ are vectors satisfying
$$\frac{d\mathbf{r}_1}{dt}=-\mu^{\frac{1}{2}}\mathbf{r}_1, \qquad \frac{d\mathbf{r}_2}{dt}=+\mu^{\frac{1}{2}}\mathbf{r}_2.$$
These can be written in the forms
$$\frac{d}{dt}(\mathbf{r}_1e^{\mu^{\frac{1}{2}}t})=0, \qquad \frac{d}{dt}(\mathbf{r}_2e^{-\mu^{\frac{1}{2}}t})=0,$$
which integrate in the forms
$$\mathbf{r}_1=e^{-\mu^{\frac{1}{2}}t}\mathbf{B}, \qquad \mathbf{r}_2=e^{+\mu^{\frac{1}{2}}t}\mathbf{C}.$$

This solution, containing two arbitrary vectors $\mathbf{B}$ and $\mathbf{C}$ satisfying $\mathbf{A}.\mathbf{B}=0$, $\mathbf{A}.\mathbf{C}=0$, involving four arbitrary constants, is thus the most general solution. We see that $\mathbf{r}$ is the sum of two vectors lying in fixed directions, of which one decreases exponentially with the time, the other increases. For t large, the path tends to become parallel to $\mathbf{C}$. The limiting length of the perpendicular from the origin to a line through P parallel to $\mathbf{C}$ is clearly

$$\lim \frac{|\mathbf{r}\wedge\mathbf{C}|}{|\mathbf{C}|}$$

which is zero. The path of the particle thus has an asymptote through the origin parallel to $\mathbf{C}$, and there is clearly a second asymptote in the direction of $\mathbf{B}$ as $t\to-\infty$. Moreover

$$|\mathbf{r}_1|\ |\mathbf{r}_2|=|\mathbf{B}|\ |\mathbf{C}|=\text{const.},$$

so that the path is a hyperbola with asymptotes through the origin.

If the particle is projected at time $t=0$ from the position $\mathbf{r}_0$ with velocity $\mathbf{V}_0$, then

$$\mathbf{r}_0=\mathbf{B}+\mathbf{C}, \qquad \mathbf{V}_0=-\mu^{\frac{1}{2}}(\mathbf{B}-\mathbf{C}),$$

whence the asymptotes, namely lines parallel to $\mathbf{B}$ and $\mathbf{C}$, lie along the vectors $\mathbf{r}_0\pm\mu^{-\frac{1}{2}}\mathbf{V}_0$.

257. *The equiangular spiral.* We have now considered the two distinct types of motion defined by

$$\frac{d\mathbf{r}}{dt}=\omega\mathbf{i}\wedge\mathbf{r}$$

and

$$\frac{d\mathbf{r}}{dt}=-k\mathbf{r}.$$

This suggests examination of the motion defined by the single vector equation

$$\frac{d\mathbf{r}}{dt}=\omega\mathbf{i}\wedge\mathbf{r}-k\mathbf{r}.$$

We notice first that this can be written

$$\frac{d}{dt}(\mathbf{r}e^{kt})=\omega\mathbf{i}\wedge\mathbf{r}e^{kt}.$$

Putting

$$\mathbf{r}e^{kt}=\boldsymbol{\rho},$$

we have

$$\frac{d\boldsymbol{\rho}}{dt}=\omega\mathbf{i}\wedge\boldsymbol{\rho},$$

so that $\boldsymbol{\rho}$ is a vector of constant modulus rotating about $\mathbf{i}$ with angular velocity ω. Hence $\mathbf{r}=\boldsymbol{\rho}e^{-kt}$ is a vector rotating about $\mathbf{i}$ and shrinking exponentially with the time, if $k>0$. The point $\mathbf{r}$ therefore always lies on the surface of a cone.

Let α be the semi-vertical angle of this cone so that $\boldsymbol{\rho}.\mathbf{i}=|\boldsymbol{\rho}|\cos\alpha$. Then the position and velocity of the particle are specified by

$$\mathbf{r}=\boldsymbol{\rho}e^{-kt}, \qquad \frac{d\mathbf{r}}{dt}=(\omega\mathbf{i}\wedge\boldsymbol{\rho}-k\boldsymbol{\rho})e^{-kt}.$$

The locus of $\mathbf{r}$ cuts the generators of the cone at an angle ψ whose tangent is given by

$$\tan\psi=\left|\frac{\mathbf{r}\wedge\frac{d\mathbf{r}}{dt}}{\mathbf{r}.\frac{d\mathbf{r}}{dt}}\right|=\frac{\omega|\boldsymbol{\rho}\wedge(\mathbf{i}\wedge\boldsymbol{\rho})|}{k\rho^2}=\frac{\omega}{k}\frac{|\mathbf{i}\wedge\boldsymbol{\rho}|}{|\boldsymbol{\rho}|}=\frac{\omega}{k}\sin\alpha.$$

When $\mathbf{r}.\mathbf{i}=0$, $\alpha=\frac{1}{2}\pi$, the locus of $\mathbf{r}$ lies in a plane through O perpendicular to $\mathbf{i}$, and the locus meets all radii from O at a constant angle ψ. The locus is therefore an equiangular spiral. In this case, $k=\omega\cot\psi$. Accordingly, an equiangular spiral of angle ψ is defined by the equation

$$\frac{d\mathbf{r}}{dt}=\omega(\mathbf{i}\wedge\mathbf{r}-\cot\psi\,\mathbf{r}), \qquad (\mathbf{r}.\mathbf{i}=0).$$

The spiral is described by the particle with uniform angular velocity ω. Writing the equation in the form

$$\frac{d}{dt}(\mathbf{r}e^{\omega t\cot\psi})=\omega\mathbf{i}\wedge\mathbf{r}e^{\omega t\cot\psi}$$

we see that if $0<\psi<\frac{1}{2}\pi$, the vector $\mathbf{r}e^{\omega t\cot\psi}$ rotates with constant angular velocity ω, and so $\mathbf{r}$ shrinks exponentially with the time.

258. *Damped motion under an attractive or repelling force varying as the distance.* The equation of motion is

$$\frac{d^2\mathbf{r}}{dt^2}=-\mu\mathbf{r}-2k\frac{d\mathbf{r}}{dt}, \qquad (k>0),$$

where we have represented the damping by a deceleration proportional to but opposite to the velocity. The natural way of seeking an integral of this vector differential equation is to attempt to determine an integrating factor $e^{\lambda t}$ and a scalar constant, a, such that the equation can be put in the form

$$\frac{d}{dt}\left(e^{\lambda t}\frac{d\mathbf{r}}{dt}\right)=a\frac{d}{dt}(e^{\lambda t}\mathbf{r}).$$

Since this reduces to

$$\frac{d^2\mathbf{r}}{dt^2}=-(\lambda-a)\frac{d\mathbf{r}}{dt}+\lambda a\mathbf{r},$$

the given equation can be put in the required form if we can find numbers λ and a such that

$$\lambda-a=2k, \qquad \lambda a=-\mu.$$

Eliminating a, we have

$$\lambda^2-2k\lambda+\mu=0,$$

whence
$$\lambda=\lambda_1,\ \lambda_2,=k\pm(k^2-\mu)^{\frac{1}{2}},$$
$$a=a_1,\ a_2,=-k\pm(k^2-\mu)^{\frac{1}{2}}.$$

The values of λ and a are accordingly real if $k^2>\mu$.

Case (i). $k^2>\mu$. In this case a first integral of the motion is obtained in either of the forms

$$\frac{d\mathbf{r}}{dt}=a_1\mathbf{r}+\mathbf{A}_1e^{-\lambda_1 t},$$

$$\frac{d\mathbf{r}}{dt}=a_2\mathbf{r}+\mathbf{A}_2e^{-\lambda_2 t},$$

where $\mathbf{A}_1$, $\mathbf{A}_2$ are vector constants. A first integral can, however, at most contain *one* vector constant, and so these two first integrals must be identical. Hence $\mathbf{r}$ must be of the form

$$\mathbf{r}=\mathbf{A}e^{-\lambda_1 t}+\mathbf{B}e^{-\lambda_2 t},$$

and as this involves two vector constants it must be the most general form of the second integral of the original differential equation.

This can be seen otherwise by writing either of the first integrals in the form

$$\frac{d}{dt}(\mathbf{r}e^{-at})=\mathbf{A}e^{-(\lambda+a)t},$$

which integrates to give

$$\mathbf{r}=-\frac{\mathbf{A}}{\lambda+a}e^{-\lambda t}+\mathbf{B}e^{+at} \qquad (\lambda+a\neq 0).$$

Whichever value we choose for λ, with the corresponding constant a, this expression for $\mathbf{r}$ reduces to

$$\mathbf{r}=\mathbf{A}_1e^{-[k+(k^2-\mu)^{\frac{1}{2}}]t}+\mathbf{B}e^{[-k+(k^2-\mu)^{\frac{1}{2}}]t},$$

in agreement with the earlier method.

We observe that if $k^2>\mu>0$, the indices of both the exponentials are negative. Thus if $k^2>\mu>0$, $r\to 0$ as $t\to\infty$, and the motion is fully damped and aperiodic. If $k^2>0>\mu$, $r\to\infty$ as $t\to\infty$.

Case (ii). $k^2=\mu$. Then $\lambda+a=0$, and the first integral becomes

$$\frac{d}{dt}(\mathbf{r}e^{kt})=\mathbf{A},$$

of which the second integral is

$$\mathbf{r}=\mathbf{A}te^{-kt}+\mathbf{B}e^{-kt}.$$

The motion is again damped and aperiodic. The damping is said to be *critical*. In all cases, the motion lies in one plane, namely the plane of the vectors **A** and **B**.

Case (iii). $k^2<\mu$. In this case λ is complex. The solution for $\mathbf{r}$ can be made real by choosing for **A** and **B** complex vector constants. This is a rapid and convenient way of obtaining formal algebraic results, but to attach a rational meaning to the steps involved we should have to define complex vectors. This would require reconstruction of our vector algebra. It is in principle simpler and more interesting to obtain real solutions as follows.

Let us seek a first integral of the form

$$\frac{d\mathbf{r}}{dt}=\omega\mathbf{i}\wedge\mathbf{r}-\lambda\mathbf{r}, \qquad (\mathbf{r}.\mathbf{i}=0).$$

Then
$$\frac{d^2\mathbf{r}}{dt^2}=\omega\mathbf{i}\wedge(\omega\mathbf{i}\wedge\mathbf{r}-\lambda\mathbf{r})-\lambda(\omega\mathbf{i}\wedge\mathbf{r}-\lambda\mathbf{r})$$
$$=(\lambda^2-\omega^2)\mathbf{r}-2\lambda\omega\mathbf{i}\wedge\mathbf{r}.$$

This will be a solution of the given vector differential equation provided that

$$(\lambda^2-\omega^2)\mathbf{r}-2\lambda\omega\mathbf{i}\wedge\mathbf{r}\equiv-\mu\mathbf{r}-2k[\omega(\mathbf{i}\wedge\mathbf{r})-\lambda\mathbf{r}].$$

This identity requires
$$\lambda=k,$$
$$\lambda^2-\omega^2=2k\lambda-\mu.$$

Hence
$$\omega^2=\mu-k^2, \qquad \omega=\pm(\mu-k^2)^{\frac{1}{2}}.$$

Since in the present case (iii), $k^2<\mu$, ω is real.

By § 257, the motion of $\mathbf{r}$, for either value of ω, is along an equiangular spiral. The most general solution is therefore of the form

$$\mathbf{r}=\mathbf{r}_1+\mathbf{r}_2,$$

where $\mathbf{r}_1$, $\mathbf{r}_2$ are two uniformly but oppositely rotating exponentially shrinking vectors of arbitrary initial values. This solution, involving *two* arbitrary vector constants, has the desired degree of generality. The motion is *damped periodic*.

The reader will have recognized that the methods we have been using are equivalent to the methods customarily used for solving scalar linear differential equations. But instead of using exponentials with complex indices, we have gained more insight by obtaining kinematic first integrals containing the motion of rotation.

259. *The pendulum*. Let a particle P of mass m be suspended by a string of length l from a fixed point O. Let $\mathbf{i}$ be a unit vector along OP, $\mathbf{z}$ a unit vector vertically downwards. Then the equation of motion of P is

$$ml\frac{d^2\mathbf{i}}{dt^2}=-T\mathbf{i}+mg\mathbf{z},$$

where T is the tension of the string. To eliminate T, multiply vectorially by $\mathbf{i}$. We get

$$l\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}=g\mathbf{i}\wedge\mathbf{z}. \tag{1}$$

This is the fundamental equation giving the motion of the pendulum.

To obtain a first integral, multiply vectorially by $\frac{d\mathbf{i}}{dt}$. Since $\mathbf{i}^2=1$, $\mathbf{i}.d\mathbf{i}/dt=0$, the continued vector products give

$$l\mathbf{i}\left(\frac{d^2\mathbf{i}}{dt^2}.\frac{d\mathbf{i}}{dt}\right)=g\mathbf{i}\left(\mathbf{z}.\frac{d\mathbf{i}}{dt}\right),$$

or

$$l\frac{d^2\mathbf{i}}{dt^2}.\frac{d\mathbf{i}}{dt}=g\mathbf{z}.\frac{d\mathbf{i}}{dt}.$$

(This follows also from the equation of motion directly.) Integrating we have

$$\tfrac{1}{2}l\left(\frac{d\mathbf{i}}{dt}\right)^2-g(\mathbf{z}.\mathbf{i})=\text{const.} \tag{2}$$

This is the energy integral: $\frac{1}{2}m\left(l\frac{d\mathbf{i}}{dt}\right)^2$ is the kinetic energy, $-mg(\mathbf{z}.l\mathbf{i})$ is the potential energy of the particle in the earth's gravitational field.

To obtain a second first integral, multiply (1) scalarly by $\mathbf{z}$ and integrate. We find

$$l\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}.\mathbf{z}=\text{const.} \tag{3}$$

This is the integral of constancy of angular momentum about the vertical line through O. It arises because all the forces acting either pass through O or are parallel to the vertical through O, and hence the component, along this vertical, of the angular momentum about O is constant.

The last relation, written in the form

$$\frac{d\mathbf{i}}{dt}.(\mathbf{z}\wedge\mathbf{i})=\text{const.},$$

determines the component of $d\mathbf{i}/dt$ along $\mathbf{z}\wedge\mathbf{i}$; its component along $\mathbf{i}$ is also known, namely zero; and (2) determines the absolute magnitude of $d\mathbf{i}/dt$. Hence, for any given position $\mathbf{i}$, the vector $d\mathbf{i}/dt$ is completely determined by (2) and (3). The kinematic behaviour of $\mathbf{i}$ is thus fully known from (2) and (3).

However, any problem of interest concerning pendulum motion is better treated *de novo*, without use of the integrals of the motion. One of the differences between the methods we are here exposing and the more conventional methods in use in current textbooks is the reduced importance here attached to *scalar integrals* of the motion and the enhanced importance here attached to a *vector description* of the motion.

Example. From the *scalar* first integrals of the motion in the forms

$$\frac{d\mathbf{i}}{dt}.(\mathbf{z}\wedge\mathbf{i})=h, \qquad \tfrac{1}{2}l^2\left(\frac{d\mathbf{i}}{dt}\right)^2-gl(\mathbf{z}.\mathbf{i})=W,$$

obtain the vector first integral for $d\mathbf{i}/dt$.

The fundamental equation (1) is not linear in **i**. It is due to this circumstance that no *simple* vector first integral of the motion is obtainable in general. But various special cases of the motion possess simple vector integrals. Amongst these are: (1) motion near the vertical; (2) motion as a conical pendulum (steady precession); (3) disturbed motion near steady precession. These we proceed to consider in turn.

We note first one further point. The equation of motion in the form (1) holds good whether the bob of the pendulum is free to move in three dimensions or is confined to a plane through the vertical. Vector methods of their own accord deal naturally with the more general, more natural case of motion in three dimensions; restriction to plane motion is essentially an arbitrary restriction.

260. *Motion near the vertical.* Put $\mathbf{i}=\mathbf{z}+\boldsymbol{\epsilon}$, where $|\boldsymbol{\epsilon}|$ is small. Then since **i** and **z** are both unit vectors, we have on squaring

$$1=1+2\mathbf{z}.\boldsymbol{\epsilon}+\boldsymbol{\epsilon}^2,$$

or, neglecting $\boldsymbol{\epsilon}^2$,

$$\mathbf{z}.\boldsymbol{\epsilon}=0.$$

The equation of motion (1) (§ 259) then reduces to

$$l\mathbf{z}\wedge\frac{d^2\boldsymbol{\epsilon}}{dt^2}=g\boldsymbol{\epsilon}\wedge\mathbf{z}$$

on neglecting products of terms involving $\boldsymbol{\epsilon}$. Multiplying vectorially by **z**, since $\mathbf{z}.d^2\boldsymbol{\epsilon}/dt^2=0$, we get

$$l\frac{d^2\boldsymbol{\epsilon}}{dt^2}=-g\boldsymbol{\epsilon}.$$

This equation is the same in form as that of a particle under an attractive force proportional to the distance, discussed in § 255. The motion is accordingly simple harmonic, and the solution is

$$\boldsymbol{\epsilon}=\boldsymbol{\epsilon}_1+\boldsymbol{\epsilon}_2,$$

where

$$\frac{d\boldsymbol{\epsilon}_1}{dt}=\left(\frac{g}{l}\right)^{\frac{1}{2}}\mathbf{z}\wedge\boldsymbol{\epsilon}_1, \qquad \frac{d\boldsymbol{\epsilon}_2}{dt}=-\left(\frac{g}{l}\right)^{\frac{1}{2}}\mathbf{z}\wedge\boldsymbol{\epsilon}_2,$$

and $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ have arbitrary amplitudes and phases. The vector $\boldsymbol{\epsilon}$ is thus the sum of two oppositely rotating vectors of constant moduli, rotating with period $2\pi(l/g)^{\frac{1}{2}}$. The period is independent of the (arbitrary) amplitudes of the two rotating vectors, provided they are sufficiently small. These amplitudes are in the present case pure numbers, and 'small' means 'small compared with unity.' If the motion is arbitrarily

disturbed and then left to itself, the particle simply begins to execute a new Lissajous's figure with different rotating vectors.

261. *Conical pendulum. Steady precession.* Let us seek an exact particular solution of the non-linear equation (1) of § 259 such that $\mathbf{i}$ rotates uniformly round the vertical. If such a motion be possible, there must exist a number ω such that

$$\frac{d\mathbf{i}}{dt}=\omega\mathbf{z}\wedge\mathbf{i}.$$

In that case,
$$\mathbf{z}.\frac{d\mathbf{i}}{dt}=0$$

or
$$\mathbf{z}.\mathbf{i}=\text{const.}=\cos\alpha,$$

say. Then

$$\frac{d^2\mathbf{i}}{dt^2}=\omega\mathbf{z}\wedge(\omega\mathbf{z}\wedge\mathbf{i})=\omega^2[-\mathbf{i}+\mathbf{z}\cos\alpha].$$

Substituting in equation (1), § 257, we require that

$$\omega^2 l(\mathbf{i}\wedge\mathbf{z})\cos\alpha=g(\mathbf{i}\wedge\mathbf{z}).$$

This is satisfied identically for all values of $\mathbf{i}$ occurring during the motion if we choose ω so that

$$\omega^2=(g\sec\alpha)/l.$$

When ω and α are connected by this relation, a motion is possible in which $\mathbf{i}$ rotates round $\mathbf{z}$ with uniform angular velocity ω and making an angle α with $\mathbf{z}$. The system is then described as a conical pendulum, and the motion is said to be one of steady precession.

The idea of seeking a special solution described by means of a rotating vector is fundamental in a variety of problems, especially gyroscopic problems.

262. *Disturbed motion about a state of steady precession.* Suppose that the pendulum is describing a motion of steady precession, and that it is then slightly disturbed. We wish to analyse the resulting motion. In particular, we are interested in a kinematic description of the motion of the particle relative to the configuration of steady precession, i.e. in the motion in a frame rotating with the undisturbed motion. This example will illustrate the genuine use of rotating frames of reference, as opposed to the mere use of moving vectors. In customary presentations of dynamics, moving frames of reference (so called 'moving axes') are often used to determine a motion relative to a stationary frame, but the use of moving vectors often obviates this unnecessary employment of 'moving axes.' Here, however, we encounter a case where a moving frame of reference is genuinely required.

The equation of motion of the pendulum being of the form

$$l\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}=g(\mathbf{i}\wedge\mathbf{z}), \tag{1}$$

the state of steady precession is defined by a moving unit vector $\mathbf{i}_0$ satisfying

$$l\mathbf{i}_0 \wedge \frac{d^2\mathbf{i}_0}{dt^2} = g(\mathbf{i}_0 \wedge \mathbf{z}).$$

We have seen that this motion is specified by

$$\frac{d\mathbf{i}_0}{dt} = \omega\mathbf{z} \wedge \mathbf{i}_0, \qquad \frac{d^2\mathbf{i}_0}{dt^2} = \omega^2[-\mathbf{i}_0 + \mathbf{z} \cos \alpha],$$

provided $$l\omega^2 \cos \alpha = g.$$

Now put $$\mathbf{i} = \mathbf{i}_0 + \boldsymbol{\epsilon},$$

where $|\boldsymbol{\epsilon}|$ is small. Then since $\mathbf{i}$, $\mathbf{i}_0$ are unit vectors, we have $\mathbf{i}_0 . \boldsymbol{\epsilon} = 0$ on neglect of $\boldsymbol{\epsilon}^2$. Substitution in the equation of motion (1) gives, again neglecting terms of the second order in $\boldsymbol{\epsilon}$,

$$l\mathbf{i}_0 \wedge \frac{d^2\boldsymbol{\epsilon}}{dt^2} + l\boldsymbol{\epsilon} \wedge \omega^2(-\mathbf{i}_0 + \mathbf{z} \cos \alpha) = g\boldsymbol{\epsilon} \wedge \mathbf{z}.$$

Using the relation $l\omega^2 \cos \alpha = g$, we get

$$\mathbf{i}_0 \wedge \frac{d^2\boldsymbol{\epsilon}}{dt^2} + \omega^2 \mathbf{i}_0 \wedge \boldsymbol{\epsilon} = 0. \tag{2}$$

This simple equation determines the disturbance $\boldsymbol{\epsilon}$ as a function of the time. If $\mathbf{i}_0$ were constant in time, it would, of course, possess the integral $d\boldsymbol{\epsilon}/dt = \omega\mathbf{i}_0 \wedge \boldsymbol{\epsilon}$. But $\mathbf{i}_0$ is itself a function of the time. A possible mode of progress would be to write $d\boldsymbol{\epsilon}/dt = \omega\mathbf{i}_0 \wedge \boldsymbol{\epsilon} + \mathbf{X}$, and ascertain the behaviour of $\mathbf{X}$. But this would give us little insight into the motion. A better plan is to attempt to ascertain the behaviour of $\boldsymbol{\epsilon}$ *relative to the moving frame defined by the rotation of* $\mathbf{i}_0$. We therefore use our former notation (§§ 203 ff.),

$$\frac{d\boldsymbol{\epsilon}}{dt} = \frac{\partial \boldsymbol{\epsilon}}{\partial t} + \omega\mathbf{z} \wedge \boldsymbol{\epsilon}$$

where the operator $\partial/\partial t$ refers to motion relative to the moving frame. Under the operator $\partial/\partial t$, $\mathbf{i}_0$ and all other vectors rigidly attached to the moving frame behave as constants. We have then

$$\frac{d^2\boldsymbol{\epsilon}}{dt^2} = \left(\frac{\partial}{\partial t} + \omega\mathbf{z} \wedge\right)\left(\frac{\partial \boldsymbol{\epsilon}}{\partial t} + \omega\mathbf{z} \wedge \boldsymbol{\epsilon}\right) = \frac{\partial^2 \boldsymbol{\epsilon}}{\partial t^2} + 2\omega\mathbf{z} \wedge \frac{\partial \boldsymbol{\epsilon}}{\partial t} + \omega^2 \mathbf{z} \wedge (\mathbf{z} \wedge \boldsymbol{\epsilon}),$$

$$\mathbf{i}_0 \wedge \frac{d^2\boldsymbol{\epsilon}}{dt^2} = \mathbf{i}_0 \wedge \frac{\partial^2 \boldsymbol{\epsilon}}{\partial t^2} - 2\omega \cos \alpha \frac{\partial \boldsymbol{\epsilon}}{\partial t} + \omega^2 \mathbf{i}_0 \wedge [-\boldsymbol{\epsilon} + \mathbf{z}(\boldsymbol{\epsilon}.\mathbf{z})],$$

since $\mathbf{i}_0 . \partial\boldsymbol{\epsilon}/\partial t = (\partial/\partial t)(\mathbf{i}_0 . \boldsymbol{\epsilon}) = 0$. Hence the equation for $\boldsymbol{\epsilon}$, namely (2), becomes

$$\mathbf{i}_0 \wedge \frac{\partial^2 \boldsymbol{\epsilon}}{\partial t^2} - 2\omega \cos \alpha \frac{\partial \boldsymbol{\epsilon}}{\partial t} + \omega^2(\boldsymbol{\epsilon}.\mathbf{z})(\mathbf{i}_0 \wedge \mathbf{z}) = 0. \tag{3}$$

Now introduce a unit vector $\mathbf{k}$ in the plane of $\mathbf{i}_0$ and $\mathbf{z}$ perpendicular to $\mathbf{i}_0$ (Fig. 65). Then since

$$\mathbf{z}=\mathbf{i}_0\cos\alpha+\mathbf{k}\sin\alpha,$$

we have
$$\mathbf{i}_0\wedge\mathbf{z}=-\mathbf{j}\sin\alpha,$$

and
$$\boldsymbol{\epsilon}.\mathbf{z}=(\boldsymbol{\epsilon}.\mathbf{k})\sin\alpha,$$

where $\mathbf{j}=\mathbf{k}\wedge\mathbf{i}_0$. Thus (3) becomes

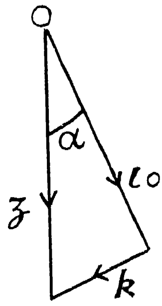

Fig. 65

$$\mathbf{i}_0\wedge\frac{\partial^2\boldsymbol{\epsilon}}{\partial t^2}-2\omega\cos\alpha\frac{\partial\boldsymbol{\epsilon}}{\partial t}-\omega^2\sin^2\alpha(\boldsymbol{\epsilon}.\mathbf{k})\mathbf{j}=0,$$

or again, on multiplying vectorially by $\mathbf{i}_0$,

$$\frac{\partial^2\boldsymbol{\epsilon}}{\partial t^2}+2\omega\cos\alpha\mathbf{i}_0\wedge\frac{\partial\boldsymbol{\epsilon}}{\partial t}+\omega^2\sin^2\alpha(\boldsymbol{\epsilon}.\mathbf{k})\mathbf{k}=0. \tag{4}$$

The vector $\boldsymbol{\epsilon}$ is always perpendicular to $\mathbf{i}_0$ and so rotates in some fashion round $\mathbf{i}_0$, but it does not remain of constant length. The equation governing $\boldsymbol{\epsilon}$, in the form (4), shows that the behaviour of $\boldsymbol{\epsilon}$ is specially related to the $\mathbf{k}$ direction. The next simplest motion to the rotation of a constant vector is motion in which the extremity of the rotating vector describes an ellipse, and equation (4) is in fact the vector description of motion in an ellipse, whose auxiliary circle is described by the corresponding point with uniform angular velocity, and whose major axis is oriented along $\mathbf{k}$.

To establish this, we seek a solution of (4) of the form

$$\boldsymbol{\epsilon}=(\boldsymbol{\rho}.\mathbf{k})\mathbf{k}+\lambda(\boldsymbol{\rho}.\mathbf{j})\mathbf{j} \tag{5}$$

where $\boldsymbol{\rho}$ is a rotating vector of constant length determined by

$$\frac{\partial\boldsymbol{\rho}}{\partial t}=\omega'\mathbf{i}_0\wedge\boldsymbol{\rho}, \tag{6}$$

and ω' and λ are constants to be determined. By direct differentiation, we have

$$\frac{\partial\boldsymbol{\epsilon}}{\partial t}=\omega'(\mathbf{i}_0\wedge\boldsymbol{\rho}.\mathbf{k})\mathbf{k}+\lambda\omega'(\mathbf{i}_0\wedge\boldsymbol{\rho}.\mathbf{j})\mathbf{j}=\omega'(\boldsymbol{\rho}.\mathbf{j})\mathbf{k}-\lambda\omega'(\boldsymbol{\rho}.\mathbf{k})\mathbf{j},$$

$$\frac{\partial^2\boldsymbol{\epsilon}}{\partial t^2}=\omega'^2(\mathbf{i}_0\wedge\boldsymbol{\rho}.\mathbf{j})\mathbf{k}-\lambda\omega'^2(\mathbf{i}_0\wedge\boldsymbol{\rho}.\mathbf{k})\mathbf{j}=-\omega'^2(\boldsymbol{\rho}.\mathbf{k})\mathbf{k}-\lambda\omega'^2(\boldsymbol{\rho}.\mathbf{j})\mathbf{j},$$

$\mathbf{k}$ and $\mathbf{j}$ behaving as constants under $\partial/\partial t$. Inserting in (4), we see that (5) will be a solution provided λ and ω' can be chosen so that

$$-\omega'^2[(\boldsymbol{\rho}.\mathbf{k})\mathbf{k}+\lambda(\boldsymbol{\rho}.\mathbf{j})\mathbf{j}]+2\omega\omega'\cos\alpha[-(\boldsymbol{\rho}.\mathbf{j})\mathbf{j}-\lambda(\boldsymbol{\rho}.\mathbf{k})\mathbf{k}]+\omega^2\sin^2\alpha(\boldsymbol{\rho}.\mathbf{k})\mathbf{k}=0.$$

This requires
$$-\omega'^2-2\lambda\omega\omega'\cos\alpha+\omega^2\sin^2\alpha=0,$$
$$-\lambda\omega'^2-2\omega\omega'\cos\alpha=0.$$

From the second of these,

$$\omega'=-(2\omega\cos\alpha)/\lambda,$$

and the first then gives

$$\lambda^2 = \frac{4\cos^2\alpha}{1+3\cos^2\alpha},$$

whence, taking $\lambda > 0$,

$$\omega' = -\omega(1+3\cos^2\alpha)^{\frac{1}{2}} = -\left(\frac{g}{l}\right)^{\frac{1}{2}}\left(\frac{1+3\cos^2\alpha}{\cos\alpha}\right)^{\frac{1}{2}}.$$

(We can take $\lambda > 0$ without loss of generality, since changing the sign of λ simply changes the sign of ω', and so leads merely to a different description of the same motion.) We note that $\lambda < 1$, and accordingly the locus defined by (5), (6) is an ellipse, from the known properties of the auxiliary circle of an ellipse. The equation of the ellipse, putting $x = \boldsymbol{\epsilon}.\mathbf{k}$, $y = \boldsymbol{\epsilon}.\mathbf{j}$ is, since $x = \boldsymbol{\rho}.\mathbf{k}$, $y = \lambda(\boldsymbol{\rho}.\mathbf{j})$, just

$$\frac{x^2}{\rho^2} + \frac{y^2}{\lambda^2\rho^2} = 1.$$

The major axis $|\boldsymbol{\rho}|$ is arbitrary but the ratio of the axes is determinate. The ellipse lies in the plane normal to $\mathbf{i}_0$, and relative to the moving frame rotating with $\mathbf{i}_0$ is described with period

$$2\pi\left(\frac{l}{g}\right)^{\frac{1}{2}}\left(\frac{\cos\alpha}{1+3\cos^2\alpha}\right)^{\frac{1}{2}},$$

the associated auxiliary circle being described with the constant *retrograde* angular velocity $-\omega'\mathbf{i}_0$.

It follows from our analysis that the motion of steady precession is *stable*.

263. *Note on a vector differential equation.* The reader should observe that we have in effect obtained the solution of the vector differential equation

$$\frac{d^2\mathbf{X}}{dt^2} + a\mathbf{i}\wedge\frac{d\mathbf{X}}{dt} + b(\mathbf{X}.\mathbf{k})\mathbf{k} = 0,$$

where $\qquad \mathbf{X}.\mathbf{i} = 0, \qquad \mathbf{k}.\mathbf{i} = 0,$

$\mathbf{i}$, $\mathbf{k}$ being given unit vectors and a, b given constants. The solution is

$$\mathbf{X} = (\boldsymbol{\rho}.\mathbf{k})\mathbf{k} + \lambda(\boldsymbol{\rho}.\mathbf{j})\mathbf{j},$$

where $$\frac{d\boldsymbol{\rho}}{dt} = \omega\mathbf{i}\wedge\boldsymbol{\rho},$$

and λ and ω are given by

$$\omega = -(a^2+b)^{\frac{1}{2}}, \qquad \lambda = \frac{a}{(a^2+b)^{\frac{1}{2}}}.$$

The solution thus corresponds to a real motion provided $b > -a^2$. The discussion of the case $b < -a^2$ is left to the reader.

264. *Example on relative motion in two dimensions connected with pendulum motion* (Besant and Ramsay, p. 207, *M.T.* 1908). A string OPQ has one point O fixed. The point P of the string is constrained to move with uniform angular velocity ω in a circle of radius OP$=$a and centre O. A massive particle is attached at Q, and is free to move in the plane of motion of OP. Prove that the motion of PQ relative to OP is the same as that of a simple pendulum of length $gb/a\omega^2$, where PQ$=$b; and that for the string to remain taut we must have $a>4b$.

Take unit vectors $\mathbf{i}$ and $\mathbf{j}$ in OP and PQ respectively (Fig. 66). If T is the tension in PQ, the equation of motion of the particle is

$$m\frac{d^2}{dt^2}(a\mathbf{i}+b\mathbf{j})=-T\mathbf{j}.$$

The motion of $\mathbf{i}$ is determined by the relations

$$\frac{d\mathbf{i}}{dt}=\omega\mathbf{z}\wedge\mathbf{i},\qquad \frac{d^2\mathbf{i}}{dt^2}=-\omega^2\mathbf{i},$$

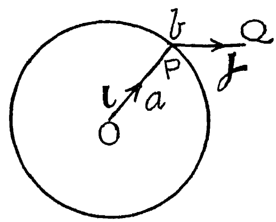

Fig. 66

where $\mathbf{z}$ is a unit vector normal to the plane. If $\partial\mathbf{j}/\partial t$ denotes the apparent motion of $\mathbf{j}$ in the frame rotating with $\mathbf{i}$, then

$$\frac{d\mathbf{j}}{dt}=\frac{\partial\mathbf{j}}{\partial t}+\omega\mathbf{z}\wedge\mathbf{j},\qquad \frac{d^2\mathbf{j}}{dt^2}=\frac{\partial^2\mathbf{j}}{\partial t^2}+2\omega\mathbf{z}\wedge\frac{\partial\mathbf{j}}{\partial t}-\omega^2\mathbf{j}.$$

Hence, substituting in the equation of motion,

$$-\omega^2 a\mathbf{i}+b\left[\frac{\partial^2\mathbf{j}}{\partial t^2}+2\omega\mathbf{z}\wedge\frac{\partial\mathbf{j}}{\partial t}-\omega^2\mathbf{j}\right]=-\frac{T}{m}\mathbf{j}. \qquad (1)$$

The equation of motion of a simple pendulum, of length l, whose direction at any instant is along a vector $\mathbf{j}$ is

$$l\frac{\partial^2\mathbf{j}}{\partial t^2}=-\frac{T'}{m}\mathbf{j}+g\mathbf{i},$$

where T' is the tension, and $\mathbf{i}$ defines the downward vertical. This is of the form of the preceding equation, for the vector $\mathbf{z}\wedge\dfrac{\partial\mathbf{j}}{\partial t}$, being perpendicular to $\mathbf{z}$ and to $\dfrac{\partial\mathbf{j}}{\partial t}$, must be along $\mathbf{j}$. The coefficients of $\mathbf{i}$ and $\partial^2\mathbf{j}/\partial t^2$ in the two equations are now proportional to one another provided

$$\frac{\omega^2 a}{g}=\frac{b}{l},$$

which is the desired result.

The details of the second part will be left to the reader. It may be pointed out, however, that if θ is the angle between $\mathbf{i}$ and $\mathbf{j}$, then

$$\frac{\partial\mathbf{j}}{\partial t}=\dot\theta\mathbf{z}\wedge\mathbf{j},\qquad \frac{\partial^2\mathbf{j}}{\partial t^2}=-\dot\theta^2\mathbf{j}+\ddot\theta\mathbf{z}\wedge\mathbf{j},$$

and
$$\mathbf{z}\wedge\frac{\partial\mathbf{j}}{\partial t}=-\dot{\theta}\mathbf{j}.$$

The tension T is given by multiplying the equation (1) scalarly by $\mathbf{j}$, when we get
$$T/m=a\omega^2\cos\theta+b(\omega^2+\dot{\theta}^2+2\omega\dot{\theta}).$$
The behaviour of θ is given by multiplying (1) scalarly by $\partial\mathbf{j}/\partial t$,
$$-\omega^2 a\mathbf{i}.\frac{\partial\mathbf{j}}{\partial t}+b\frac{\partial^2\mathbf{j}}{\partial t^2}.\frac{\partial\mathbf{j}}{\partial t}=0.$$
Since $\mathbf{i}$ is constant under $\partial/\partial t$, this integrates at once in the form
$$-\omega^2 a\mathbf{i}.\mathbf{j}+\tfrac{1}{2}b\left(\frac{\partial\mathbf{j}}{\partial t}\right)^2=\text{const.}$$
or
$$-\omega^2 a\cos\theta+\tfrac{1}{2}b\dot{\theta}^2=\text{const.}$$

Combination of this with the formula for the tension in terms of θ and $\dot{\theta}$ will yield the desired result.

265. *Central orbits.* The preceding set of kinematical and dynamical situations arise out of consideration of *simple harmonic motion*, damped or undamped, in its most general sense, that is to say motion in which the force is proportional to the distance of the particle from a fixed point. Problems arising from motions under more complicated types of central forces can often be handled by similar methods. It is not proposed to give here a general account of orbits under central forces, but the following sections will derive the fundamental properties of central orbits, and in particular of orbits under an inverse square law.

266. *Constancy of angular momentum.* The acceleration in any central orbit is directed along the radius vector, inwards or outwards. The equation of motion may therefore be written in the form
$$\frac{d^2\mathbf{r}}{dt^2}=-f(|\mathbf{r}|)\mathbf{r}, \tag{1}$$
where $|\mathbf{r}|f(|\mathbf{r}|)$ is the scalar value of the acceleration at distance $|\mathbf{r}|$, reckoned positive inwards. As previously, on multiplying (1) vectorially by $\mathbf{r}$ we have
$$\mathbf{r}\wedge\frac{d^2\mathbf{r}}{dt^2}=0,$$
or integrating
$$\mathbf{r}\wedge\frac{d\mathbf{r}}{dt}=\text{const.}=\mathbf{h}, \tag{2}$$
say. This integral expresses the constancy of angular momentum about the origin. It gives at once
$$\mathbf{h}.\mathbf{r}=0, \qquad \mathbf{h}.\frac{d\mathbf{r}}{dt}=0,$$

of which the former shows that $\mathbf{r}$ is always perpendicular to the constant vector $\mathbf{h}$, and hence that the orbit lies in a plane normal to $\mathbf{h}$, passing through the centre of force. The scalar areal velocity about the origin is $\frac{1}{2}|\mathbf{h}|$.

267. *Energy integral.* Multiplying (1) scalarly by $d\mathbf{r}/dt$, we have

$$\frac{d\mathbf{r}}{dt}.\frac{d^2\mathbf{r}}{dt^2}=-f(|\mathbf{r}|)\mathbf{r}.\frac{d\mathbf{r}}{dt}.$$

Since $\mathbf{r}^2=|\mathbf{r}|^2$, $\mathbf{r}.d\mathbf{r}=|\mathbf{r}|d|\mathbf{r}|$, whence integrating

$$\tfrac{1}{2}\left(\frac{d\mathbf{r}}{dt}\right)^2=-\int^{|\mathbf{r}|}f(|\mathbf{r}|)|\mathbf{r}|d|\mathbf{r}|+\text{const.}$$

It is customary to write this integral in the form

$$\tfrac{1}{2}\left(\frac{d\mathbf{r}}{dt}\right)^2-\int_{|\mathbf{r}|}^{\infty}f(|\mathbf{r}|)|\mathbf{r}|d|\mathbf{r}|=W, \tag{3}$$

and to call W the energy of the orbit.

268. *Hamilton's transformation.* From the equation of motion (1) and the angular momentum integral (2), we have on vector multiplication of appropriate sides,

$$\mathbf{h}\wedge\frac{d^2\mathbf{r}}{dt^2}=-f(|\mathbf{r}|)\left[\left(\mathbf{r}\wedge\frac{d\mathbf{r}}{dt}\right)\wedge\mathbf{r}\right].$$

But
$$\left(\mathbf{r}\wedge\frac{d\mathbf{r}}{dt}\right)\wedge\mathbf{r}=-\mathbf{r}\left(\mathbf{r}.\frac{d\mathbf{r}}{dt}\right)+\mathbf{r}^2\frac{d\mathbf{r}}{dt}$$
$$=|\mathbf{r}|^3\left[\frac{1}{|\mathbf{r}|}\frac{d\mathbf{r}}{dt}-\frac{\mathbf{r}}{|\mathbf{r}|^2}\frac{d|\mathbf{r}|}{dt}\right]$$
$$=|\mathbf{r}|^3\frac{d}{dt}\left(\frac{\mathbf{r}}{|\mathbf{r}|}\right).$$

Hence
$$\mathbf{h}\wedge\frac{d^2\mathbf{r}}{dt^2}=-|\mathbf{r}|^3f(|\mathbf{r}|)\frac{d}{dt}\left(\frac{\mathbf{r}}{|\mathbf{r}|}\right). \tag{4}$$

This transformation is due to Hamilton.

The physical meaning of the equation preceding (4) is immediate: putting $\mathbf{r}/|\mathbf{r}|=\mathbf{i}$, a unit vector, and writing $\mathbf{h}$ for $\mathbf{r}\wedge\frac{d\mathbf{r}}{dt}$, we have

$$\frac{d\mathbf{i}}{dt}=\frac{\mathbf{h}}{|\mathbf{r}|^2}\wedge\mathbf{i},$$

which shows that the unit vector $\mathbf{i}$ rotates with angular velocity $\mathbf{h}/|\mathbf{r}|^2$. This is the familiar integral $r^2\dot{\theta}=|\mathbf{h}|$ of the usual theory.

269. *Inverse square attraction. Hamilton's integral.* When the force is attractive and varying as the inverse square of the distance, the inward acceleration may be written $\mu/|\mathbf{r}|^2$, and the function $f(|\mathbf{r}|)$ is defined by

$$f(|\mathbf{r}|)=\mu/|\mathbf{r}|^3. \tag{1}$$

We have as usual the integral of angular momentum

$$\mathbf{r}\wedge\frac{d\mathbf{r}}{dt}=\mathbf{h}, \qquad (2)$$

and the integral of energy takes the form

$$\tfrac{1}{2}\left(\frac{d\mathbf{r}}{dt}\right)^2-\frac{\mu}{|\mathbf{r}|}=W. \qquad (3)$$

Of these integrals, the former determines the component of velocity perpendicular to $\mathbf{r}$ at any point of the orbit, and the latter determines the absolute value of the velocity. Hence, given one point on the orbit, the constants $|\mathbf{h}|$ and W completely determine the orbit. But considerable simplification in the determination of orbital properties results if we follow a brilliant artifice due to Hamilton, which consists in noticing that equation (4) § 268, integrates as it stands when $f(|\mathbf{r}|)$ is of the form (1) above. The integral is in fact

$$\mathbf{h}\wedge\frac{d\mathbf{r}}{dt}=-\mu\frac{\mathbf{r}}{|\mathbf{r}|}+\mathbf{B}, \qquad (4)$$

where $\mathbf{B}$ is a vector constant. This new vector integral of the motion is equivalent to the determination of the orbit, as we shall now see.

270. *Equation of the orbit.* Multiplying (4), § 269, scalarly by $\mathbf{h}$, we have

$$\mathbf{B}.\mathbf{h}=\mu\frac{\mathbf{r}.\mathbf{h}}{|\mathbf{r}|}=0.$$

Hence $\mathbf{B}$, being perpendicular to $\mathbf{h}$, lies in the plane of the orbit.

Now multiply (4) scalarly by $\mathbf{r}$. We find

$$\mathbf{h}.\left(\frac{d\mathbf{r}}{dt}\wedge\mathbf{r}\right)=-\mu|\mathbf{r}|+\mathbf{B}.\mathbf{r},$$

or, using (2),

$$-\mathbf{h}^2=-\mu|\mathbf{r}|+\mathbf{B}.\mathbf{r}.$$

Hence

$$\frac{\mathbf{h}^2/\mu}{|\mathbf{r}|}=1-\frac{\mathbf{B}}{\mu}.\frac{\mathbf{r}}{|\mathbf{r}|}. \qquad (5)$$

Since $\mathbf{r}/|\mathbf{r}|$ is a unit vector, this may be written in polar co-ordinates (r, θ)

$$\frac{l}{r}=1-e\cos\theta, \qquad (6)$$

where

$$l=\mathbf{h}^2/\mu, \qquad e=|\mathbf{B}|/\mu. \qquad (7)$$

This is the equation of a conic, of eccentricity e and semi-latus rectum l, whose major axis lies in the direction of the vector $\mathbf{B}$; the angle θ is the angle between $\mathbf{r}$ and $\mathbf{B}$, and the centre O is a focus ; $\mathbf{B}$ points *away from* the direction corresponding to O.

271. The physical meaning of (4) can now be seen. The vector $\mathbf{h}\wedge d\mathbf{r}/dt$ is a vector of magnitude $|\mathbf{h}|\,|\mathbf{v}|$ along the inward normal PG ; the vector $\mu\mathbf{r}/|\mathbf{r}|$ is a vector of magnitude μ along the radius vector OP (Fig. 67). Equation (4) now asserts that the sum of these vectors is along OG (the direction of $\mathbf{B}$) and of amount $|\mathbf{B}|$. Hence the magnitudes of the vectors concerned must be proportional to the sides of the triangle OPG, i.e.

$$\frac{|\mathbf{B}|}{OG}=\frac{|\mathbf{h}|\,|\mathbf{v}|}{PG}=\frac{\mu}{OP}.$$

The first and third of these then give, by (7),

$$OG=eOP,$$

and the second and third give, again using (7),

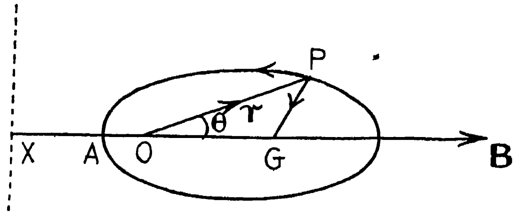

Fig. 67

$$|\mathbf{v}|=\frac{\mu}{|\mathbf{h}|}\frac{PG}{OP}=\frac{|\mathbf{h}|}{l}\frac{PG}{OP}.$$

272. A further interpretation of (4) is of interest. Multiplying it vectorially by $\mathbf{h}$, we have since $\mathbf{h}.d\mathbf{r}/dt=0$

$$\mathbf{h}^2\frac{d\mathbf{r}}{dt}=\mathbf{B}\wedge\mathbf{h}-\mu\frac{\mathbf{r}\wedge\mathbf{h}}{|\mathbf{r}|},$$

or

$$\frac{d\mathbf{r}}{dt}=\frac{\mathbf{B}\wedge\mathbf{h}}{\mathbf{h}^2}+\frac{\mu}{\mathbf{h}^2}\mathbf{h}\wedge\frac{\mathbf{r}}{|\mathbf{r}|}.$$

This exhibits the velocity $d\mathbf{r}/dt$ as the sum of two vectors in the plane of the orbit : of these the first is a constant, and the second is of constant modulus. The hodograph of the motion is accordingly a circle, centre at $\mathbf{B}\wedge\mathbf{h}/\mathbf{h}^2$ and radius $\mu/|\mathbf{h}|$.

273. *Energy in terms of the geometry of the orbit.* If we take the modulus of each side of equation (4), § 269, we get

$$\mathbf{h}^2\left(\frac{d\mathbf{r}}{dt}\right)^2=\mu^2+\mathbf{B}^2-2\mu\frac{\mathbf{B}.\mathbf{r}}{|\mathbf{r}|},$$

or, using (5) and (7),

$$\mathbf{h}^2\left(\frac{d\mathbf{r}}{dt}\right)^2=\mu^2\left[1+e^2-2\left(1-\frac{\mathbf{h}^2/\mu}{|\mathbf{r}|}\right)\right],$$

i.e.

$$\tfrac{1}{2}\left(\frac{d\mathbf{r}}{dt}\right)^2=\frac{\mu}{|\mathbf{r}|}-\tfrac{1}{2}\frac{\mu^2(1-e^2)}{\mathbf{h}^2}.$$

This relation must be identical with (3). Hence

$$W=-\frac{\mu^2(1-e^2)}{2\mathbf{h}^2}=-\frac{\mu(1-e^2)}{2l}=-\frac{\mu}{2a},\quad (0<e<1), \qquad (8)$$

where $a = l/(1-e^2)$ is the semi-axis major of the ellipse when $0 < e < 1$. Introducing this in (3) we have for the velocity in an elliptic orbit

$$v^2 = \mu\left(\frac{2}{|\mathbf{r}|} - \frac{1}{a}\right).$$

When $e > 1$, we have $a = l/(e^2-1)$, and so

$$W = +\frac{\mu}{2a},$$

and then
$$v^2 = \mu\left(\frac{2}{|\mathbf{r}|} + \frac{1}{a}\right).$$

The student may find it worth while to remember that the two fundamental formulæ giving the dynamics of an inverse square orbit in terms of the geometry are

$$|W| = \frac{\mu}{2a}, \qquad \mathbf{h}^2 = \mu l.$$

274. *Asymptotes for* $e > 1$, $W > 0$. Near a linear asymptote, the velocity of the particle $d\mathbf{r}/dt$ is ultimately parallel to $\mathbf{r}$ (Fig. 68). If for this limiting direction we put $\mathbf{r}/|\mathbf{r}| = \mathbf{i}$, $\mathbf{v} = v_\infty \mathbf{i}$, then Hamilton's integral (4) gives

$$v_\infty(\mathbf{h} \wedge \mathbf{i}) = -\mu\mathbf{i} + \mathbf{B}.$$

Hence
$$\mathbf{B.i} = \mu,$$

and
$$\mathbf{B} \wedge \mathbf{i} = v_\infty(\mathbf{h} \wedge \mathbf{i}) \wedge \mathbf{i} = -v_\infty \mathbf{h}.$$

Hence if θ is the angle between the direction of $\mathbf{B}$ and the asymptote,

$$\tan\theta = \frac{|\mathbf{B} \wedge \mathbf{i}|}{|\mathbf{B.i}|} = \frac{v_\infty |\mathbf{h}|}{\mu}.$$

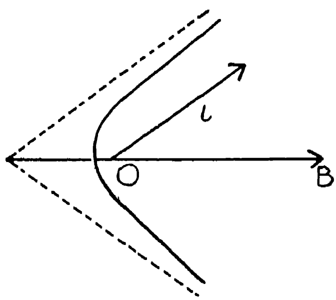

Fig. 68

But
$$|\mathbf{h}| = p v_\infty,$$

where p is the perpendicular from the centre of force O on to the asymptote, since each is equal to the angular momentum about O. Hence

$$\tan\theta = \frac{p v_\infty^2}{\mu}.$$

This is a fundamental formula in the theory of atomic collisions.

We notice also that the relation $\mathbf{B.i} = \mu$ gives at once $|\mathbf{B}| \cos\theta = \mu$, or, by (7), $\sec\theta = e$, another important result.

275. *Mean values.* Hamilton's integral (4) of § 269 can be used to evaluate certain time-means connected with the orbit. If we multiply both sides vectorially by $\mathbf{r}$, we have

$$\left(\mathbf{h} \wedge \frac{d\mathbf{r}}{dt}\right) \wedge \mathbf{r} = \mathbf{B} \wedge \mathbf{r},$$

or, since $\mathbf{h}.\mathbf{r}=0$, $$\mathbf{B}\wedge\mathbf{r}=-\mathbf{h}\left(\mathbf{r}.\frac{d\mathbf{r}}{dt}\right).$$

Integrating with respect to t along any arc of the orbit,

$$\mathbf{B}\wedge\int_{t_0}^{t_1}\mathbf{r}dt=-\tfrac{1}{2}\mathbf{h}[\mathbf{r}^2]_{t_0}^{t_1}.$$

At two points P_0, P_1 on the orbit for which $\mathbf{r}$ takes the same values, the right-hand side is zero, and hence, for such a pair

$$\mathbf{B}\wedge\int_{t_0}^{t_1}\mathbf{r}dt=0.$$

Thus $\int_{t_0}^{t_1}\mathbf{r}dt$ is parallel to $\mathbf{B}$, i.e. to the axis-major. In particular, for a periodic orbit ($e<1$), the time-mean of the radius vector is parallel to the axis-major.

Again, if we integrate Hamilton's integral with regard to the time as it stands, we get

$$\mathbf{h}\wedge[\mathbf{r}]_{t_0}^{t_1}=-\mu\int_{t_0}^{t_1}\frac{\mathbf{r}}{|\mathbf{r}|}dt+\mathbf{B}(t_1-t_0).$$

Multiply this scalarly by $\mathbf{B}$. Then

$$(\mathbf{B}\wedge\mathbf{h}).[\mathbf{r}]_{t_0}^{t_1}=-\mu\int_{t_0}^{t_1}\frac{\mathbf{r}.\mathbf{B}}{|\mathbf{r}|}dt+\mathbf{B}^2(t_1-t_0).$$

Next, integrate (5), § 270, with respect to t and use the last result to eliminate the integral in $\mathbf{r}.\mathbf{B}$. We find

$$\frac{\mathbf{h}^2}{\mu}\int_{t_0}^{t_1}\frac{1}{|\mathbf{r}|}dt=(t_1-t_0)-\frac{\mathbf{B}^2}{\mu^2}(t_1-t_0)+\frac{\mathbf{B}\wedge\mathbf{h}}{\mu^2}.[\mathbf{r}]_{t_0}^{t_1}.$$

or
$$\frac{1}{t_1-t_0}\int_{t_0}^{t_1}\frac{1}{|\mathbf{r}|}dt=\frac{1-e^2}{l}+\frac{\mathbf{B}\wedge\mathbf{h}}{\mu\mathbf{h}^2}.\frac{\mathbf{r}_1-\mathbf{r}_0}{t_1-t_0}$$

$$=\frac{1}{a}+\frac{e}{l^{\frac{1}{2}}\mu^{\frac{1}{2}}}\mathbf{j}.\frac{\mathbf{r}_1-\mathbf{r}_0}{t_1-t_0},$$

where $\mathbf{j}$ is a unit vector perpendicular to the major axis. For a complete orbit, for which $\mathbf{r}_1=\mathbf{r}_0$, $t_1-t_0=T$, we have

$$\frac{1}{T}\int_0^T\frac{1}{|\mathbf{r}|}dt=\frac{1}{a}.$$

We get still another relation on multiplying (5), § 270, by $|\mathbf{r}|$, and integrating, when we get

$$\mathbf{h}^2=\mu\frac{1}{T}\int_0^T|\mathbf{r}|dt-\mathbf{B}.\frac{1}{T}\int_0^T\mathbf{r}dt.$$

276. It is not here suggested that *all* properties of inverse square orbits are most readily obtained by vector methods. It is sufficient to have illustrated the power of Hamilton's integral, which avoids many otherwise tedious integrations.

277. *Motion under an inverse cube acceleration.* As a further example, consider the inverse cube attractive acceleration $-\mu/|\mathbf{r}|^3$. The equation of motion is

$$\ddot{\mathbf{r}} = -\mu\frac{\mathbf{r}}{|\mathbf{r}|^4}, \tag{1}$$

the integral of angular momentum is

$$\mathbf{r}\wedge\dot{\mathbf{r}} = \mathbf{h}, \tag{2}$$

and the integral of energy is

$$\dot{\mathbf{r}}^2 = \frac{\mu}{|\mathbf{r}|^2} + W, \tag{3}$$

where W now denotes *twice* the orbital energy. Scalar multiplication of (1) by $\mathbf{r}$ gives

$$\mathbf{r}.\ddot{\mathbf{r}} = -\frac{\mu}{|\mathbf{r}|^2},$$

by adding which to (3) we obtain

$$\mathbf{r}.\ddot{\mathbf{r}} + \dot{\mathbf{r}}^2 = W.$$

This integrates at once in the form

$$\mathbf{r}.\dot{\mathbf{r}} = Wt, \tag{4}$$

on reckoning t from the apse ($\dot{\mathbf{r}} = 0$). This again integrates in the form

$$\mathbf{r}^2 = Wt^2 + C. \tag{5}$$

Squaring (2) we get

$$\mathbf{r}^2\dot{\mathbf{r}}^2 - (\mathbf{r}.\dot{\mathbf{r}})^2 = \mathbf{h}^2,$$

and from (3)

$$\begin{aligned}\mathbf{r}^2\dot{\mathbf{r}}^2 &= \mu + W\mathbf{r}^2\\ &= \mu + W(Wt^2 + C).\end{aligned}$$

Hence

$$\mu + W(Wt^2 + C) - (Wt)^2 = \mathbf{h}^2$$

or

$$\mu + CW = \mathbf{h}^2 \tag{6}$$

The orbit is now determined. For (5) fixes $\mathbf{r}$ as a function of t, and (3) determines the rate of rotation of the radius vector at any $\mathbf{r}$.

MOTION IN THE VICINITY OF THE ROTATING EARTH

278. *Calculation of relative acceleration.* We propose in the next few sections to examine certain motions in the neighbourhood of a given point O of the earth's surface, taking into account the earth's rotation.

Let C (Fig. 69) be the earth's centre. Take an origin O on the earth's surface, of position vector $\mathbf{r}_0$ with respect to C. Take a frame of reference rigidly attached to the earth at O, and let $\mathbf{r}$ be the position vector of a particle P, relative to O. The earth is supposed to have an angular velocity ω, a constant, about an axis in the direction of a unit vector $\mathbf{z}$, We shall use d/dt to denote a rate of change relative to a frame in which the earth has the angular velocity $\omega\mathbf{z}$, and $\partial/\partial t$ to denote a rate of change relative to a frame fixed to the earth at O.

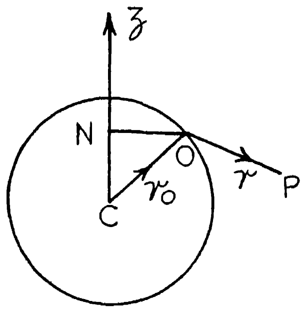

Fig. 69

The velocity of P is given by

$$\frac{d\mathbf{P}}{dt}=\frac{d}{dt}(\mathbf{r}_0+\mathbf{r})=\omega\mathbf{z}\wedge\mathbf{r}_0+\frac{\partial\mathbf{r}}{\partial t}+\omega\mathbf{z}\wedge\mathbf{r}, \tag{1}$$

and the acceleration of P is given by

$$\frac{d^2\mathbf{P}}{dt^2}=\left(\frac{\partial}{\partial t}+\omega\mathbf{z}\wedge\right)\left(\omega\mathbf{z}\wedge\mathbf{r}_0+\frac{\partial\mathbf{r}}{\partial t}+\omega\mathbf{z}\wedge\mathbf{r}\right)$$

$$=\omega^2\mathbf{z}\wedge(\mathbf{z}\wedge\mathbf{r}_0)+\frac{\partial^2\mathbf{r}}{\partial t^2}+2\omega\mathbf{z}\wedge\frac{\partial\mathbf{r}}{\partial t}+\omega^2\mathbf{z}\wedge(\mathbf{z}\wedge\mathbf{r}). \tag{2}$$

279. *Relative gravity at* O. The value at O of the ' apparent acceleration due to gravity ' is defined to be the acceleration, relative to a frame fixed to the earth, of a free particle at O at rest relative to the earth. Let this acceleration be $\mathbf{g}$ (a vector). Then $\mathbf{g}$ is defined by

$$\mathbf{g}=\frac{\partial^2\mathbf{r}}{\partial t^2}\qquad \text{for } \mathbf{r}=0,\qquad \frac{\partial\mathbf{r}}{\partial t}=0. \tag{3}$$

It follows that if $\mathbf{g}'$ is the actual acceleration due to gravity at O (in the frame in which the earth is rotating), of the same particle, then putting $\mathbf{r}=0$ and $\partial\mathbf{r}/\partial t=0$ in (2),

$$\mathbf{g}'=\mathbf{g}+\omega^2\mathbf{z}\wedge(\mathbf{z}\wedge\mathbf{r}_0). \tag{4}$$

Subtracting (2) and (4), we have

$$\frac{d^2\mathbf{P}}{dt^2}-\mathbf{g}'=\frac{\partial^2\mathbf{r}}{\partial t^2}+2\omega\mathbf{z}\wedge\frac{\partial\mathbf{r}}{\partial t}+\omega^2\mathbf{z}\wedge(\mathbf{z}\wedge\mathbf{r})-\mathbf{g}. \tag{5}$$

But if $\mathbf{R}$ is the external force acting on P in addition to gravity, the equation of motion of the particle P is

$$\frac{d^2\mathbf{P}}{dt^2}=\mathbf{g}'+\frac{\mathbf{R}}{m}, \tag{6}$$

m being its mass. Hence by (5),

$$\frac{\mathbf{R}}{m}+\mathbf{g}=\frac{\partial^2\mathbf{r}}{\partial t^2}+2\omega\mathbf{z}\wedge\frac{\partial\mathbf{r}}{\partial t}+\omega^2\mathbf{z}\wedge(\mathbf{z}\wedge\mathbf{r}). \tag{7}$$

This equation determines the motion relative to the rotating earth in terms of the externally applied force $\mathbf{R}$ and *apparent* gravity $\mathbf{g}$.

For the earth, the term in ω^2 can in general be neglected relative to that in ω, and (7) becomes approximately

$$\frac{\mathbf{R}}{m}+\mathbf{g}=\frac{\partial^2\mathbf{r}}{\partial t^2}+2\omega\mathbf{z}\wedge\frac{\partial\mathbf{r}}{\partial t}. \tag{8}$$

280. We pause to note the physical meaning of (4). It states that

$$\mathbf{g}=\mathbf{g}'+\omega^2[\mathbf{r}_0-\mathbf{z}(\mathbf{z}.\mathbf{r}_0)]$$
$$=\mathbf{g}'+\omega^2\text{NO},$$

where N is the foot of the perpendicular from O on to the polar axis. Thus apparent gravity $\mathbf{g}$ is the vector sum of true gravity $\mathbf{g}'$ and a vector proportional to ω^2 and to the cosine of the latitude, approximately. The direction of $\mathbf{g}$ is, of course, the direction of a plumb-bob at O.

281. *Free trajectory under gravity.* For a free trajectory the applied force $\mathbf{R}$ additional to gravity is zero. The equation of motion (8) accordingly becomes

$$\frac{\partial^2\mathbf{r}}{\partial t^2}+2\omega\mathbf{z}\wedge\frac{\partial\mathbf{r}}{\partial t}=\mathbf{g}. \tag{1}$$

This has a first integral

$$\frac{\partial\mathbf{r}}{\partial t}+2\omega\mathbf{z}\wedge\mathbf{r}=\mathbf{g}t+\mathbf{V}_0, \tag{2}$$

where $\mathbf{V}_0$, a constant vector, is the apparent velocity, relative to the earth, at $t=0$, $\mathbf{r}=0$, namely the velocity of projection $(\partial\mathbf{r}/\partial t)_0$.

To obtain a second integral,* substitute for $\partial\mathbf{r}/\partial t$ from (2) in (1). We get

$$\frac{\partial^2\mathbf{r}}{\partial t^2}+2\omega\mathbf{z}\wedge[\mathbf{g}t+\mathbf{V}_0-2\omega\mathbf{z}\wedge\mathbf{r}]=\mathbf{g}.$$

As we are neglecting terms in ω^2, this becomes

$$\frac{\partial^2\mathbf{r}}{\partial t}+2\omega\mathbf{z}\wedge(\mathbf{g}t+\mathbf{V}_0)=\mathbf{g}, \tag{3}$$

which integrates in the form

$$\frac{\partial\mathbf{r}}{\partial t}+2\omega\mathbf{z}\wedge(\tfrac{1}{2}\mathbf{g}t^2+\mathbf{V}_0t)=\mathbf{g}t+\mathbf{V}_0, \tag{4}$$

and then integrates a second time in the form

$$\mathbf{r}+\omega\mathbf{z}\wedge[\tfrac{1}{3}\mathbf{g}t^3+\mathbf{V}_0t^2]=\tfrac{1}{2}\mathbf{g}t^2+\mathbf{V}_0t. \tag{5}$$

* This device is due to a pupil of S. C.

282. *Use of local geographical axes.* To see the meaning of (5), take local axes at O (Fig. 70) moving with the earth as follows:

$O\zeta$ vertically upwards, i.e. in the direction opposite to apparent gravity $\mathbf{g}$;

$O\eta$ perpendicular to $O\zeta$, to the North;

$O\xi$ to the East.

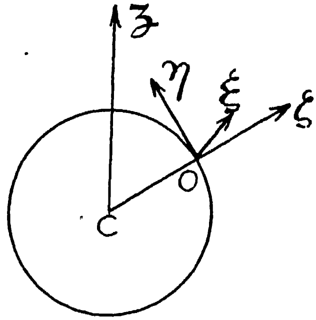

Fig. 70

Take associated unit vectors $\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}$ forming an orthogonal positive triad. Then

$$\mathbf{z} = \boldsymbol{\eta} \cos \lambda + \boldsymbol{\zeta} \sin \lambda,$$

where λ is the apparent latitude measured by the plumb-bob, i.e. $\frac{1}{2}\pi - \lambda$ is the angle between the apparent zenith and the earth's axis. We can now put

$$\mathbf{g} = -g\boldsymbol{\zeta},$$

where g is the scalar apparent gravity. Equation (5) then becomes

$$\mathbf{r} + \omega(\boldsymbol{\eta} \cos \lambda + \boldsymbol{\zeta} \sin \lambda) \wedge (-\tfrac{1}{3}gt^3\boldsymbol{\zeta} + \mathbf{V}_0 t^2) = -\tfrac{1}{2}gt^2\boldsymbol{\zeta} + \mathbf{V}_0 t.$$

or $$\mathbf{r} = \tfrac{1}{3}\omega g t^3 \cos \lambda \boldsymbol{\xi} - \omega t^2(\boldsymbol{\eta} \cos \lambda + \boldsymbol{\zeta} \sin \lambda) \wedge \mathbf{V}_0 + \mathbf{V}_0 t - \tfrac{1}{2}gt^2\boldsymbol{\zeta}. \qquad (6)$$

It follows that the correction to the position vector $\mathbf{r}$ after time t, due to the earth's rotation, is given by

$$\delta\mathbf{r} = \tfrac{1}{3}\omega g t^3 \cos \lambda \boldsymbol{\xi} - \omega t^2(\boldsymbol{\eta} \cos \lambda + \boldsymbol{\zeta} \sin \lambda) \wedge \mathbf{V}_0. \qquad (7)$$

Now let the vertical plane of projection make an angle φ to the North of East, and let $\mathbf{i}$ be a unit horizontal vector in this plane, $\mathbf{j}$ a unit horizontal vector perpendicular to $\mathbf{i}$. Then $\mathbf{i}, \mathbf{j}, \boldsymbol{\zeta}$ form a positive triad, and

$$\boldsymbol{\xi} = \mathbf{i} \cos \varphi - \mathbf{j} \sin \varphi,$$
$$\boldsymbol{\eta} = \mathbf{i} \sin \varphi + \mathbf{j} \cos \varphi,$$
$$\mathbf{V}_0 = u_0\mathbf{i} + w_0\boldsymbol{\zeta},$$

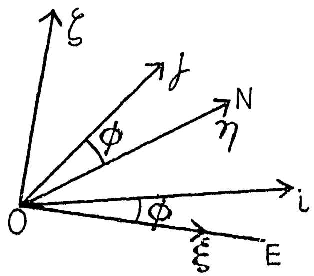

Fig. 71

where u_0, w_0 are the initial horizontal and vertical components of the velocity of projection (Fig. 71). Hence

$$\begin{aligned}\delta\mathbf{r} &= \tfrac{1}{3}\omega g t^3 \cos \lambda(\mathbf{i} \cos \varphi - \mathbf{j} \sin \varphi) \\ &\quad - \omega t^2(\boldsymbol{\zeta} \sin \lambda + \mathbf{i} \sin \varphi \cos \lambda + \mathbf{j} \cos \varphi \cos \lambda) \wedge (u_0\mathbf{i} + w_0\boldsymbol{\zeta}) \\ &= \mathbf{i}[\tfrac{1}{3}\omega g t^3 \cos \lambda \cos \varphi - \omega t^2 w_0 \cos \varphi \cos \lambda] \\ &\quad + \mathbf{j}[-\tfrac{1}{3}\omega g t^3 \cos \lambda \sin \varphi - \omega t^2 u_0 \sin \lambda + \omega t^2 w_0 \sin \varphi \cos \lambda] \\ &\quad + \boldsymbol{\zeta}[\omega t^2 u_0 \cos \varphi \cos \lambda].\end{aligned} \qquad (8)$$

In this expression the coefficients of $\mathbf{i}$, $\mathbf{j}$ and $\boldsymbol{\zeta}$ give respectively the deflexions along the horizontal in the plane of projection, to the left of the plane of projection, and vertically upwards, after a given time of flight t.

If w_0 is small compared with u_0, i.e. for nearly horizontal projection, the left deflexion for small t has the sign of $-\sin\lambda$, i.e. is to the *right* in the Northern hemisphere. This is obvious physically, since the earth is turning round underneath the flying projectile with component angular velocity $\omega \sin \lambda \zeta$.

283. *Examples on motion over the rotating earth.*

Example (1). To calculate the total deflexion of the point of fall for a complete (unresisted) trajectory. We note that the time of flight is approximately $2w_0/g$, and hence the deflexion to the *right* of the plane of projection is approximately

$$\omega\left(\frac{2w_0}{g}\right)^2[u_0 \sin\lambda - w_0 \sin\varphi \cos\lambda] + \tfrac{1}{3}\omega g\left(\frac{2w_0}{g}\right)^3 \cos\lambda \sin\varphi$$

$$= \frac{4w_0{}^2\omega}{g^2}[u_0 \sin\lambda - \tfrac{1}{3}w_0 \sin\varphi \cos\lambda].$$

For w_0/u_0 small, this is approximately

$$(4w_0{}^2\omega u_0 \sin\lambda)/g^2$$

to the right, which may be checked by noting that the deflexion is necessarily equal to

$$(\text{time of flight}) \times (\omega \sin\lambda) \times (\text{horizontal range})$$

$$= \frac{2w_0}{g} \times \omega \sin\lambda \times \frac{2w_0}{g}u_0.$$

Example (2). *Vertical projection.* Here $\mathbf{V}_0 = w_0\zeta$, and inserting this in (7) we get

$$\delta\mathbf{r} = \xi[\tfrac{1}{3}\omega g t^3 - \omega t^2 w_0] \cos\lambda,$$

so that the deflexion is in the E.–W. direction; when t is $<3w_0/g$, the deflexion is negative, i.e. to the W. For the complete up-and-down trajectory $t = 2w_0/g$, and

$$\delta\mathbf{r} = -\frac{4}{3}\frac{w_0{}^3\omega}{g^2} \cos\lambda\xi.$$

The deflexion is thus to the W. in both hemispheres.

284. *Particle in motion on a smooth horizontal plane.* For motion on a smooth horizontal plane, $\partial\mathbf{r}/\partial t$ is horizontal and the external force $\mathbf{R}$ is vertical. Putting $\mathbf{R} = R\zeta$, $\mathbf{g} = -g\zeta$, we have from (7), § 279, on neglect of ω^2,

$$\frac{\partial^2\mathbf{r}}{\partial t^2} + 2\omega(\eta \cos\lambda + \zeta \sin\lambda) \wedge \frac{\partial\mathbf{r}}{\partial t} = \left(\frac{R}{m} - g\right)\zeta. \qquad (1)$$

To eliminate R, multiply vectorially by ζ. We get, since

$$\left(\eta \wedge \frac{\partial\mathbf{r}}{\partial t}\right) \wedge \zeta = -\eta\left(\frac{\partial\mathbf{r}}{\partial t}.\zeta\right) + \frac{\partial\mathbf{r}}{\partial t}(\eta.\zeta) = 0, \qquad (\mathbf{r}.\zeta = 0)$$

and since
$$\left(\zeta\wedge\frac{\partial\mathbf{r}}{\partial t}\right)\wedge\zeta=+\frac{\partial\mathbf{r}}{\partial t},$$
the equation of motion in the form
$$\frac{\partial^2\mathbf{r}}{\partial t^2}\wedge\zeta+2\omega\sin\lambda\frac{\partial\mathbf{r}}{\partial t}=0.$$
Multiply again * vectorially by $\boldsymbol{\zeta}$. We get
$$\frac{\partial^2\mathbf{r}}{\partial t^2}+2\omega\sin\lambda\zeta\wedge\frac{\partial\mathbf{r}}{\partial t}=0. \tag{2}$$

Equation (2) integrates in the form
$$\frac{\partial\mathbf{r}}{\partial t}+2\omega\sin\lambda\zeta\wedge\mathbf{r}=\mathbf{V}_0, \tag{3}$$
where $\mathbf{V}_0$ is the velocity at $\mathbf{r}=0$. To interpret (3), express the horizontal vector $\mathbf{V}_0$ in the form
$$\mathbf{V}_0=2\omega\sin\lambda(\zeta\wedge\mathbf{a}), \qquad (\zeta.\mathbf{a}=0)$$
where, by vectorial multiplication by $\boldsymbol{\zeta}$, $\mathbf{a}$ is a vector given by
$$\mathbf{a}=\frac{\mathbf{V}_0\wedge\zeta}{2\omega\sin\lambda}.$$
Then (3) may be written
$$\frac{\partial\mathbf{r}}{\partial t}=-2\omega\sin\lambda\,\zeta\wedge(\mathbf{r}-\mathbf{a}) \tag{4}$$

This states that the motion is in a circle, centre **a**, with angular velocity $-2\omega\sin\lambda\boldsymbol{\zeta}$. The path on the horizontal plane relative to the earth is accordingly a circle (Fig. 72). At any time t, the angular arc described is of magnitude θ, where
$$\theta=(2\omega\sin\lambda)t,$$
and the deflexion to the *right* of the direction of projection is
$$|\mathbf{a}|(1-\cos\theta),$$
or, for θ small,
$$\frac{|\mathbf{V}_0|}{2\omega\sin\lambda}\tfrac{1}{2}\theta^2$$
$$=|\mathbf{V}_0|\omega t^2\sin\lambda,$$
approximately.

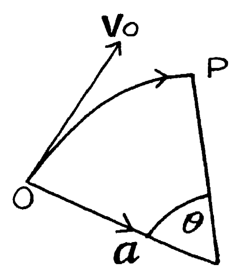

Fig. 72

Equation (3) can, of course, be integrated further by the device previously used. Substituting in (2) from (3) for $\partial\mathbf{r}/\partial t$, and neglecting ω^2, we get
$$\frac{\partial^2\mathbf{r}}{\partial t^2}+2\omega\sin\lambda(\zeta\wedge\mathbf{V}_0)=0,$$

* The result of this multiplication could have been written down directly from (1), by equating non-vertical vectors. But it is usually advisable to follow a routine, and eliminate an unknown like R by means of vectorial multiplication.

which integrates twice to give

$$\mathbf{r}=-\omega \sin \lambda(\boldsymbol{\zeta}\wedge\mathbf{V}_0)t^2+\mathbf{V}_0 t.$$

Reaction of the plane. The vertical component of (1) gives

$$2\omega \cos \lambda\left(\boldsymbol{\eta}\wedge\frac{\partial \mathbf{r}}{\partial t}\right)=\left(\frac{R}{m}-g\right)\boldsymbol{\zeta}.$$

Substituting from (3) for $\partial\mathbf{r}/\partial t$, we have

$$\left(\frac{R}{m}-g\right)\boldsymbol{\zeta}=2\omega \cos \lambda[\boldsymbol{\eta}\wedge(\mathbf{V}_0-2\omega \sin \lambda\boldsymbol{\zeta}\wedge\mathbf{r})].$$

Neglecting the term in ω^2 and writing

$$\mathbf{V}_0=u_0\boldsymbol{\xi}+v_0\boldsymbol{\eta},$$

$\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ being unit vectors to E. and N., we have

$$\left(\frac{R}{m}-g\right)\boldsymbol{\zeta}=-2\omega \cos \lambda u_0\boldsymbol{\zeta}.$$

The reaction is therefore *reduced* by the rotation, for $u_0>0$, by the amount $2\omega \cos \lambda\, u_0$.

285. *Motion on a smooth horizontal plane under an applied force.* Let **P** be the applied horizontal force. In the notation of § 284, the equation of motion is

$$\frac{\partial^2\mathbf{r}}{\partial t^2}+2\omega(\boldsymbol{\eta} \cos \lambda+\boldsymbol{\zeta} \sin \lambda)\wedge\frac{\partial \mathbf{r}}{\partial t}=\left(\frac{R}{m}-g\right)\boldsymbol{\zeta}+\frac{\mathbf{P}}{m}. \qquad \begin{matrix}(\mathbf{r}.\boldsymbol{\zeta}=0)\\(\mathbf{P}.\boldsymbol{\zeta}=0)\end{matrix}$$

Multiplying vectorially by $\boldsymbol{\zeta}$,

$$\frac{\partial^2\mathbf{r}}{\partial t^2}\wedge\boldsymbol{\zeta}+2\omega \sin \lambda\frac{\partial \mathbf{r}}{\partial t}=\frac{\mathbf{P}}{m}\wedge\boldsymbol{\zeta},$$

or, again,

$$\frac{\partial^2\mathbf{r}}{\partial t^2}+2\omega \sin \lambda\, \boldsymbol{\zeta}\wedge\frac{\partial \mathbf{r}}{\partial t}=\frac{\mathbf{P}}{m}.$$

Comparing with § 219, we see that this is of the same form as the equation of motion relative to axes rotating in their own plane with angular velocity $\omega \sin \lambda$ about the normal to the plane. In scalar components, the equations are

$$\ddot{x}-2\omega\dot{y} \sin \lambda=P_x/m,$$

$$\ddot{y}+2\omega\dot{x} \sin \lambda=P_y/m.$$

286. *Geostrophic wind.* We can seek to determine the horizontal force **P** required in order to secure that the particle shall move in a straight line relative to axes fixed in the rotating frame. In this case, $\partial^2\mathbf{r}/\partial t^2$ is parallel to $\partial\mathbf{r}/\partial t$, and hence has no component perpendicular to $\partial\mathbf{r}/\partial t$. Hence, by § 285, the required force component must be just

$$2m\omega\boldsymbol{\zeta} \sin \lambda\wedge\frac{\partial \mathbf{r}}{\partial t},$$

which is a force perpendicular to the velocity of the particle and of modulus $2m\omega \sin\lambda\, v$. This has a fundamental application in meteorology, which it is interesting to consider here.

Consider air in motion over the surface of the rotating earth. In order that the motion may be rectilinear, there must be a force normal to the direction of motion, and this must be provided by the pressure gradient. A possible motion is thus one in which the direction of motion is perpendicular to the pressure gradient. The velocity v occasioned in this way by a pressure gradient dp/dr is given by

$$v = \frac{1}{2\omega \sin\lambda} \frac{1}{\rho}\left|\frac{dp}{dr}\right|,$$

and its direction in the northern hemisphere is such that to an observer facing in the direction of the air-motion the high-pressure region is to the right.

This air-velocity is called the *geostrophic wind.* Any pressure gradient in excess of that corresponding to the actual wind velocity will be accompanied by curvature of the air-track. The radius of curvature R is given by

$$\frac{1}{\rho}\left|\frac{dp}{dr}\right| = 2\omega v \sin\lambda + \frac{v^2}{R}.$$

To see the order of magnitude of the geostrophic wind, consider a pressure difference of 5 millibars in 500 kms., and take $\rho = 0.0013$ gram. cm.$^{-3}$, $\lambda = 45°$. Then since $\omega = 2\pi/(24 \times 60 \times 60)$ we have

$$v = \frac{24 \times 60 \times 60}{2 \times 0.707 \times 2\pi} \times \frac{1}{0.0013} \times \frac{5 \times 10^3}{500 \times 10^5} \text{cm. sec.}^{-1}.$$

$$= 760 \text{ cm. sec.}^{-1} = 25 \text{ ft. sec.} = 17 \text{ m.p.h.}$$

In practice the geostrophic wind is only realized at some height (say 1–2 kms.) above the surface, as the friction of the ground contributes a force tending to reduce the geostrophic deflexion.

287. *Foucault's pendulum.* We have seen by numerous examples that motion near a given point O of the earth, relative to the earth, is equivalent to motion above a plane set of horizontal axes rotating with angular velocity $\omega \sin\lambda$ (the normal component of the earth's angular velocity) about the normal at O. The ground is in fact rotating in its own plane at this rate. It is plausible to infer from this that if a pendulum freely pivoted about its upper end is set in motion in a vertical plane, this vertical plane will appear to rotate (relative to the ground) with angular velocity $-\omega \sin\lambda$. This phenomenon was predicted by Foucault, and used to demonstrate the reality of the earth's rotation: a long pendulum was suspended inside one of the towers of Notre Dame at Paris, and the rotation of its plane observed.

The above description of the phenomenon suggests the following detailed analysis.

Let **i** be a unit vector along the string OP (see diagram, Fig. 73), $\boldsymbol{\zeta}$ a unit vector vertically upwards, l the length of the string. Putting l**i** for the position vector of P with respect to O, we have for the equation of motion of the particle P, approximately

$$l\frac{\partial^2 \mathbf{i}}{\partial t^2}+2\,\omega l\mathbf{z}\wedge\frac{\partial \mathbf{i}}{\partial t}=-g\boldsymbol{\zeta}-\frac{T}{m}\mathbf{i}, \qquad (1)$$

by (8), § 279, T being the tension and g as usual apparent gravity. Eliminating T by vectorial multiplication by **i**, we have

$$l\frac{\partial^2 \mathbf{i}}{\partial t^2}\wedge\mathbf{i}+2\,\omega l\frac{\partial \mathbf{i}}{\partial t}(\mathbf{z}.\mathbf{i})=-g\boldsymbol{\zeta}\wedge\mathbf{i}. \qquad (2)$$

In this equation put $\mathbf{i}=-\boldsymbol{\zeta}+\boldsymbol{\rho}$, (3)

where $|\boldsymbol{\rho}|$ is small compared with unity. Then neglecting $\boldsymbol{\rho}^2$, we have

$$\boldsymbol{\rho}.\boldsymbol{\zeta}=0,$$

and (2) becomes, to the same approximation,

$$-l\frac{\partial^2 \boldsymbol{\rho}}{\partial t^2}\wedge\boldsymbol{\zeta}-2\,\omega l\frac{\partial \boldsymbol{\rho}}{\partial t}(\mathbf{z}.\boldsymbol{\zeta})=-g\boldsymbol{\zeta}\wedge\boldsymbol{\rho}.$$

But $\mathbf{z}.\boldsymbol{\zeta}=\sin\lambda$,

where λ is the latitude. Hence

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial t^2}\wedge\boldsymbol{\zeta}+2\,\omega\sin\lambda\frac{\partial \boldsymbol{\rho}}{\partial t}=-\frac{g}{l}\boldsymbol{\rho}\wedge\boldsymbol{\zeta},$$

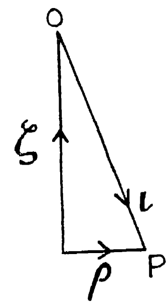

Fig. 73

or, on vector multiplication by $\boldsymbol{\zeta}$,

$$-\frac{\partial^2 \boldsymbol{\rho}}{\partial t^2}+2\,\omega\sin\lambda\frac{\partial \boldsymbol{\rho}}{\partial t}\wedge\boldsymbol{\zeta}=\frac{g}{l}\boldsymbol{\rho}. \qquad (4)$$

This is the equation governing $\boldsymbol{\rho}$, $\boldsymbol{\rho}$ being a measure of the small displacement of the bob from the equilibrium position.

We see at once that this equation describes the motion of a simple pendulum relative to axes rotating with angular velocity $2\,\omega\sin\lambda$ about the vertical; for the equation of a simple pendulum for motion near the vertical was seen in § 260 to be

$$-\frac{d^2\boldsymbol{\rho}}{dt^2}=\frac{g}{l}\boldsymbol{\rho},$$

and here $$\frac{d^2\boldsymbol{\rho}}{dt^2}=\frac{\partial^2\boldsymbol{\rho}}{\partial t^2}+2\,\omega\sin\lambda\boldsymbol{\zeta}\wedge\frac{\partial\boldsymbol{\rho}}{\partial t}.$$

This immediately suggests the solution of (4). Seek a solution

$$\frac{\partial\boldsymbol{\rho}}{\partial t}=\tilde{\omega}\boldsymbol{\zeta}\wedge\boldsymbol{\rho}$$

where $\tilde{\omega}$ is to be determined. Then

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial t^2} = +\tilde{\omega}^2 \boldsymbol{\zeta} \wedge (\boldsymbol{\zeta} \wedge \boldsymbol{\rho}) = -\tilde{\omega}^2 \boldsymbol{\rho}, \qquad \frac{\partial \boldsymbol{\rho}}{\partial t} \wedge \boldsymbol{\zeta} = \tilde{\omega} \boldsymbol{\rho}$$

so that the satisfaction of (4) requires

$$\tilde{\omega}^2 + 2\tilde{\omega}\omega \sin \lambda = g/l,$$

whence

$$\tilde{\omega}_1, \tilde{\omega}_2 = -\omega \sin \lambda \pm \left(\frac{g}{l}\right)^{\frac{1}{2}} \tag{5}$$

on neglect again of ω^2. It follows that the path ρ of the bob of the pendulum is given by

$$\boldsymbol{\rho} = \boldsymbol{\rho}_1 + \boldsymbol{\rho}_2,$$

where $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$ are vectors of constant modulus rotating with the angular velocities (5) respectively. The loci are thus Lissajous's figures and in a frame rotating with angular velocity $-\omega \sin \lambda$ are just the ellipses corresponding to the equal and opposite angular velocities $\pm(g/l)^{\frac{1}{2}}$. Consequently the motion of the bob, projected on to the ground, may be described as a rotating ellipse, rotating with period $2\pi/\omega \sin \lambda$ in the retrograde direction. In particular cases the ellipse reduces to a rotating straight line, traced and re-traced.

288. *Motion of a particle relative to a rotating frame of reference. The 'modified potential.'* Consider a particle of mass m, in motion relative to a frame of reference rotating with constant angular velocity $\omega\mathbf{z}$ about an axis in the direction of the unit vector $\mathbf{z}$. Let $\mathbf{r}$ be its position vector with respect to a point on the axis of rotation, $\mathbf{F}$ the force acting on it. The equation of motion of the particle is

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{\mathbf{F}}{m},$$

or, relative to the frame rotating with angular velocity $\omega\mathbf{z}$ (§ 219),

$$\frac{\partial^2 \mathbf{r}}{\partial t^2} + 2\omega\mathbf{z} \wedge \frac{\partial \mathbf{r}}{\partial t} + \omega^2 \mathbf{z} \wedge (\mathbf{z} \wedge \mathbf{r}) = \frac{\mathbf{F}}{m}.$$

Now let the position of P be specified by a co-ordinate ζ measured along the $\mathbf{z}$-axis, together with a vector $\boldsymbol{\rho}$ normal to the $\mathbf{z}$-axis, so that

$$\mathbf{r} = \zeta\mathbf{z} + \boldsymbol{\rho}. \qquad (\boldsymbol{\rho}.\mathbf{z} = 0).$$

Then

$$\omega^2 \mathbf{z} \wedge (\mathbf{z} \wedge \mathbf{r}) = \omega^2 \mathbf{z} \wedge (\mathbf{z} \wedge \boldsymbol{\rho}) = -\omega^2 \boldsymbol{\rho},$$

and the equation of motion becomes

$$\frac{\partial^2 \mathbf{r}}{\partial t^2} + 2\omega\mathbf{z} \wedge \frac{\partial \mathbf{r}}{\partial t} - \omega^2 \boldsymbol{\rho} = \frac{\mathbf{F}}{m}. \tag{1}$$

Multiply this scalarly by $\partial\mathbf{r}/\partial t$. Then since

$$\boldsymbol{\rho}.\frac{\partial \mathbf{r}}{\partial t} = \boldsymbol{\rho}.\frac{\partial \boldsymbol{\rho}}{\partial t},$$

we get

$$\frac{\partial^2 \mathbf{r}}{\partial t^2}.\frac{\partial \mathbf{r}}{\partial t} - \omega^2 \boldsymbol{\rho}.\frac{\partial \boldsymbol{\rho}}{\partial t} = \frac{\mathbf{F}}{m}.\frac{\partial \mathbf{r}}{\partial t}. \tag{2}$$

Now suppose that $\mathbf{F}$ arises from an external field of force, of potential V per unit mass, symmetrical with respect to the axis of rotation, together with any external reactions arising from surfaces, etc., moving with the rotating frame of reference. Then

$$\mathbf{F}=-m\frac{\partial V}{\partial \mathbf{r}}+\mathbf{R},$$

where we have

$$\mathbf{R}.\frac{\partial \mathbf{r}}{\partial t}=0,$$

since the relative motion $\partial\mathbf{r}/\partial t$ is necessarily perpendicular to the reactions, being over the surfaces giving rise to the reactions. Hence equation (2) above becomes

$$\frac{\partial^2\mathbf{r}}{\partial t^2}.\frac{\partial \mathbf{r}}{\partial t}+\frac{\partial V}{\partial \mathbf{r}}.\frac{\partial \mathbf{r}}{\partial t}-\omega^2\boldsymbol{\rho}.\frac{\partial \boldsymbol{\rho}}{\partial t}=0.$$

This integrates as it stands in the form

$$\tfrac{1}{2}\left(\frac{\partial \mathbf{r}}{\partial t}\right)^2+V-\tfrac{1}{2}\omega^2\rho^2=\text{const.} \tag{3}$$

This is the same in form as the energy integral of a particle moving in a field of potential

$$V-\tfrac{1}{2}\omega^2\rho^2,$$

ρ being the perpendicular distance of the particle from the axis of rotation. The expression $V-\frac{1}{2}\omega^2\rho^2$ in this context is called the *modified potential.*

It follows that if the moving particle has only one degree of freedom, for example if it is constrained to move on a smooth curve fixed relative to the rotating system, then its motion is fully determined by the integral (3). The actual motion is the same as if the rotation is ignored, and the potential replaced by the modified potential. This result holds good whether the smooth curve on which the particle is constrained to move is either (*a*) wholly in a plane normal to the axis of rotation ; or (*b*) wholly in a plane passing through the axis of rotation ; or (*c*) a skew curve.

289. *Driving couple.* It follows from the 'modified' energy integral (3) that the true energy

$$m\left[\tfrac{1}{2}\left(\frac{d\mathbf{r}}{dt}\right)^2+V\right]$$

is not constant. For

$$\frac{d\mathbf{r}}{dt}=\frac{\partial \mathbf{r}}{\partial t}+\omega\mathbf{z}\wedge\mathbf{r}=\frac{\partial \mathbf{r}}{\partial t}+\omega\mathbf{z}\wedge\boldsymbol{\rho},$$

and so

$$\left(\frac{d\mathbf{r}}{dt}\right)^2=\left(\frac{\partial \mathbf{r}}{\partial t}\right)^2+2\omega\mathbf{z}\wedge\boldsymbol{\rho}.\frac{\partial \boldsymbol{\rho}}{\partial t}+\omega^2\rho^2.$$

Hence the energy W is given by

$$W=m\left[\tfrac{1}{2}\left(\frac{\partial \mathbf{r}}{\partial t}\right)^2+\omega\mathbf{z}\wedge\boldsymbol{\rho}.\frac{\partial\boldsymbol{\rho}}{\partial t}+\tfrac{1}{2}\omega^2\rho^2+V\right],$$

or, using (3)

$$W=\text{const.}+m\omega\mathbf{z}\wedge\boldsymbol{\rho}.\frac{\partial\boldsymbol{\rho}}{\partial t}+m\omega^2\rho^2. \tag{4}$$

The change of energy occurring during the motion is provided through the work done on the particle by the reaction **R** of the moving surface in contact with it. For, though $\mathbf{R}.\frac{\partial \mathbf{r}}{\partial t}$ is zero, the actual rate of doing work, namely $\mathbf{R}.\frac{d\mathbf{r}}{dt}$, is not zero. In order to maintain the motion $\omega\mathbf{z}$ of the system under the opposing reaction $-\mathbf{R}$, work must be done on the rotating system from outside. The couple $G\mathbf{z}$ necessary to maintain the rotation is necessarily given by

$$(G\mathbf{z}).(\omega\mathbf{z})=\frac{dW}{dt},$$

since the left-hand side (§ 189) is the rate of performance of work by the couple. Hence

$$G=\frac{1}{\omega}\frac{dW}{dt},$$

or, by (4),

$$G=\frac{1}{\omega}\frac{d}{dt}\left[m\omega\mathbf{z}\wedge\boldsymbol{\rho}.\frac{\partial\boldsymbol{\rho}}{\partial t}+m\omega^2\rho^2\right].$$

Now the operator d/dt acting on a scalar is the same as the operator $\partial/\partial t$. Hence

$$G=m\mathbf{z}\wedge\boldsymbol{\rho}.\frac{\partial^2\boldsymbol{\rho}}{\partial t^2}+2m\omega\boldsymbol{\rho}.\frac{\partial\boldsymbol{\rho}}{\partial t}. \tag{5}$$

This evaluates the couple G necessary to drive the system in terms of the instantaneous relative acceleration and velocity, and position, of the moving particle.

Expression (5) may be checked by calculating the rate of performance of work by the reaction **R**. For

$$\mathbf{R}.\frac{d\mathbf{r}}{dt}=m\left(\frac{d^2\mathbf{r}}{dt^2}+\frac{\partial V}{\partial \mathbf{r}}\right).\frac{d\mathbf{r}}{dt}$$

$$=m\frac{d}{dt}\left[\tfrac{1}{2}\left(\frac{d\mathbf{r}}{dt}\right)^2+V\right]=\frac{dW}{dt}.$$

If the curve on which the particle is constrained to move lies entirely in a meridian plane through the axis of rotation, $\boldsymbol{\rho}\wedge\frac{\partial^2\boldsymbol{\rho}}{\partial t^2}$ is a vector normal

to this meridian plane, and hence normal to $\mathbf{z}$. Hence in this case G reduces to

$$2m\omega\rho.\frac{\partial\rho}{\partial t}.$$

But in the general case, G exceeds this by the term

$$m\mathbf{z}\wedge\rho.\frac{\partial^2\rho}{\partial t^2}.$$

290. The actual reaction $\mathbf{R}$ on the particle differs from that which would be calculated from the modified potential $V-\frac{1}{2}\omega^2\rho^2$, the rotation being otherwise ignored. For if $\mathbf{R}'$ is the latter fictitious reaction, we have

$$m\frac{\partial^2\mathbf{r}}{\partial t^2}=-m\frac{\partial}{\partial\mathbf{r}}[V-\tfrac{1}{2}\omega^2\rho^2]+\mathbf{R}',$$

whilst
$$m\frac{d^2\mathbf{r}}{dt^2}=-m\frac{\partial V}{\partial\mathbf{r}}+\mathbf{R}.$$

Hence
$$\mathbf{R}-\mathbf{R}'=m\left[\frac{d^2\mathbf{r}}{dt^2}-\frac{\partial^2\mathbf{r}}{\partial t^2}\right]+m\frac{\partial}{\partial\mathbf{r}}(\tfrac{1}{2}\omega^2\rho^2)$$
$$=m\left[2\omega\mathbf{z}\wedge\frac{\partial\mathbf{r}}{\partial t}\right]=m\left[2\omega\mathbf{z}\wedge\frac{\partial\rho}{\partial t}\right].$$

This only vanishes when the direction of relative motion $\frac{\partial\rho}{\partial t}$ is parallel to the axis of rotation.

Larmor's theorem, and the magnetic properties of electronic orbits.

291. One of the most beautiful examples of the theory of motion in rotating frames of reference is provided by Larmor's theorem relating to the rotation of electronic orbits under the influence of a magnetic field. Though the theory of the motion of electrons in magnetic fields is outside the general scope of this volume, it seems desirable to assume sufficient of the theory to enable us to prove Larmor's theorem in this its logical place. We first establish a lemma.

292. *Lemma.* Consider a system of material particles in motion in any manner. Let m be the mass of a typical particle, $\mathbf{r}$ its position vector, $\mathbf{F}$ the force acting on it. Let the motion of each particle be perturbed by the addition of a small force

$$\alpha\frac{d\mathbf{r}}{dt}\wedge\mathbf{z},$$

$\mathbf{z}$ being a given constant unit vector and α a scalar constant.

Let us examine the resulting motion relative to a frame rotating round an axis parallel to $\mathbf{z}$ with some constant angular velocity ω. The equation of the perturbed motion, namely

$$m\frac{d^2\mathbf{r}}{dt^2}=\mathbf{F}+\alpha\frac{d\mathbf{r}}{dt}\wedge\mathbf{z},$$

is equivalent to

$$m\left(\frac{\partial}{\partial t}+\omega\mathbf{z}\wedge\right)\left(\frac{\partial\mathbf{r}}{\partial t}+\omega\mathbf{z}\wedge\mathbf{r}\right)=\mathbf{F}+\alpha\left(\frac{\partial\mathbf{r}}{\partial t}+\omega\mathbf{z}\wedge\mathbf{r}\right)\wedge\mathbf{z},$$

i.e. to $$m\left[\frac{\partial^2\mathbf{r}}{\partial t^2}+2\omega\mathbf{z}\wedge\frac{\partial\mathbf{r}}{\partial t}+\omega^2\mathbf{z}\wedge(\mathbf{z}\wedge\mathbf{r})\right]=\mathbf{F}+\alpha\frac{\partial\mathbf{r}}{\partial t}\wedge\mathbf{z}+\alpha\omega(\mathbf{z}\wedge\mathbf{r})\wedge\mathbf{z}.$$

Choose ω so that the terms in $\mathbf{z}\wedge\frac{\partial\mathbf{r}}{\partial t}$ cancel. This requires

$$\omega=-\frac{1}{2}\,\frac{\alpha}{m}.$$

The equation then becomes

$$m\frac{\partial^2\mathbf{r}}{\partial t^2}=\mathbf{F}+\frac{1}{4}\,\frac{\alpha^2}{m}\mathbf{z}\wedge(\mathbf{z}\wedge\mathbf{r}).$$

If α is so small that the term in α^2 may be neglected, this equation reduces to

$$m\frac{\partial^2\mathbf{r}}{\partial t^2}=\mathbf{F}.$$

Hence, provided the forces $\mathbf{F}$ rotate with the motion, the motion in the rotating frame, after the addition of the perturbing forces, is the same as the original motion in a fixed frame. Hence the *effect of the addition of the force* $\alpha(d\mathbf{r}/dt)\wedge\mathbf{z}$ *to the forces acting on each particle is to cause approximately a uniform precession of the whole system about an axis parallel to* $\mathbf{z}$ *at the angular velocity* $-\frac{1}{2}\alpha/m$. The new motion will be *exactly* the same as the old if a force $-\frac{1}{4}\,\frac{\alpha^2}{m}\mathbf{z}\wedge(\mathbf{z}\wedge\mathbf{r})$ is added also.

For the result to be as stated, ω must be the same for each particle of the system. Hence α/m must be the same for each particle.

293. *Larmor's theorem.* Now suppose that the particles are electrons, of charge $-e$, $(e>0)$, describing orbits under an assigned field of force arising in any manner, together with their mutual interactions. Then let a magnetic field of intensity H in the $\mathbf{z}$-direction be superposed. Each electron is now subject to an additional force due to its motion in the magnetic field. This force is equal to the charge divided by c, the velocity of light, times the vector product of the velocity of the electron and the magnetic field. In the present case, this force is given by

$$\frac{-e}{c}\,\frac{d\mathbf{r}}{dt}\wedge H\mathbf{z}.$$

Hence we may apply the preceding lemma, and deduce that the effect of the addition of the magnetic field is to cause a uniform precession of the orbits about an axis in the direction of the field, of angular velocity ω given by

$$\omega=-\frac{1}{2}\left(-\frac{eH}{c}\right)\frac{1}{m},$$

or
$$\omega=eH/2cm.$$

The corresponding precessional frequency is $\omega/2\pi$, whence this frequency, say σ, is given by

$$\sigma=\frac{eH}{4\pi mc}.$$

This is called the *Larmor precession.* Its approximate validity depends upon H being small ; and e/m must be the same for all the particles concerned.

294. *Energy of the perturbed, precessing system.* The excess of the energy of the perturbed system over that of the original system is

$$\Sigma\left[\tfrac{1}{2}m\left(\frac{\partial \mathbf{r}}{\partial t}+\omega\mathbf{z}\wedge\mathbf{r}\right)^2-\tfrac{1}{2}m\left(\frac{\partial \mathbf{r}}{\partial t}\right)^2\right]$$

or, neglecting ω^2,

$$\tfrac{1}{2}\Sigma 2m\omega(\mathbf{z}\wedge\mathbf{r}).\frac{\partial \mathbf{r}}{\partial t}$$

$$=\Sigma m\omega\left(\mathbf{r}\wedge\frac{\partial \mathbf{r}}{\partial t}\right).\mathbf{z}.$$

But
$$\Sigma m\mathbf{r}\wedge\frac{\partial \mathbf{r}}{\partial t}$$

is the angular momentum about the origin of the original non-precessing system, or the apparent angular momentum about the origin in the precessing system taken relative to the rotating frame. Call this angular momentum $\mathbf{p}$. Then the increase in kinetic energy is

$$\Delta W=\omega\mathbf{p}.\mathbf{z}=\frac{eH}{2mc}\mathbf{p}.\mathbf{z}.$$

If, as in the original quantum theory of Bohr, $\mathbf{p}.\mathbf{z}$ is a multiple of $h/2\pi$, where h is Planck's constant, then

$$\Delta W=n\frac{eHh}{4\pi mc}=n\sigma h.$$

This relation is used in the theory of atomic spectra.

CHAPTER XIII

THE DYNAMICS OF SYSTEMS OF PARTICLES

295. *The momentum of a system of particles.* We have seen that a particle of mass m, moving with velocity $\mathbf{v}$, is defined to possess momentum $m\mathbf{v}$. The vector $m\mathbf{v}$ may be considered as a line vector located in the line through the particle m in the direction of $\mathbf{v}$. Now consider a system of particles, in motion in any manner. At any given instant the set of momenta of the particles constitutes a system of line vectors. This system of line vectors is described as the *momentum* of the system. It is particularly to be noted that the momentum of a system is not a vector; it is neither a free vector nor a line vector, but a *system* of line vectors. As such, it can be reduced, by the methods developed in Chapter VI, to a vector at any given point O and a couple. By the general theory of systems of line vectors, the value of the vector at O is independent of the base point O chosen. This vector is called the *linear momentum* of the system. Also by the general theory, the couple depends on the base point O chosen. This couple is called the *angular momentum* of the system about O, or sometimes the moment of momentum of the system about O.

The momentum of any system of particles can thus be specified by the linear momentum of the system together with its angular momentum about any point.

We express this in symbols. If $\mathbf{r}$ is the position vector of a typical particle of the system, of mass m, with respect to an origin O, so that $\dot{\mathbf{r}}$ is its velocity, the linear momentum $\mathbf{L}$ is given by

$$\mathbf{L} = \Sigma m\dot{\mathbf{r}},$$

and the angular momentum about O, say $\mathbf{H}(O)$, is given by

$$\mathbf{H}(O) = \Sigma \mathbf{r} \wedge m\dot{\mathbf{r}}.$$

296. *Equations of motion. Rate of change of momentum.* Before we consider further the properties of the momentum of a system of particles, it is instructive to see why the momentum is so important in dynamics. If $\mathbf{P}$ is the force acting on a particle m, the equation of motion of m is

$$\mathbf{P} = m\ddot{\mathbf{r}}.$$

Hence, summing for the system,

$$\Sigma\mathbf{P} = \Sigma m\ddot{\mathbf{r}},$$

and $$\Sigma\mathbf{r}\wedge\mathbf{P} = \Sigma\mathbf{r}\wedge m\ddot{\mathbf{r}}.$$

But since the reactions between the particles occur in equal and opposite pairs, or in the form of nul-concurrent systems, the system of forces (**P**) is equivalent to the system of external forces. This system of external forces is in turn equivalent to a force at the base point O, together with a couple $\mathbf{\Gamma}(O)$. Hence, by the conditions of equivalence of systems of line vectors,

$$\Sigma\mathbf{P}=\mathbf{R}, \qquad \Sigma\mathbf{r}\wedge\mathbf{P}=\mathbf{\Gamma}(O).$$

Hence $$\mathbf{R}=\Sigma m\ddot{\mathbf{r}}, \qquad \mathbf{\Gamma}(O)=\Sigma\mathbf{r}\wedge m\ddot{\mathbf{r}}.$$

But $$\Sigma m\ddot{\mathbf{r}}=\frac{d}{dt}[\Sigma m\dot{\mathbf{r}}]=\frac{d\mathbf{L}}{dt},$$

and $$\Sigma\mathbf{r}\wedge m\ddot{\mathbf{r}}=\frac{d}{dt}[\Sigma\mathbf{r}\wedge m\dot{\mathbf{r}}]=\frac{d\mathbf{H}(O)}{dt}.$$

Thus we have the following:

Theorem: If O is any fixed point, if $\mathbf{R}$, $\mathbf{\Gamma}(O)$ are the force and couple constituting the external forces acting on the system of particles when O is taken as base point, and if $\mathbf{L}$, $\mathbf{H}(O)$ are the linear momentum and angular momentum about O, then

$$\frac{d\mathbf{L}}{dt}=\mathbf{R}, \qquad \frac{d\mathbf{H}(O)}{dt}=\mathbf{\Gamma}(O).$$

297. *Principles of linear and angular momentum.* If the resultant external force **R** is zero, then we have

$$\mathbf{L}=\text{const.}$$

If the external couple $\mathbf{\Gamma}(O)$ about O is zero, then we have

$$\mathbf{H}(O)=\text{const.}$$

Thus we have the following theorems:

If the system of external forces reduces to zero or to a couple, the linear momentum remains constant.

If the moment of the system of external forces about a fixed point O is steadily zero, the angular momentum of the system about this point is constant.

298. *Motion of the centre of mass of a system of particles.* We have

$$\mathbf{L}=\Sigma m\dot{\mathbf{r}}=\Sigma m\frac{\Sigma m\dot{\mathbf{r}}}{\Sigma m}=(\Sigma m)\frac{d}{dt}\frac{\Sigma m\mathbf{r}}{\Sigma m}=M\dot{\bar{\mathbf{r}}},$$

where $\bar{\mathbf{r}}$ is the position vector of the centre of mass of the system and M is the total mass. From this we have

$$\mathbf{R}=M\ddot{\bar{\mathbf{r}}}.$$

These results are conveniently expressed in words thus:

Theorem: The linear momentum of a system of particles is equal to the momentum of a particle of mass equal to the total mass of the system moving with the velocity of the centre of mass of the system.

Theorem: The motion of the centre of mass of a system of particles is the same as the motion of a particle of mass equal to the total mass of the system acted on by a force equal to the vector sum of the external forces.

299. *Angular momentum.* To analyse the motion of a system of particles *relative to* the centre of mass of the system, it is convenient first to establish certain useful theorems about the angular momentum of a system of particles.

300. *Determination of the angular momentum of a system of particles about a point O′ in terms of the angular momentum about some other point* O. Take O′ as origin (Fig. 74), and let O have position vector $\mathbf{r}_0$ with respect to O′. Then if any particle P has position vectors $\mathbf{r}$, $\mathbf{r}'$ with respect to O, O′, we have

$$\mathbf{r}' = \mathbf{r}_0 + \mathbf{r}.$$

If $\mathbf{v}$ is the velocity of the particle P, m its mass, then the angular momentum about O′ is given by

$$\begin{aligned}\mathbf{H}(O') &= \Sigma \mathbf{r}' \wedge m\mathbf{v}\\ &= \Sigma(\mathbf{r}_0 + \mathbf{r}) \wedge m\mathbf{v}\\ &= \mathbf{r}_0 \wedge \Sigma m\mathbf{v} + \Sigma \mathbf{r} \wedge m\mathbf{v}\\ &= \mathbf{r}_0 \wedge \mathbf{L} + \mathbf{H}(O).\end{aligned}$$

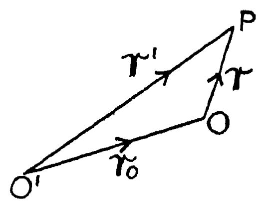

Fig. 74

Hence the following:

Theorem: The angular momentum of a system of particles about a point O′ exceeds the angular momentum about any other given point O by the moment about O′ of the linear momentum acting at O.

Corollary. If O coincides with G, the centre of mass of the system of particles, then

$$\mathbf{H}(O') = \bar{\mathbf{r}} \wedge \mathbf{L} + \mathbf{H}(G),$$

where $\bar{\mathbf{r}}$ is the position vector of the centre of mass with respect to O′.

301. *Determination of the angular momentum about a moving point* O *in terms of the angular momentum about* O *of the motion of the system relative to* O. Let O be a point, not necessarily a particle of the system, moving with velocity $\mathbf{V}$ in some fixed frame. Let $\mathbf{v}$ be the velocity, in this frame, of a typical particle P of the system, $\mathbf{v}'$ its velocity *relative* to O. Then

$$\mathbf{v} = \mathbf{V} + \mathbf{v}'.$$

Let P have a position vector $\mathbf{r}$ with respect to a fixed origin with which O momentarily coincides. Then

$$\begin{aligned}\mathbf{H}(O) = \Sigma \mathbf{r} \wedge m\mathbf{v} &= \Sigma \mathbf{r} \wedge m(\mathbf{V} + \mathbf{v}')\\ &= (\Sigma m\mathbf{r}) \wedge \mathbf{V} + \Sigma \mathbf{r} \wedge m\mathbf{v}'\\ &= M\bar{\mathbf{r}} \wedge \mathbf{V} + \mathbf{H}_r(O)\\ &= \bar{\mathbf{r}} \wedge M\mathbf{V} + \mathbf{H}_r(O),\end{aligned}$$

where $\bar{\mathbf{r}}$ is the position vector with respect to O of the centre of mass of the system and $\mathbf{H}_r(O)$ is the angular momentum about O of the motion relative to O. Hence the following:

Theorem: The angular momentum of a system of particles about any moving point O is equal to the angular momentum about O of the motion *relative to* O, together with the moment about O of a mass equal to the total mass of the system placed at the centre of mass and moving with the velocity of O.

Corollary. If O is itself the centre of mass of the system, then $\bar{\mathbf{r}}=0$, and we have

$$\mathbf{H}(G)=\mathbf{H}_r(G).$$

This is sufficiently important to be catalogued as a separate theorem.

Theorem: The angular momentum of a system about its centre of mass G is equal to the angular momentum about G of the motion of the system *relative to* G.

Thus, whatever be the motion of the centre of mass, the angular momentum about the centre of mass depends only on the motion *relative* to the centre of mass. We do not need to know the velocity of the centre of mass in order to know the angular momentum about G; we need only to know the motions relative to G.

302. *The relation between the rate of change of angular momentum about a moving point and that about a fixed point with which the moving point momentarily coincides.* In forming the equations of motion of a system in terms of the external force system, it is necessary to equate the moment of the external forces about any chosen point to the rate of change of the angular momentum about that point. It is important to recognize that in calculating the rate of change, the point in question must be kept fixed—that is to say, fixed in the frame in which the velocities are measured. This is so because $\frac{d}{dt}(\mathbf{r}\wedge m\dot{\mathbf{r}})$ is simply a convenient way of writing the more fundamental expression $\mathbf{r}\wedge m\ddot{\mathbf{r}}$, and the two are only equal in virtue of the vanishing of the vector product $\dot{\mathbf{r}}\wedge\dot{\mathbf{r}}$, wherein the second $\dot{\mathbf{r}}$ is the velocity of a material particle, and the first $\dot{\mathbf{r}}$ is the rate of change of the position vector of the particle relative to the point about which moments are being taken.

It is, however, frequently more convenient to calculate the rate of change of angular momentum about a moving point, following its motion. We therefore require to know how the two rates of change are related.

303. Let O′ be any fixed point, O a moving point whose position vector with respect to O′ at time t is $\mathbf{r}_0$. Then we have seen that

$$\mathbf{H}(O')=\mathbf{H}(O)+\mathbf{r}_0\wedge\mathbf{L},$$

where $\mathbf{H}(O')$, $\mathbf{H}(O)$ are the angular momenta about O′ and O, and $\mathbf{L}$ is the linear momentum. Differentiate with respect to the time. We obtain

$$\frac{d}{dt}\mathbf{H}(O') = \frac{d}{dt}\mathbf{H}(O) + \dot{\mathbf{r}}_0 \wedge \mathbf{L} + \mathbf{r}_0 \wedge \frac{d\mathbf{L}}{dt}.$$

Now put $\mathbf{r}_0 = 0$, $\dot{\mathbf{r}}_0 = \mathbf{V}$, so that $\mathbf{V}$ is the velocity with which O is instantaneously passing through O′. Then

$$\frac{d}{dt}\mathbf{H}(O') = \left[\frac{d}{dt}\mathbf{H}(O)\right]_{O=O'} + \mathbf{V} \wedge \mathbf{L}.$$

Hence the following :

Theorem : The rate of change of angular momentum about any fixed point O exceeds the rate of change of angular momentum about a moving point passing through O by the vector product of the velocity of the moving point and the linear momentum.

Corollary (1). The term $\mathbf{V} \wedge \mathbf{L}$ vanishes when $\mathbf{V} = 0$ or when $\mathbf{V}$ is parallel to $\mathbf{L}$ or when $\mathbf{L} = 0$. Thus the two rates of change are equal when the moving point is instantaneously at rest, or when it is moving parallel to the velocity of the centre of mass or when the centre of mass is at rest.

Corollary (2). The moving point is always moving parallel to the velocity of the centre of mass when it moves in coincidence with the centre of mass. In symbols, when $\mathbf{V} = \dot{\bar{\mathbf{r}}}$, since the linear momentum $\mathbf{L}$ is given by $\mathbf{L} = M\dot{\bar{\mathbf{r}}}$, we have $\mathbf{V} \wedge \mathbf{L} = 0$. Hence

$$\frac{d}{dt}\mathbf{H}(G') = \left[\frac{d}{dt}\mathbf{H}(G)\right]_{G=G'}.$$

This important corollary we enunciate as a separate theorem.

Theorem : The rate of change of the angular momentum of a system of particles about its centre of mass is equal to the rate of change of angular momentum about the fixed point through which the centre of mass is instantaneously passing.

304. This last theorem is so constantly used that we give a direct proof. From § 300, the corollary to the theorem gives

$$\mathbf{H}(O') = \bar{\mathbf{r}} \wedge \mathbf{L} + \mathbf{H}(G).$$

Differentiate and put $\bar{\mathbf{r}} = 0$. Then since $\dot{\bar{\mathbf{r}}} \wedge \mathbf{L} = \dot{\bar{\mathbf{r}}} \wedge M\dot{\bar{\mathbf{r}}} = 0$, we have at once

$$\frac{d\mathbf{H}(G')}{dt} = \frac{d\mathbf{H}(G)}{dt}.$$

305. By combination of the last theorem with the corollary to the theorem of § 301, we have

$$\frac{d}{dt}\mathbf{H}(G') = \frac{d}{dt}\mathbf{H}_r\,(G).$$

This gives us the following :

Theorem : The rate of change of angular momentum about a fixed point with which the centre of mass G momentarily coincides is equal to

the rate of change, following the motion, of the angular momentum about G of the motion relative to G.

306. By combination of the theorems of §§ 303, 301, we have

$$\frac{d}{dt}\mathbf{H}(O') = \left[\frac{d}{dt}\mathbf{H}_r(O)\right]_{O=O'} + \mathbf{V}\wedge\mathbf{L} + \frac{d}{dt}[\bar{\mathbf{r}}\wedge M\mathbf{V}]_{O=O'}.$$

Here $\bar{\mathbf{r}}$ is the position vector of the centre of mass with regard to O, which is itself moving with velocity $\mathbf{V}$ relative to O′. But

$$\mathbf{L} = M\left(\mathbf{V} + \frac{d\bar{\mathbf{r}}}{dt}\right).$$

Hence
$$M\frac{d\bar{\mathbf{r}}}{dt} = \mathbf{L} - M\mathbf{V},$$

and so
$$\frac{d}{dt}[\bar{\mathbf{r}}\wedge M\mathbf{V}] = (\mathbf{L} - M\mathbf{V})\wedge\mathbf{V} + \bar{\mathbf{r}}\wedge M\frac{d\mathbf{V}}{dt}.$$

Hence
$$\frac{d}{dt}\mathbf{H}(O') = \left[\frac{d}{dt}\mathbf{H}_r(O)\right]_{O=O'} + \bar{\mathbf{r}}\wedge M\frac{d\mathbf{V}}{dt}.$$

In this formula, $\bar{\mathbf{r}}$ is now the position vector of G with respect to O′. In words, we have the following theorem.

Theorem: The rate of change of angular momentum of a system of particles about any fixed point O′ exceeds the rate of change, following the motion, of the angular momentum, about a point O momentarily coinciding with O′, of the motion relative to O, by the moment about O′ of the total mass M at G taken as if moving with the acceleration of O.

307. This theorem is more simply proved directly. If $\mathbf{r}'$, $\mathbf{r}$ are the position vectors of a typical particle P of mass m with regard to O′ and O, and if $O'O = \mathbf{r}_0$, then

$$\mathbf{r}' = \mathbf{r}_0 + \mathbf{r},$$

and
$$\mathbf{H}(O') = \Sigma\mathbf{r}'\wedge m\dot{\mathbf{r}}' = \Sigma(\mathbf{r}_0 + \mathbf{r})\wedge m(\dot{\mathbf{r}}_0 + \dot{\mathbf{r}}),$$

and so
$$\frac{d}{dt}\mathbf{H}(O') = \Sigma(\mathbf{r}_0 + \mathbf{r})\wedge m(\ddot{\mathbf{r}}_0 + \ddot{\mathbf{r}}).$$

In this put
$$\mathbf{r}_0 = 0, \qquad \dot{\mathbf{r}}_0 = \mathbf{V}, \qquad \ddot{\mathbf{r}}_0 = d\mathbf{V}/dt.$$

Then
$$\frac{d}{dt}\mathbf{H}(O') = \Sigma m\mathbf{r}\wedge\frac{d\mathbf{V}}{dt} + \Sigma\mathbf{r}\wedge m\ddot{\mathbf{r}}$$

$$= \bar{\mathbf{r}}\wedge M\frac{d\mathbf{V}}{dt} + \frac{d}{dt}[\Sigma\mathbf{r}\wedge m\dot{\mathbf{r}}]$$

$$= \bar{\mathbf{r}}\wedge M\frac{d\mathbf{V}}{dt} + \left[\frac{d}{dt}\mathbf{H}_r(O)\right]_{O=O'}$$

308. The student may find the following list of results convenient for reference :

$$\mathbf{H}(O')=\mathbf{H}(O)+\mathbf{r}_0\wedge\mathbf{L}. \tag{1}$$

$$\mathbf{H}(O')=\mathbf{H}(G)+\bar{\mathbf{r}}\wedge\mathbf{L}. \tag{2}$$

$$\mathbf{H}(O)=\mathbf{H}_r(O)+\bar{\mathbf{r}}\wedge M\mathbf{V}. \tag{3}$$

$$\mathbf{H}(G)=\mathbf{H}_r(G). \tag{4}$$

$$\frac{d}{dt}\mathbf{H}(O')=\left[\frac{d}{dt}\mathbf{H}(O)\right]_{O=O'}+\mathbf{V}\wedge\mathbf{L}. \tag{5}$$

$$\frac{d}{dt}\mathbf{H}(G')=\frac{d}{dt}\mathbf{H}(G)=\frac{d}{dt}\mathbf{H}_r(G). \tag{6}$$

$$\frac{d}{dt}\mathbf{H}(O')=\left[\frac{d}{dt}\mathbf{H}_r(O)\right]_{O=O'}+\bar{\mathbf{r}}\wedge M\frac{d\mathbf{V}}{dt}. \tag{7}$$

By combination of (1) and (3) we have also

$$\mathbf{H}(O')=\mathbf{H}_r(O)+\mathbf{r}_0\wedge\mathbf{L}+\bar{\mathbf{r}}\wedge M\mathbf{V}. \tag{8}$$

309. *The kinetic energy of a system of particles.* Since the equation of motion of a single particle is

$$\mathbf{P}=m\ddot{\mathbf{r}};$$

the total rate of doing work on the system by all the forces, internal as well as external, is

$$\Sigma\mathbf{P}.\dot{\mathbf{r}}=\Sigma m\dot{\mathbf{r}}.\ddot{\mathbf{r}}=\frac{d}{dt}(\Sigma\tfrac{1}{2}m\dot{\mathbf{r}}^2).$$

The scalar $\Sigma\frac{1}{2}m\dot{\mathbf{r}}^2$ is called the kinetic energy of the system and is denoted by T.

310. *Expression of the kinetic energy of a system of particles in terms of motion relative to a moving point.* Let O be a point moving with velocity $\mathbf{V}$. Then if $\mathbf{v}$ is the velocity of a typical particle of the system in a fixed frame, $\mathbf{v}'$ its velocity relative to O, we have

$$\mathbf{v}=\mathbf{v}'+\mathbf{V}.$$

Hence

$$\begin{aligned} T&=\tfrac{1}{2}\Sigma m\mathbf{v}^2=\tfrac{1}{2}\Sigma m(\mathbf{v}'^2+2\mathbf{v}'.\mathbf{V}+\mathbf{V}^2)\\ &=\tfrac{1}{2}M\mathbf{V}^2+\mathbf{V}.\Sigma m\mathbf{v}'+\tfrac{1}{2}\Sigma m\mathbf{v}'^2\\ &=\tfrac{1}{2}M\mathbf{V}^2+M\mathbf{V}.\bar{\mathbf{v}}_r+T_r(O), \end{aligned}$$

where $T_r(O)$ is the kinetic energy of the motion relative to O, and $\bar{\mathbf{v}}_r$ is the velocity of the centre of mass of the system relative to O.

When O coincides with G, $\bar{\mathbf{v}}_r=0$ and so

$$T=\tfrac{1}{2}M\bar{\mathbf{v}}^2+T_r(G).$$

This an important result, and may be stated in words thus :

Theorem : The kinetic energy of a system of particles is equal to the kinetic energy of the motion relative to the centre of mass, together with the kinetic energy of the total mass moving with the velocity of the centre of mass.

CHAPTER XIV

RIGID BODIES IN MOTION. THE INERTIA TENSOR

311. When the system of particles contemplated in the preceding chapter constitutes a rigid body, the rigid body possesses an angular velocity $\mathbf{\Omega}$, and the angular momentum and kinetic energy may be expressed in forms involving $\mathbf{\Omega}$. The angular velocity $\mathbf{\Omega}$ usually appears in association with a certain tensor depending on the geometrical distribution of mass in the system considered. This tensor is called the *inertia tensor* of the system. Any finite system of particles possesses an inertia tensor, but the determination of the inertia tensor is chiefly useful when the system constitutes a rigid body. The components of the inertia tensor in any frame, i.e. with respect to any given triad, are called the *inertia constants* of the body in this frame, and are usually classified as *moments* and *products* of inertia. They remain constant in any triad of reference which moves with the rigid body. The inertia tensor may be reduced, by proper choice of the triad of reference, to a *diagonal* tensor. The corresponding triad defines the *principal axes of inertia* of the system.

We shall first show how the inertia tensor emerges from discussion of the angular momentum and kinetic energy of the rigid body. We shall then discuss its properties.

312. *Angular momentum of a rigid body.* Let $\mathbf{H}(\mathrm{O})$ be the angular momentum of a system of particles about a point O. Then if $\mathbf{v}$ is the velocity of a typical particle of the system, of mass m and position vector $\mathbf{r}$ with respect to O, we have by definition

$$\mathbf{H}(\mathrm{O})=\Sigma\mathbf{r}\wedge \mathrm{m}\mathbf{v}.$$

Now let the system be a rigid body, possessing an angular velocity $\mathbf{\Omega}$. If the point O is a particle of the rigid body, the velocity $\mathbf{v}$ of any particle is given by

$$\mathbf{v}=\mathbf{V}+\mathbf{\Omega}\wedge\mathbf{r},$$

where $\mathbf{V}$ is the velocity of the particle O. Hence

$$\begin{aligned}\mathbf{H}(\mathrm{O})&=\Sigma\mathbf{r}\wedge \mathrm{m}(\mathbf{V}+\mathbf{\Omega}\wedge\mathbf{r})\\ &=\Sigma \mathrm{m}\mathbf{r}\wedge\mathbf{V}+\Sigma \mathrm{m}\mathbf{r}\wedge(\mathbf{\Omega}\wedge\mathbf{r})\\ &=\mathrm{M}\bar{\mathbf{r}}\wedge\mathbf{V}+\Sigma \mathrm{m}[\mathbf{\Omega}\mathbf{r}^2-\mathbf{r}(\mathbf{r}.\mathbf{\Omega})],\end{aligned}$$

where $\bar{\mathbf{r}}$ is the position vector of the centre of mass, M the total mass.

Since $\mathbf{\Omega}$ is the same for all particles of the rigid body, it suggests itself that we should endeavour to take $\mathbf{\Omega}$ outside the last bracket as a factor. To do this, we write, in tensor notation,

$$\mathbf{r}(\mathbf{r}.\mathbf{\Omega})=(\mathbf{rr}).\mathbf{\Omega},$$

thus introducing the dyad $\mathbf{rr}$. And we write similarly

$$\mathbf{\Omega}=\mathbf{U}.\mathbf{\Omega},$$

where $\mathbf{U}$ is the idem tensor. We have then

$$\mathbf{H}(\mathrm{O})=\bar{\mathbf{r}}\wedge\mathrm{M}\mathbf{V}+[(\Sigma\mathrm{m}\mathbf{r}^2)\mathbf{U}-\Sigma\mathrm{m}\mathbf{rr}].\mathbf{\Omega}.$$

Define a tensor $\mathbf{I}(\mathrm{O})$ by the formula

$$\mathbf{I}(\mathrm{O})=(\Sigma\mathrm{m}\mathbf{r}^2)\mathbf{U}-\Sigma\mathrm{m}\mathbf{rr}.$$

We call $\mathbf{I}(\mathrm{O})$ the *inertia tensor of the rigid body about* O. Then

$$\mathbf{H}(\mathrm{O})=\bar{\mathbf{r}}\wedge\mathrm{M}\mathbf{V}+\mathbf{I}(\mathrm{O}).\mathbf{\Omega}.$$

When $\mathbf{V}=0$, the angular momentum about O is given by

$$\mathbf{H}(\mathrm{O})=\mathbf{I}(\mathrm{O}).\mathbf{\Omega}.$$

This must give the angular momentum about O of the motion relative to O. Accordingly

$$\mathbf{H}_r(\mathrm{O})=\mathbf{I}(\mathrm{O}).\mathbf{\Omega},$$

whence, in general, $$\mathbf{H}(\mathrm{O})=\bar{\mathbf{r}}\wedge\mathrm{M}\mathbf{V}+\mathbf{H}_r(\mathrm{O}),$$

in accordance with (3) of § 308. The results obtained may be stated in words as follows:

Theorem: If a rigid body is in motion with angular velocity $\mathbf{\Omega}$ with one particle O fixed, the angular momentum about O is $\mathbf{I}(\mathrm{O}).\mathbf{\Omega}$, where $\mathbf{I}(\mathrm{O})$ is the inertia tensor about O. If the particle O is in motion with velocity $\mathbf{V}$, the angular momentum about O exceeds $\mathbf{I}(\mathrm{O}).\mathbf{\Omega}$ by the moment about O of a mass equal to the total mass at the centre of mass moving with velocity $\mathbf{V}$.

When $\bar{\mathbf{r}}=0$, G coincides with O, and we have

$$\mathbf{H}(\mathrm{G})=\mathbf{I}(\mathrm{G}).\mathbf{\Omega},$$

whatever the velocity of G.

313. *Kinetic energy of a rigid body.* We shall consider first the case of a rigid body with one particle O fixed. Let $\mathbf{\Omega}$ be its angular velocity. The kinetic energy is given by

$$\mathrm{T}=\tfrac{1}{2}\Sigma\mathrm{m}\mathbf{v}^2,$$

where $$\mathbf{v}=\mathbf{\Omega}\wedge\mathbf{r}.$$

Hence
$$\begin{aligned}\mathrm{T}&=\tfrac{1}{2}\Sigma\mathrm{m}(\mathbf{\Omega}\wedge\mathbf{r}).(\mathbf{\Omega}\wedge\mathbf{r})\\&=\tfrac{1}{2}\Sigma\mathrm{m}(\mathbf{r}^2\mathbf{\Omega}^2-(\mathbf{r}.\mathbf{\Omega})^2)\\&=\tfrac{1}{2}\Sigma\mathrm{m}[\mathbf{r}^2\mathbf{\Omega}-\mathbf{r}(\mathbf{r}.\mathbf{\Omega})].\mathbf{\Omega}\\&=\tfrac{1}{2}[\{(\Sigma\mathrm{m}\mathbf{r}^2)\mathbf{U}-\Sigma\mathrm{m}\mathbf{rr}.\}.\mathbf{\Omega}].\mathbf{\Omega}\\&=\tfrac{1}{2}[\mathbf{I}(\mathrm{O}).\mathbf{\Omega}].\mathbf{\Omega}\\&=\tfrac{1}{2}\mathbf{I}(\mathrm{O}):\mathbf{\Omega\Omega}.\end{aligned}$$

This can also be written in the form

$$T=\tfrac{1}{2}\mathbf{H}(O).\mathbf{\Omega}.$$

Next, consider the case of a rigid body in general motion. Let $\mathbf{V}$ be the velocity of a particle O of the body, $\mathbf{\Omega}$ the angular velocity. Then

$$\begin{aligned}T&=\tfrac{1}{2}\Sigma m\mathbf{v}^2=\tfrac{1}{2}\Sigma m\mathbf{v}.(\mathbf{V}+\mathbf{\Omega}\wedge\mathbf{r})\\&=\tfrac{1}{2}\mathbf{V}.\Sigma m\mathbf{v}+\tfrac{1}{2}(\Sigma m\mathbf{r}\wedge\mathbf{v}).\mathbf{\Omega}\\&=\tfrac{1}{2}\mathbf{V}.\mathbf{L}+\tfrac{1}{2}\mathbf{H}(O).\mathbf{\Omega},\end{aligned}$$

where $\mathbf{L}$ is the linear momentum. Using a result of § 312, this may be put in the form

$$\begin{aligned}T&=\tfrac{1}{2}\mathbf{V}.\mathbf{L}+\tfrac{1}{2}[\bar{\mathbf{r}}\wedge M\mathbf{V}+\mathbf{I}(O).\mathbf{\Omega}].\mathbf{\Omega}\\&=\tfrac{1}{2}\mathbf{V}.\mathbf{L}+\tfrac{1}{2}(\mathbf{\Omega}\wedge\bar{\mathbf{r}}).M\mathbf{V}+\tfrac{1}{2}\mathbf{I}(O):\mathbf{\Omega}\mathbf{\Omega}.\end{aligned}$$

Now the velocity of the centre of mass is given by

$$\bar{\mathbf{v}}=\mathbf{V}+\mathbf{\Omega}\wedge\bar{\mathbf{r}}.$$

Hence the linear momentum is given by

$$\mathbf{L}=M\mathbf{V}+M(\mathbf{\Omega}\wedge\bar{\mathbf{r}}).$$

Hence*
$$T=\tfrac{1}{2}M\mathbf{V}^2+M(\mathbf{\Omega}\wedge\bar{\mathbf{r}}).\mathbf{V}+\tfrac{1}{2}\mathbf{I}(O):\mathbf{\Omega}\mathbf{\Omega}.$$

(This formula is readily proved from first principles.)

When O is the centre of mass itself, $\bar{\mathbf{r}}=0$ and $\mathbf{V}=\bar{\mathbf{v}}$, and we have

$$T=\tfrac{1}{2}M\bar{\mathbf{v}}^2+\tfrac{1}{2}\mathbf{I}(G):\mathbf{\Omega}\mathbf{\Omega}.$$

In words, this is the following :

Theorem : The kinetic energy of a rigid body is equal to the kinetic energy of the motion of the whole mass moving with the velocity of the centre of mass together with the kinetic energy of the motion relative to the centre of mass.

(This is, of course, a particular case of the theorem of § 310.)

314. It is worth while giving a separate proof of the last theorem. If the centre of mass is taken as origin, we have

$$\begin{aligned}T&=\Sigma\tfrac{1}{2}m\mathbf{v}^2=\tfrac{1}{2}\Sigma m(\bar{\mathbf{v}}+\mathbf{\Omega}\wedge\mathbf{r})^2\\&=\tfrac{1}{2}M\bar{\mathbf{v}}^2+\bar{\mathbf{v}}.\mathbf{\Omega}\wedge\Sigma m\mathbf{r}+\tfrac{1}{2}\Sigma m(\mathbf{\Omega}\wedge\mathbf{r})^2.\end{aligned}$$

But now
$$\Sigma m\mathbf{r}=0,$$

and as before
$$\Sigma m(\mathbf{\Omega}\wedge\mathbf{r})^2=\mathbf{I}(G):\mathbf{\Omega}\mathbf{\Omega}.$$

Hence
$$T=\tfrac{1}{2}M\bar{\mathbf{v}}^2+\tfrac{1}{2}\mathbf{I}(G):\mathbf{\Omega}\mathbf{\Omega}.$$

315. It is convenient now to summarize the various formulæ for the angular momentum and kinetic energy of a rigid body. In the following, $\mathbf{V}$ is the velocity of the origin O, which is itself a particle of the rigid body ;

* This result is used by Lamb, *Hydrodynamics*, 4th edition, p. 156.

$\bar{\mathbf{r}}$ is the position vector of the centre of mass G with respect to O ; $\bar{\mathbf{v}}$ is the velocity of G in a fixed frame. Then

$$\mathbf{H}(\mathrm{O})=\bar{\mathbf{r}}\wedge \mathrm{M}\mathbf{V}+\mathbf{I}(\mathrm{O}).\boldsymbol{\Omega}. \tag{1}$$

$$\mathbf{H}(\mathrm{G})=\mathbf{I}(\mathrm{G}).\boldsymbol{\Omega}. \tag{2}$$

$$\mathrm{T}=\tfrac{1}{2}\mathrm{M}\mathbf{V}^2+\mathrm{M}(\boldsymbol{\Omega}\wedge\bar{\mathbf{r}}).\mathbf{V}+\tfrac{1}{2}\mathbf{I}(\mathrm{O}):\boldsymbol{\Omega}\boldsymbol{\Omega}. \tag{3}$$

$$=\tfrac{1}{2}\mathrm{M}\bar{\mathbf{v}}^2+\tfrac{1}{2}\mathbf{I}(\mathrm{G}):\boldsymbol{\Omega}\boldsymbol{\Omega}. \tag{4}$$

If the origin of reference O is a particle of the body at rest,

$$\mathbf{H}(\mathrm{O})=\mathbf{I}(\mathrm{O}).\boldsymbol{\Omega}. \tag{5}$$

$$\mathrm{T}=\tfrac{1}{2}\mathbf{I}(\mathrm{O}):\boldsymbol{\Omega}\boldsymbol{\Omega} \tag{6}$$

$$=\tfrac{1}{2}\mathbf{H}(\mathrm{O}).\boldsymbol{\Omega}. \tag{7}$$

We have proved also (§ 313),

$$\mathrm{T}=\tfrac{1}{2}\mathbf{V}.\mathbf{L}+\tfrac{1}{2}\boldsymbol{\Omega}.\mathbf{H}(\mathrm{O}). \tag{8}$$

316. *Rate of performance of work on a rigid body by a system of forces.* Since the kinetic energy T is given by

$$\mathrm{T}=\Sigma\tfrac{1}{2}m\dot{\mathbf{r}}^2$$

we have
$$\frac{d\mathrm{T}}{dt}=\Sigma m\dot{\mathbf{r}}.\ddot{\mathbf{r}},$$
$$=\Sigma m\ddot{\mathbf{r}}.(\mathbf{V}+\boldsymbol{\Omega}\wedge\mathbf{r}),$$

where as usual $\mathbf{V}$ is the velocity of the particle O of the rigid body taken as origin and $\boldsymbol{\Omega}$ is the angular velocity. If $(\mathbf{R}, \boldsymbol{\Gamma}(\mathrm{O}))$ is the force system and couple about O, we shall see in Chapter XV that the equations of motion of the rigid body can be written in the form

$$\mathbf{R}=\Sigma\mathrm{M}\ddot{\mathbf{r}}, \qquad \boldsymbol{\Gamma}(\mathrm{O})=\Sigma m\mathbf{r}\wedge\mathbf{r}.$$

Hence
$$\frac{d\mathrm{T}}{dt}=\mathbf{V}.\mathbf{R}+\boldsymbol{\Omega}.\boldsymbol{\Gamma}(\mathrm{O}).$$

317. *Relations between kinetic energy and linear and angular momentum.* Consider the motion of the rigid body during a short time dt. Since the equations of motion are of the form

$$\frac{d\mathbf{L}}{dt}=\mathbf{R}, \qquad \frac{d\mathbf{H}(\mathrm{O})}{dt}=\boldsymbol{\Gamma}(\mathrm{O}),$$

it follows that the change of kinetic energy in time dt is given by

$$d\mathrm{T}=\mathbf{V}.(\mathbf{R}dt)+\boldsymbol{\Omega}.[\boldsymbol{\Gamma}(\mathrm{O})dt]$$
$$=\mathbf{V}.d\mathbf{L}+\boldsymbol{\Omega}.d\mathbf{H}(\mathrm{O}).$$

It follows that if we take $\mathbf{L}$ and $\mathbf{H}(\mathrm{O})$ as independent vector variables describing the motion of the body in place of $\mathbf{V}$ and $\boldsymbol{\Omega}$, then

$$\frac{\partial\mathrm{T}}{\partial\mathbf{L}}=\mathbf{V}, \qquad \frac{\partial\mathrm{T}}{\partial\mathbf{H}(\mathrm{O})}=\boldsymbol{\Omega}.$$

But since $$T=\tfrac{1}{2}\mathbf{V}.\mathbf{L}+\tfrac{1}{2}\mathbf{\Omega}.\mathbf{H}(O),$$
it follows that
$$2dT=\mathbf{V}.d\mathbf{L}+\mathbf{L}.d\mathbf{V}+\mathbf{\Omega}.d\mathbf{H}(O)+\mathbf{H}(O).d\mathbf{\Omega}.$$
But $$dT=\mathbf{V}.d\mathbf{L}+\mathbf{\Omega}.d\mathbf{H}(O).$$
Hence, subtracting, $$dT=\mathbf{L}.d\mathbf{V}+\mathbf{H}(O).d\mathbf{\Omega}.$$
Hence when $\mathbf{V}$ and $\mathbf{\Omega}$ are taken as independent vector variables describing the motion of the body, we have
$$\frac{\partial T}{\partial \mathbf{V}}=\mathbf{L}, \qquad \frac{\partial T}{\partial \mathbf{\Omega}}=\mathbf{H}(O).$$

THE INERTIA TENSOR

318. *Components of the inertia tensor.* Having seen the emergence of a definite tensor, the inertia tensor $\mathbf{I}(O)$ defined by
$$\mathbf{I}(O)=(\Sigma m\mathbf{r}^2)\mathbf{U}-\Sigma m\mathbf{rr},$$
in the calculation of the angular momentum and kinetic energy of a rigid body, we proceed to the study of this tensor.

We note that $\Sigma m\mathbf{r}^2$ is a 'weighted' sum of scalars, $\Sigma m\mathbf{rr}$ a 'weighted' sum of dyads. Either can also be written as an integral. In the suffix notation, if x_α ($\alpha=1, 2, 3$) or (x, y, z) are the components of the position vector $\mathbf{r}$, reckoned from O as origin, then
$$\begin{aligned}[\mathrm{I}(O)]_{\alpha\beta}&=(\Sigma m r^2)U_{\alpha\beta}-\Sigma m r_\alpha r_\beta\\ &=\Sigma m(x^2+y^2+z^2)U_{\alpha\beta}-\Sigma m x_\alpha x_\beta.\end{aligned}$$
Thus the components of $\mathbf{I}(O)$ are given by the scheme
$$\mathbf{I}(O)=\begin{array}{c|ccc} & \beta=1 & \beta=2 & \beta=3\\ \hline \alpha=1 & A & -H & -G\\ \alpha=2 & -H & B & -F\\ \alpha=3 & -G & -F & C\end{array}$$
where
$$\begin{aligned}[\mathrm{I}(O)]_{11}&=\Sigma m(x^2+y^2+z^2)-\Sigma m x^2=\Sigma m(y^2+z^2)=A,\\ [\mathrm{I}(O)]_{22}&=\Sigma m(x^2+y^2+z^2)-\Sigma m y^2=\Sigma m(z^2+x^2)=B,\\ [\mathrm{I}(O)]_{33}&=\Sigma m(x^2+y^2+z^2)-\Sigma m z^2=\Sigma m(x^2+y^2)=C,\\ [\mathrm{I}(O)]_{23}&=[\mathrm{I}(O)]_{32}=-\Sigma m yz=-F,\\ [\mathrm{I}(O)]_{31}&=[\mathrm{I}(O)]_{13}=-\Sigma m zx=-G,\\ [\mathrm{I}(O)]_{12}&=[\mathrm{I}(O)]_{21}=-\Sigma m xy=-H.\end{aligned}$$

Of the six quantities A, B, C, F, G, H the first three, A, B, C (which are essentially positive), are called *moments of inertia*; the second three, F, G, H, are called *products of inertia*—in each case with respect to the

axes of the triad of reference. The reason for the name *moments of inertia* will be apparent shortly.

Clearly $\mathbf{I}(O)$ is a symmetrical or self-conjugate tensor. Hence, if $\mathbf{X}$ is any vector,

$$\mathbf{I}(O).\mathbf{X}=\mathbf{X}.\mathbf{I}(O),$$

and
$$\mathbf{X}.\mathbf{I}(O).\mathbf{X}=\mathbf{XX}:\mathbf{I}(O)=\mathbf{I}(O):\mathbf{XX}.$$

If the components of $\mathbf{\Omega}$ in a given triad are ω_1, ω_2, ω_3, the components of $\mathbf{I}(O).\mathbf{\Omega}$ are

$$(A\omega_1-H\omega_2-G\omega_3, \quad -H\omega_1+B\omega_2-F\omega_3, \quad -G\omega_1-F\omega_2+C\omega_3).$$

When O coincides with a particle of the body at rest, these are accordingly the components of the angular momentum about the origin.

The value of the scalar $\mathbf{I}(O):\mathbf{\Omega\Omega}$ is

$$\omega_1(A\omega_1-H\omega_2-G\omega_3)+\omega_2(-H\omega_1+B\omega_2-F\omega_3)+\omega_3(-G\omega_1-F\omega_2+C\omega_3)$$
$$=A\omega_1{}^2+B\omega_2{}^2+C\omega_3{}^2-2F\omega_2\omega_3-2G\omega_3\omega_1-2H\omega_1\omega_2.$$

When O is a particle at rest, this is accordingly twice the kinetic energy of the body.

When the triad of reference is chosen in such a way that $\mathbf{I}(O)$ reduces to its principal diagonal, i.e. when the tensor $\mathbf{I}(O)$ is referred to its principal axes, the components of $\mathbf{I}(O).\mathbf{\Omega}$ reduce to

$$A\omega_1, \qquad B\omega_2, \qquad C\omega_3,$$

and the value of $\mathbf{I}(O):\mathbf{\Omega\Omega}$ reduces to

$$A\omega_1{}^2+B\omega_2{}^2+C\omega_3{}^2.$$

The principal axes of the tensor $\mathbf{I}(O)$ are called the *principal axes of inertia* of the system. They are, of course, the axes of the quadric

$$\mathbf{I}(O):\mathbf{rr}=\text{const.},$$

i.e.
$$Ax^2+By^2+Cz^2-2Fyz-2Gzx-2Hxy=\text{const.}$$

Since sca $\mathbf{I}(O)=A+B+C$, the quantity $A+B+C$ takes the same value in all triads of reference. This is otherwise obvious since

$$A+B+C=\text{sca }[(\Sigma m\mathrm{r}^2)\mathbf{U}-\Sigma m\mathbf{rr}]=3\Sigma m\mathrm{r}^2-\Sigma m\mathrm{r}^2=2\Sigma m\mathrm{r}^2,$$

or, in components,

$$A+B+C=\Sigma m(y^2+z^2)=2m\Sigma(x^2+y^2+z^2).$$

Other invariants are $\mathbf{I}(O):\mathbf{I}(O)$, which expands to

$$A^2+B^2+C^2+2(F^2+G^2+H^2),$$

and the determinant

$$\begin{vmatrix} A & -H & -G \\ -H & B & -F \\ -G & -F & C \end{vmatrix}$$

Example. Prove that $\mathbf{I}(O):\mathbf{I}(O)=(\Sigma m\mathrm{r}^2)^2+(\Sigma m\mathbf{rr}:\Sigma m\mathbf{rr})$.

319. *Determination of the inertia tensor about* O *in terms of the inertia tensor about the centre of mass* G. Let $\bar{\mathbf{r}}$ be the position vector of G with respect to O, and $\mathbf{r}$, $\mathbf{r}'$ the position vectors of any particle P with respect to O and G respectively. Then

$$\mathbf{r}=\bar{\mathbf{r}}+\mathbf{r}',$$

where

$$\Sigma m\mathbf{r}'=0.$$

Hence

$$\begin{aligned}\mathbf{I}(\mathrm{O})&=(\Sigma m\mathbf{r}^2)\mathbf{U}-\Sigma m\mathbf{rr}\\&=\mathbf{U}\Sigma m(\bar{\mathbf{r}}+\mathbf{r}')^2-\Sigma m(\bar{\mathbf{r}}+\mathbf{r}')(\bar{\mathbf{r}}+\mathbf{r}')\\&=\mathbf{U}[(\Sigma m)\bar{\mathbf{r}}^2+\Sigma m\mathbf{r}'^2]-(\Sigma m)\bar{\mathbf{r}}\bar{\mathbf{r}}-\Sigma m\mathbf{r}'\mathbf{r}'\\&=\mathrm{M}[\bar{\mathbf{r}}^2\mathbf{U}-\bar{\mathbf{r}}\bar{\mathbf{r}}]+\mathbf{I}(\mathrm{G}).\end{aligned}$$

This result is of fundamental importance in the calculation of inertia tensors. We state it as the following :

Theorem : The inertia tensor about any point O is equal to the inertia tensor about the centre of mass G together with the inertia tensor about O of a particle at G of mass equal to the total mass.

Corollary. If two dynamical systems have the same inertia tensors about some given point O and if they have the same mass and the same centre of mass, then their inertia tensors about any point are equal.

Such systems are called equi-momental.

Example. Establish the equivalence of the two formulæ (3) and (4) of § 315 for T by direct transformation of the inertia tensor.

We have as formula (3) the equality

$$\mathrm{T}=\tfrac{1}{2}\mathrm{M}\mathbf{V}^2+\mathrm{M}(\boldsymbol{\Omega}\wedge\bar{\mathbf{r}}).\mathbf{V}+\tfrac{1}{2}\mathbf{I}(\mathrm{O}):\boldsymbol{\Omega}\boldsymbol{\Omega},$$

and we know that

$$\mathbf{I}(\mathrm{O})=\mathbf{I}(\mathrm{G})+\mathrm{M}(\bar{\mathbf{r}}^2\mathbf{U}-\bar{\mathbf{r}}\bar{\mathbf{r}}).$$

But

$$\begin{aligned}\mathrm{M}(\bar{\mathbf{r}}^2\mathbf{U}-\bar{\mathbf{r}}\bar{\mathbf{r}}):\boldsymbol{\Omega}\boldsymbol{\Omega}&=\mathrm{M}(\bar{\mathbf{r}}^2\boldsymbol{\Omega}^2-(\bar{\mathbf{r}}.\boldsymbol{\Omega})^2)\\&=\mathrm{M}(\boldsymbol{\Omega}\wedge\bar{\mathbf{r}})^2\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{T}&=\tfrac{1}{2}\mathrm{M}[\mathbf{V}+\boldsymbol{\Omega}\wedge\bar{\mathbf{r}}]^2+\tfrac{1}{2}\mathbf{I}(\mathrm{G}):\boldsymbol{\Omega}\boldsymbol{\Omega}\\&=\tfrac{1}{2}\mathrm{M}\mathbf{v}^2+\tfrac{1}{2}\mathbf{I}(\mathrm{G}):\boldsymbol{\Omega}\boldsymbol{\Omega},\end{aligned}$$

which is formula (4).

320. *Moments of inertia. Definition.* We have already described A, B, C, the elements of the principal diagonal of the inertia tensor in any triad, as the moments of inertia about the members of the triad. Now

$$\mathrm{A}=\Sigma m(y^2+z^2)=\Sigma m p_x{}^2,$$

where p_x is the perpendicular from a typical particle upon the x-axis. This suggests the following general definition. Let $\mathbf{i}$ be a unit vector in any given line l through a point O, p the perpendicular distance of a typical particle from l. Then we define the moment of inertia of the system about l as Σmp^2, and we denote it by the symbol $\mu(\mathrm{O}, \mathbf{i})$.

321. *Relation of moments of inertia to the inertia tensor.* We proceed to determine the moment of inertia of a system about any line l in terms of the inertia tensor about any point O in the line l.

Let $\mathbf{i}$ be a unit vector in the line l (Fig. 75). Then if $\mathbf{r}$ is the position vector of any point with respect to O, p its perpendicular distance from l,

$$p^2=(\mathbf{r}\wedge\mathbf{i})^2.$$

Hence
$$\mu(O, \mathbf{i})=\Sigma m(\mathbf{r}\wedge\mathbf{i})^2$$
$$=\Sigma m(\mathbf{r}^2-(\mathbf{r}.\mathbf{i})^2)$$
$$=\Sigma m\mathbf{r}^2-\Sigma m\mathbf{r}(\mathbf{r}.\mathbf{i}).\mathbf{i}.$$

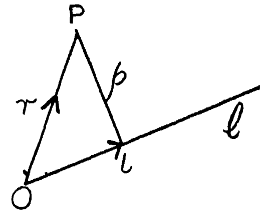

Fig. 75

But
$$1=\mathbf{i}^2=\mathbf{i}.\mathbf{i}=(\mathbf{U}.\mathbf{i}).\mathbf{i}=\mathbf{U}:\mathbf{ii}$$
and
$$\mathbf{r}(\mathbf{r}.\mathbf{i}).\mathbf{i}=[(\mathbf{rr}).\mathbf{i}].\mathbf{i}=\mathbf{rr}:\mathbf{ii}.$$
Hence
$$\mu(O, \mathbf{i})=(\Sigma m\mathbf{r}^2\mathbf{U}-\Sigma m\mathbf{rr}):\mathbf{ii}$$
$$=\mathbf{I}(O):\mathbf{ii}.$$

This is the desired relation.

It follows that if, in any given triad, the direction cosines of $\mathbf{i}$ are (l, m, n) then

$$\mu(O, \mathbf{i})=Al^2+Bm^2+Cn^2-2Fmn-2Gnl-2Hlm.$$

322. *Kinetic energy of a rigid body in terms of the moment of inertia about the instantaneous axis.* Consider a rigid body in motion about a fixed particle O. Let $\mathbf{\Omega}$ be its angular velocity, $\mathbf{i}$ a unit vector in the direction of $\mathbf{\Omega}$. If $\mathbf{v}$ is the velocity of a typical particle of mass m, we have

$$T=\Sigma\tfrac{1}{2}m\mathbf{v}^2=\Sigma\tfrac{1}{2}m(\mathbf{\Omega}\wedge\mathbf{r})^2=\tfrac{1}{2}\Omega^2\Sigma m(\mathbf{i}\wedge\mathbf{r})^2$$
$$=\tfrac{1}{2}\mathbf{\Omega}^2\mu(O, \mathbf{i}).$$

This also follows from the two formulæ

$$T=\tfrac{1}{2}\mathbf{I}(O):\mathbf{\Omega\Omega}$$
$$\mu(O, \mathbf{i})=\mathbf{I}(O):\mathbf{ii}.$$

323. *Determination of moment of inertia about an axis in terms of that about a parallel axis through* G. Let O be any given point, $\mathbf{i}$ a unit vector, G the centre of mass. Then the moment of inertia about an axis in the direction of $\mathbf{i}$, through O, is given by

$$\mu(O, \mathbf{i})=\mathbf{I}(O):\mathbf{ii}$$

or, by § 319,
$$\mu(O, \mathbf{i})=M[\bar{\mathbf{r}}^2\mathbf{U}-\bar{\mathbf{r}}\bar{\mathbf{r}}]:\mathbf{ii}+\mathbf{I}(G):\mathbf{ii}$$
$$=M[\bar{\mathbf{r}}^2-(\bar{\mathbf{r}}.\mathbf{i})^2]+\mathbf{I}(G):\mathbf{ii}$$
$$=Mp^2+\mu(G, \mathbf{i}),$$

where p is the perpendicular from G on the axis $\mathbf{i}$ through O.

324. *Determination of the moment of inertia about an axis of line co-ordinates* (**i**, **a**) *with respect to* O, *in terms of* **I**(O). The foot of the perpendicular from O to (**i**, **a**), say N, has a position vector $\mathbf{i}\wedge\mathbf{a}$ with respect to O (§ 124). Hence by the definition of the inertia tensor,

$$\begin{aligned}\mathbf{I}(\mathrm{N}) &= \Sigma m(\mathbf{r}-\mathbf{i}\wedge\mathbf{a})^2\mathbf{U}-\Sigma m(\mathbf{r}-\mathbf{i}\wedge\mathbf{a})(\mathbf{r}-\mathbf{i}\wedge\mathbf{a})\\ &= \mathbf{I}(\mathrm{O})-2\mathrm{M}(\bar{\mathbf{r}}.\mathbf{i}\wedge\mathbf{a})\mathbf{U}+\mathrm{M}\mathbf{a}^2\mathbf{U}+\mathrm{M}[(\mathbf{i}\wedge\mathbf{a})\bar{\mathbf{r}}+\bar{\mathbf{r}}(\mathbf{i}\wedge\mathbf{a})]\\ &\quad -\mathrm{M}(\mathbf{i}\wedge\mathbf{a})(\mathbf{i}\wedge\mathbf{a}).\end{aligned}$$

Now, if $\mu(\mathbf{i}, \mathbf{a})$ denotes the moment of inertia about the line (**i**, **a**),

$$\mu(\mathbf{i}, \mathbf{a}) = \mathbf{I}(\mathrm{N}):\mathbf{ii}.$$

But
$$\begin{aligned}[(\mathbf{i}\wedge\mathbf{a})\bar{\mathbf{r}}]:\mathbf{ii} &= (\mathbf{i}\wedge\mathbf{a})(\bar{\mathbf{r}}.\mathbf{i}).\mathbf{i} = 0,\\ [\bar{\mathbf{r}}(\mathbf{i}\wedge\mathbf{a})]:\mathbf{ii} &= 0,\\ (\mathbf{i}\wedge\mathbf{a})(\mathbf{i}\wedge\mathbf{a}):\mathbf{ii} &= 0,\\ \mathbf{U}:\mathbf{ii} = \mathbf{i}.\mathbf{i} &= 1.\end{aligned}$$

Hence
$$\begin{aligned}\mu(\mathbf{i}, \mathbf{a}) &= \mathbf{I}(\mathrm{O}):\mathbf{ii}-2\mathrm{M}\bar{\mathbf{r}}.(\mathbf{i}\wedge\mathbf{a})+\mathrm{M}\mathbf{a}^2\\ &= \mu(\mathrm{O}, \mathbf{i})-2\mathrm{M}\bar{\mathbf{r}}.(\mathbf{i}\wedge\mathbf{a})+\mathrm{M}\mathbf{a}^2.\end{aligned}$$

Corollary. If $\bar{\mathbf{r}} = 0$, we have

$$\mu(\mathbf{i}, \mathbf{a}) = \mu(\mathrm{G}, \mathbf{i})+\mathrm{M}\mathbf{a}^2$$

and we recover the result of § 323, for now $\mathbf{a}^2$ is the square of the perpendicular from G to the line (**i**, **a**). Stated in words, we have the following:

Theorem: The moment of inertia of a system about any given line is equal to the moment of inertia about a parallel line through G, together with the moment of inertia of the whole mass concentrated at G about the given line.

325. *Momental ellipsoid.* Let O be a fixed point, **I**(O) the inertia tensor of a given system about O. Then if **r** is the position vector of any point P with respect to O, **I**(O):**rr** is an invariant, independent of the triad of reference. Now consider the locus of P such that

$$\mathbf{I}(\mathrm{O}):\mathbf{rr} = \text{const.}$$

This locus has for its equation, in a triad of reference in which **r** is (x, y, z),

$$\mathrm{A}x^2+\mathrm{B}y^2+\mathrm{C}z^2-2\mathrm{F}yz-2\mathrm{G}zx-2\mathrm{H}xy = \text{const.},$$

A, ... H being the inertia constants about O with respect to this triad of reference. If the triad of reference is such that **I**(O) reduces to its principal diagonal, the locus reduces to

$$\mathrm{A}x^2+\mathrm{B}y^2+\mathrm{C}z^2 = \text{const.},$$

and since A, B, C are essentially positive, this locus is an ellipsoid. It is called the *momental ellipsoid* at O.

Let $\mathbf{i}$ be a unit vector in the direction of $\mathbf{r}$. Then $\mathbf{r}=|\mathbf{r}|\mathbf{i}$, and the equation of the momental ellipsoid at O may be written

$$\mathbf{I}(O):\mathbf{ii}=\frac{\text{const.}}{\mathbf{r}^2},$$

whence $$\mu(O,\mathbf{i})=\frac{\text{const.}}{\mathbf{r}^2}.$$

Hence the following:

Theorem: The moment of inertia about any axis through O is inversely proportional to the square of the corresponding radius vector of the momental ellipsoid.

326. Given the momental ellipsoid at G, we can determine the momental ellipsoid at any other point O. For

$$\mathbf{I}(O)=\mathbf{I}(G)+M(\bar{\mathbf{r}}^2\mathbf{U}-\bar{\mathbf{r}}\bar{\mathbf{r}}),$$

where $\bar{\mathbf{r}}$ is the position vector of G with respect to O, and accordingly the momental ellipsoid at O has for its equation

$$\mathbf{I}(G):\mathbf{rr}+M[\bar{\mathbf{r}}^2\mathbf{U}-\bar{\mathbf{r}}\bar{\mathbf{r}}]:\mathbf{rr}=\text{const.}$$

It is of interest to see the Cartesian form of this. Let $\mathbf{r}$ be (x, y, z) and let $\bar{\mathbf{r}}$ be (f, g, h). Then

$$\bar{\mathbf{r}}^2\mathbf{U}:\mathbf{rr}=\bar{\mathbf{r}}^2\mathbf{r}^2=(f^2+g^2+h^2)(x^2+y^2+z^2),$$
$$\bar{\mathbf{r}}\bar{\mathbf{r}}:\mathbf{rr}=(\bar{\mathbf{r}}.\mathbf{r})^2=(fx+gy+hz)^2.$$

Hence the momental ellipsoid at O has for its Cartesian equation

$$\Sigma Ax^2-2\Sigma Fyz+M[(f^2+g^2+h^2)(x^2+y^2+z^2)-(fx+gy+hz)^2]=\text{const.}$$

i.e. $$\Sigma x^2[A+M(g^2+h^2)]-2\Sigma yz[F+Mgh]=\text{const.}$$

327. *Ellipsoid of gyration.* The inertia tensor $\mathbf{I}(O)$ possesses an inverse $\mathbf{I}^{-1}(O)$. The quadric

$$\mathbf{I}^{-1}(O):\mathbf{rr}=\frac{1}{M}$$

is called the ellipsoid of gyration at O.

If the moment of inertia $\mu(O,\mathbf{i})$ about an axis in the direction of $\mathbf{i}$, through O, is written in the form Mp^2, p is called the *radius of gyration* of the system about $\mathbf{i}$. The ellipsoid of gyration possesses the property enunciated as follows:

Theorem: The length of the perpendicular from O on to a tangent plane of the ellipsoid of gyration at O is equal to the radius of gyration about an axis in the direction of this perpendicular.

For, if p is the (scalar) perpendicular from O to the tangent plane at a point P, $\mathbf{r}$ the position vector OP, $\mathbf{i}$ a unit vector in the direction of the perpendicular ON (Fig. 76), then

$$p=\mathbf{r}.\mathbf{i}.$$

But by a property of any self-conjugate tensor, the normal at $\mathbf{r}$ to the tensor quadric

$$\mathbf{I}^{-1}(\mathrm{O}):\mathbf{rr} = \text{const.}$$

is parallel to the vector $\mathbf{I}^{-1}(\mathrm{O}).\mathbf{r}$. Put then

$$\mathbf{i} = \lambda\mathbf{I}^{-1}(\mathrm{O}).\mathbf{r}.$$

Then
$$\mathrm{p} = \lambda\mathbf{I}^{-1}(\mathrm{O}):\mathbf{rr} = \lambda/\mathrm{M}.$$

Hence the moment of inertia about an axis through O along $\mathbf{i}$ is given by

$$\begin{aligned}\mu(\mathrm{O}, \mathbf{i}) &= \mathbf{I}(\mathrm{O}):\mathbf{ii}\\ &= \lambda^2\mathbf{I}(\mathrm{O}):[\mathbf{I}^{-1}(\mathrm{O}).\mathbf{r}][\mathbf{I}^{-1}(\mathrm{O}).\mathbf{r}]\\ &= \lambda^2\mathbf{r}.\mathbf{I}^{-1}(\mathrm{O}).\mathbf{r}\\ &= (\mathrm{Mp})^2/\mathrm{M}\\ &= \mathrm{Mp}^2.\end{aligned}$$

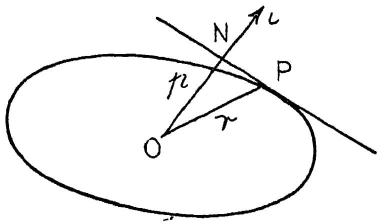

Fig. 76

Hence p is the radius of gyration about $\mathbf{i}$.

We see in particular that if the principal radii of gyration at O are α, β, γ, so that $\mathrm{A} = \mathrm{M}\alpha^2$, $\mathrm{B} = \mathrm{M}\beta^2$, $\mathrm{C} = \mathrm{M}\gamma^2$, then the ellipsoid of gyration at O has for its equation

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1.$$

Any confocal to this may be written

$$\frac{x^2}{\alpha^2+\lambda} + \frac{y^2}{\beta^2+\lambda} + \frac{z^2}{\gamma^2+\lambda} = 1,$$

or, in tensor form

$$[\mathbf{I}(\mathrm{G}) + \lambda\mathrm{M}\mathbf{U}]^{-1}:\mathbf{rr} = 1/\mathrm{M}.$$

328. *Relation of principal axes of inertia at any point to the ellipsoid of gyration at the mass centre.* We now prove* the following:

Theorem: The principal axes of inertia at any point are the three normals at that point to the three quadrics through it confocal with the ellipsoid of gyration at the mass centre.

Let G be the centre of mass, P any point. If $\bar{\mathbf{r}}$ is the position vector of P with respect to G, we have

$$\mathbf{I}(\mathrm{P}) = \mathbf{I}(\mathrm{G}) + \mathrm{M}(\bar{\mathbf{r}}^2\mathbf{U} - \bar{\mathbf{r}}\bar{\mathbf{r}}).$$

We desire to determine the principal axes of $\mathbf{I}(\mathrm{P})$, i.e. the vectors $\mathbf{i}$ for which

$$\mathbf{I}(\mathrm{P}).\mathbf{i} = \theta\mathbf{i}.$$

The corresponding values of θ will be the corresponding principal moments of inertia (§ 83).

* Proof adapted from Weatherburn, *Advanced Vector Analysis*, p. 107.

It follows from § 327 that the vector $\mathbf{i}$ must satisfy

$$[\mathbf{I}(G)+M(\bar{\mathbf{r}}^2\mathbf{U}-\bar{\mathbf{r}}\bar{\mathbf{r}})].\mathbf{i}=\theta\mathbf{i}$$

or
$$[\mathbf{I}(G)+(M\bar{\mathbf{r}}^2-\theta)\mathbf{U}].\mathbf{i}=M\bar{\mathbf{r}}(\bar{\mathbf{r}}.\mathbf{i}).$$

Hence $\mathbf{i}$ is given by

$$\mathbf{i}=M(\bar{\mathbf{r}}.\mathbf{i})[\mathbf{I}(G)+(M\bar{\mathbf{r}}^2-\theta)\mathbf{U}]^{-1}.\bar{\mathbf{r}}. \qquad (1)$$

Multiplying scalarly by $\bar{\mathbf{r}}$ and dividing by the factor $\bar{\mathbf{r}}.\mathbf{i}$, we get

$$\frac{1}{M}=[\mathbf{I}(G)+(M\bar{\mathbf{r}}^2-\theta)\mathbf{U}]^{-1}:\bar{\mathbf{r}}\bar{\mathbf{r}}.$$

This shows that the point $\bar{\mathbf{r}}$ lies on the quadric

$$[\mathbf{I}(G)+(M\bar{\mathbf{r}}^2-\theta)\mathbf{U}]^{-1}:\mathbf{r}\mathbf{r}=1/M.$$

This quadric is clearly, by § 327, confocal with the ellipsoid of gyration at G, having for its parameter λ the value given by

$$M\lambda=M\bar{\mathbf{r}}^2-\theta.$$

To each value of θ there is a value of λ, and a corresponding quadric. If we write the moment of inertia θ in the form $\theta=Mk^2$, then

$$\lambda=\bar{\mathbf{r}}^2-k^2.$$

Now the normal to the θ-confocal through $\mathbf{r}$ to the ellipsoid of gyration at G is parallel to

$$[\mathbf{I}(G)+(M\bar{\mathbf{r}}^2-0)]^{-1}.\bar{\mathbf{r}},$$

which by (1) above is parallel to the corresponding principal axis $\mathbf{i}$ at $\bar{\mathbf{r}}$ (or P). Hence the principal axes at P are parallel to the normals at P to the three quadrics through P confocal with the ellipsoid of gyration at G. The corresponding radii of gyration k_1, k_2, k_3 are connected with the parameters λ_1, λ_2, λ_3 of these quadrics by the relations

$$k_1{}^2=\bar{\mathbf{r}}^2-\lambda_1, \qquad k_2{}^2=\bar{\mathbf{r}}^2-\lambda_2, \qquad k_3{}^2=\bar{\mathbf{r}}^2-\lambda_3.$$

329. The following are solutions by tensor methods of examples given in Lamb's *Higher Mechanics*.

Example (1). Prove that the principal axes at the various points of a system form a complex of the second order, whose equation in line co-ordinates is

$$Alp+Bmq+Cnr=0,$$

the axes of Cartesian co-ordinates being the principal axes of inertia at the mass centre.

We know that the directions $\mathbf{i}$ of the principal axes of inertia at P are solutions of

$$\mathbf{I}(P).\mathbf{i}=\theta\mathbf{i}.$$

If (**i**, **a**) are the line co-ordinates of a principal axis, we have accordingly, since $\mathbf{i}.\mathbf{a} = 0$,

$$\mathbf{I}(P).\mathbf{i}.\mathbf{a} = 0.$$

But $\qquad \mathbf{I}(P) = \mathbf{I}(G) + M(\bar{\mathbf{r}}^2\mathbf{U} - \bar{\mathbf{r}}\bar{\mathbf{r}}),$

where (Fig. 77), $\qquad -\bar{\mathbf{r}} = \mathbf{i}\wedge\mathbf{a} + \lambda\mathbf{i}.$

Since $\qquad \mathbf{U}.\mathbf{i}.\mathbf{a} = \mathbf{i}.\mathbf{a} = 0$

and since $\quad (\bar{\mathbf{r}}\bar{\mathbf{r}}).\mathbf{i}.\mathbf{a} = (\mathbf{i}\wedge\mathbf{a} + \lambda\mathbf{i})[(\mathbf{i}\wedge\mathbf{a} + \lambda\mathbf{i}).\mathbf{i}].\mathbf{a} = 0,$

we have $\qquad \mathbf{I}(G).\mathbf{i}.\mathbf{a} = 0,$

or $\qquad \mathbf{I}(G):\mathbf{i}\mathbf{a} = 0.$

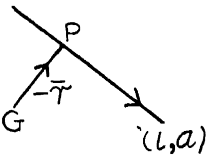

Fig. 77

If (l, m, n), (p, q, r) are the components of **i**, **a** with respect to the principal axes of inertia at G, the latter equation reduces to the form stated in the enunciation.

Example (2). The inertia tensor of a body at O is **I**. A small body is added, of inertia tensor **J** with respect to O. Determine the directions of the principal axes of inertia of the composite body with respect to the directions of the principal axes of inertia of the original body; and determine the principal moments of inertia of the composite body.

Let **i**, **j**, **k** be unit vectors along the principal axes of the original body. Then **i**, **j**, **k** satisfy

$$\mathbf{I}.\mathbf{i} = A\mathbf{i}, \qquad \mathbf{I}.\mathbf{j} = B\mathbf{j}, \qquad \mathbf{I}.\mathbf{k} = C\mathbf{k},$$

where A, B, C are the principal moments of inertia of the original body. Let $\mathbf{i}+\boldsymbol{\epsilon}$, $A+\alpha$ be the principal direction and corresponding principal value, for that axis of the composite body which is in the vicinity of **i**. Then since $\mathbf{I}+\mathbf{J}$ is the new inertia tensor,

$$(\mathbf{I}+\mathbf{J}).(\mathbf{i}+\boldsymbol{\epsilon}) = (A+\alpha)(\mathbf{i}+\boldsymbol{\epsilon}),$$

or, approximately $\qquad \mathbf{J}.\mathbf{i} + \mathbf{I}.\boldsymbol{\epsilon} = \alpha\mathbf{i} + A\boldsymbol{\epsilon}. \qquad (1)$

To this order, $\mathbf{i}.\boldsymbol{\epsilon} = 0$. Hence, multiplying scalarly by **i**, we have

$$\mathbf{J}:\mathbf{i}\mathbf{i} + \mathbf{I}.\boldsymbol{\epsilon}.\mathbf{i} = \alpha.$$

But $\qquad \mathbf{I}.\boldsymbol{\epsilon}.\mathbf{i} = (\mathbf{I}.\mathbf{i}).\boldsymbol{\epsilon} = A\mathbf{i}.\boldsymbol{\epsilon} = 0.$

Hence $\qquad \alpha = \mathbf{J}:\mathbf{i}\mathbf{i}.$

This means that if A', B', C', $-F'$, $-G'$, $-H'$ are the components of **J** referred to the triad **i**, **j**, **k**, then $\alpha = A'$. Accordingly the principal moments of inertia of the composite body are $A+A'$, $B+B'$, $C+C'$.

Again, relation (1) may be written

$$(\mathbf{J} - \alpha\mathbf{U}).\mathbf{i} = -(\mathbf{I} - A\mathbf{U}).\boldsymbol{\epsilon},$$

i.e. $\qquad [\mathbf{J} - (\mathbf{J}:\mathbf{i}\mathbf{i})\mathbf{U}].\mathbf{i} = -(\mathbf{I} - A\mathbf{U}).\boldsymbol{\epsilon}.$

Referred to the triad **i**, **j**, **k**, this relation expands into

$$[A'\mathbf{i}\mathbf{i} + B'\mathbf{j}\mathbf{j} + C'\mathbf{k}\mathbf{k} - F'(\mathbf{j}\mathbf{k}+\mathbf{k}\mathbf{j}) - G'(\mathbf{k}\mathbf{i}+\mathbf{i}\mathbf{k}) - H'(\mathbf{i}\mathbf{j}+\mathbf{j}\mathbf{i}) - A'(\mathbf{i}\mathbf{i}+\mathbf{j}\mathbf{j}+\mathbf{k}\mathbf{k})].\mathbf{i}$$

$$= -[A\mathbf{i}\mathbf{i} + B\mathbf{j}\mathbf{j} + C\mathbf{k}\mathbf{k} - A(\mathbf{i}\mathbf{i}+\mathbf{j}\mathbf{j}+\mathbf{k}\mathbf{k})].\boldsymbol{\epsilon},$$

i.e. $\qquad -G'\mathbf{k} - H'\mathbf{j} = (A-B)(\boldsymbol{\epsilon}.\mathbf{j})\mathbf{j} + (A-C)(\boldsymbol{\epsilon}.\mathbf{k})\mathbf{k}.$

Hence $$\boldsymbol{\epsilon}.\mathbf{j} = -\frac{H'}{A-B}, \qquad \boldsymbol{\epsilon}.\mathbf{k} = -\frac{G'}{A-C}.$$

Hence since $$\boldsymbol{\epsilon} = (\boldsymbol{\epsilon}.\mathbf{i})\mathbf{i} + (\boldsymbol{\epsilon}.\mathbf{j})\mathbf{j} + (\boldsymbol{\epsilon}.\mathbf{k})\mathbf{k},$$
and since $\boldsymbol{\epsilon}.\mathbf{i} = 0$, we have for the new principal axis $\mathbf{i}+\boldsymbol{\epsilon}$ the vector
$$\mathbf{i} - \frac{H'}{A-B}\mathbf{j} - \frac{G'}{A-C}\mathbf{k}.$$
Hence the direction cosines of the new $\mathbf{i}$-axis with respect to the original principal axes are approximately
$$1, \quad \frac{H'}{B-A}, \quad \frac{G'}{C-A}.$$

Example (3). The principal moments of inertia of a uniaxal body are A, A, C. Determine the components of the inertia tensor, with respect to a given triad in which the axis of the body has direction cosines l, m, n.

Let $\mathbf{k}$ be a unit vector in the direction of the axis of the body, $\mathbf{i}$ and $\mathbf{j}$ any two perpendicular unit vectors forming with $\mathbf{k}$ a positive orthogonal triad. Then the inertia tensor $\mathbf{I}$ is given by
$$\begin{aligned}\mathbf{I} &= A\mathbf{ii} + A\mathbf{jj} + C\mathbf{kk}\\ &= A(\mathbf{ii}+\mathbf{jj}+\mathbf{kk}) + (C-A)\mathbf{kk}\\ &= A\mathbf{U} + (C-A)\mathbf{kk}.\end{aligned}$$
Accordingly, if $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are unit vectors along the members of the given triad, so that
$$\mathbf{k} = l\mathbf{x} + m\mathbf{y} + n\mathbf{z},$$
then
$$\begin{aligned}\mathbf{I} &= A(\mathbf{xx}+\mathbf{yy}+\mathbf{zz}) + (C-A)(l\mathbf{x}+m\mathbf{y}+n\mathbf{z})(l\mathbf{x}+m\mathbf{y}+n\mathbf{z})\\ &= A(\mathbf{xx}+\mathbf{yy}+\mathbf{zz}) + (C-A)(l^2\mathbf{xx}+m^2\mathbf{yy}+n^2\mathbf{zz}+mn(\mathbf{yz}+\mathbf{zy})+\ldots+\ldots).\end{aligned}$$
Relative to the given triad, then, $I_{\mu\nu}$ is represented by

	$\nu=1$	$\nu=2$	$\nu=3$
$\mu=1$	$A+(C-A)l^2$	$-(A-C)lm$	$-(A-C)nl$
$\mu=2$	$-(A-C)lm$	$A+(C-A)m^2$	$-(A-C)mn$
$\mu=3$	$-(A-C)nl$	$-(A-C)mn$	$A+(C-A)n^2$.

From this the moments and products of inertia may be read off.

The following additional example is due to Professor L. Rosenhead.

Consider a closed surface V containing a density distribution ρ which is altering with the time. Referred to its principal axes of inertia as triad of reference, let the inertia tensor be
$$A\mathbf{ii} + B\mathbf{jj} + C\mathbf{kk}.$$
Then $$A\mathbf{ii} + B\mathbf{jj} + C\mathbf{kk} = \int_V \rho(\mathbf{r}^2\mathbf{U} - \mathbf{rr})\,d\tau,$$

where $d\tau$ denotes an element of volume of V. Keeping the surface V fixed, and differentiating each side with respect to the time, we have

$$\Sigma\frac{dA}{dt}\mathbf{ii}+\Sigma A\mathbf{i}\frac{d\mathbf{i}}{dt}+\Sigma A\frac{d\mathbf{i}}{dt}\mathbf{i}=\int_V \frac{d\rho}{dt}(\mathbf{r}^2\mathbf{U}-\mathbf{rr})d\tau.$$

But if $\mathbf{\Omega}$ is the angular velocity of the principal axes of the inertia tensor,

$$\frac{d\mathbf{i}}{dt}=\mathbf{\Omega}\wedge\mathbf{i},$$

etc. Thus

$$\Sigma\frac{dA}{dt}\mathbf{ii}+\Sigma A\mathbf{i}(\mathbf{\Omega}\wedge\mathbf{i})+\Sigma A(\mathbf{\Omega}\wedge\mathbf{i})\mathbf{i}=\int_V \frac{d\rho}{dt}(\mathbf{r}^2\mathbf{U}-\mathbf{rr})d\tau.$$

If
$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k},$$
the expansion of the tensor on the right-hand side with reference to the triad $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is

$$\int_V \frac{d\rho}{dt}[\Sigma(y^2+z^2)\mathbf{ii}-\Sigma yz(\mathbf{jk}+\mathbf{kj})]d\tau.$$

But if
$$\mathbf{\Omega}=\omega_1\mathbf{i}+\omega_2\mathbf{j}+\omega_3\mathbf{k},$$
$$\mathbf{\Omega}\wedge\mathbf{i}=-\omega_2\mathbf{k}+\omega_3\mathbf{j},$$
etc. Hence the expansion of the left-hand side with reference to the same triad of reference is

$$\Sigma\frac{dA}{dt}\mathbf{ii}+\Sigma A\mathbf{i}(-\omega_2\mathbf{k}+\omega_3\mathbf{j})+\Sigma A(-\omega_2\mathbf{k}+\omega_3\mathbf{j})\mathbf{i}$$
$$=\Sigma\frac{dA}{dt}\mathbf{ii}+\Sigma(\mathbf{jk}+\mathbf{kj})\omega_1(B-C).$$

Accordingly, equating tensor components, we have

$$\frac{dA}{dt}=\int_V \frac{d\rho}{dt}(y^2+z^2)d\tau,$$

and
$$\omega_1(B-C)=-\int_V \frac{d\rho}{dt}yz\,d\tau,$$

with four other relations obtained by cyclical interchange.

330. *The angular momentum of a uniaxal body.* Let O be a fixed particle of a body rotating about O with angular velocity $\mathbf{\Omega}$. Let A, A, C be the principal inertial constants about O. Let $\mathbf{i}$ be a unit vector in the axis of the body. Then (§ 231), the angular velocity $\mathbf{\Omega}$ of the body is of the form

$$\mathbf{\Omega}=n\mathbf{i}+\mathbf{i}\wedge\frac{d\mathbf{i}}{dt},$$

whilst the inertia tensor $\mathbf{I}(\mathrm{O})$ is of the form

$$\mathbf{I}(\mathrm{O})=\mathrm{C}\mathbf{ii}+\mathrm{A}\mathbf{jj}+\mathrm{A}\mathbf{kk}$$
$$=\mathrm{A}\mathbf{U}+(\mathrm{C}-\mathrm{A})\mathbf{ii}.$$

Hence the angular momentum $\mathbf{H}(\mathrm{O})$ about O is given by

$$\mathbf{H}(\mathrm{O})=\boldsymbol{\Omega}.\mathbf{I}(\mathrm{O})=\left(\mathrm{n}\mathbf{i}+\mathbf{i}\wedge\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}\right).((\mathrm{C}-\mathrm{A})\mathbf{ii}+\mathrm{A}\mathbf{U})$$
$$=(\mathrm{C}-\mathrm{A})\mathrm{n}\mathbf{i}+\mathrm{An}\mathbf{i}+\mathrm{A}\mathbf{i}\wedge\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}$$
$$=\mathrm{Cn}\mathbf{i}+\mathrm{A}\mathbf{i}\wedge\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}.$$

This is a standard form for the angular momentum of a uniaxal body in terms of the motion of its axis $\mathbf{i}$, the spin about its axis n, and the inertia constants C, A. It is of fundamental importance in the discussion of gyroscopic problems.

An alternative derivation of this formula is of interest. Avoiding the introduction of the idem tensor $\mathbf{U}$, we have

$$\mathbf{H}(\mathrm{O})=\boldsymbol{\Omega}.\mathbf{I}(\mathrm{O})=\left(\mathrm{n}\mathbf{i}+\mathbf{i}\wedge\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}\right).(\mathrm{A}\mathbf{jj}+\mathrm{A}\mathbf{kk}+\mathrm{C}\mathbf{ii})$$
$$=\mathrm{Cn}\mathbf{i}+\mathrm{A}\left[\left(\mathbf{i}\wedge\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}.\mathbf{j}\right)\mathbf{j}+\left(\mathbf{i}\wedge\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}.\mathbf{k}\right)\mathbf{k}\right].$$

Since $\mathbf{i}\wedge\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}$ has zero component along $\mathbf{i}$, the coefficient of A is just $\mathbf{i}\wedge\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}$ itself. Alternatively it may be reduced as

$$-\left(\mathbf{k}.\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}\right)\mathbf{j}+\left(\mathbf{j}.\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}\right)\mathbf{k}$$
$$=(\mathbf{j}\wedge\mathbf{k})\wedge\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}=\mathbf{i}\wedge\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}.$$

From one point of view, the formula

$$\mathbf{H}(\mathrm{O})=\mathrm{Cn}\mathbf{i}+\mathrm{A}\mathbf{i}\wedge\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}$$

can be written down at sight. For to obtain $\mathbf{H}(\mathrm{O})$ we have to construct the vector whose components are the products of the components of angular velocity by the principal components of the inertia tensor, in the principal triad of reference of the inertia tensor. Such a triad may be chosen as $\mathbf{i}$ and any pair of vectors $\mathbf{j}$, $\mathbf{k}$ perpendicular to $\mathbf{i}$ and to one another ; choosing $\mathbf{j}$ along the vector $\mathbf{i}\wedge\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}$ we have the formula at once. The writer finds the detailed methods given above more convincing.

CALCULATION OF THE INERTIA TENSOR FOR GIVEN BODIES

331. *Scope of the appendix.* Treatises on rigid dynamics usually develop a number of special formulæ for the *moments of inertia* of various rigid bodies. These are not easy to remember. In the following treatment we calculate the *inertia tensors* outright; the formulæ obtained are very easy to remember, and moments of inertia can be derived from them at once. Moreover the formulæ for the inertia tensors indicate at once what systems of particles, rods, or laminæ are equimomental with the rigid body without further calculations. Lastly, the procedures followed involve only the simplest of integrations.

The fundamental idea in the calculations is to proceed step by step, building up in turn the inertia tensors of rods, discs, and solid bodies by means of repeated use of the formula connecting the inertia tensor about the centre of mass with that about any other point. It is hoped that the student will find this appendix of considerable interest as compared with its rather dreary setting in the usual presentations.

332. *The inertia tensor of a uniform rod.* Let μ be the line density, M the mass, 2a the length, of a uniform straight thin rod. Then $M = 2\mu a$. Let G be the centre of mass. Then the inertia tensor about G is given by

$$\mathbf{I}(G) = \int_{-a}^{a} (x^2\mathbf{U} - x^2\mathbf{ii})\mu\, dx,$$

where $\mathbf{r} = x\mathbf{i}$ is the position vector of a typical particle of the rod with respect to G (Fig. 78), and $\mathbf{i}$ is a unit vector along the rod. Hence

$$\mathbf{I}(G) = \mu(\mathbf{U} - \mathbf{ii})\frac{2a^3}{3} = \frac{1}{3}Ma^2(\mathbf{U} - \mathbf{ii}). \tag{1}$$

This is a fundamental formula. It may also be written, if $\mathbf{j}$, $\mathbf{k}$ are two unit vectors making an orthogonal triad with $\mathbf{i}$,

$$\mathbf{I}(G) = \tfrac{1}{3}Ma^2(\mathbf{jj} + \mathbf{kk}),$$

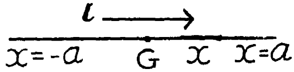

Fig. 78

but in this form it is less useful. Formula (1) shows at once that the moments of inertia about the $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ axes are 0, $\frac{1}{3}Ma^2$, $\frac{1}{3}Ma^2$, and that $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the principal axes at G.

333. *The inertia tensor of a uniform parallelogram.* As in all cases which follow, let M denote the total mass. Let σ be the surface density.

Let **i**, **j** be unit vectors along the sides, and let 2a, 2b be the lengths of the corresponding sides (Fig. 79). Then the inertia tensor of an elementary strip, centred at P and parallel to the side **i**, *about* P, is by § 332

$$\frac{\sigma}{3}(\text{area of strip})a^2(\mathbf{U}-\mathbf{ii}).$$

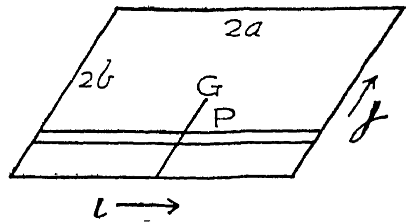

Fig. 79

Let G be the mass centre of the parallelogram, and let GP = y. If A is the total area of the parallelogram, then

$$\frac{\text{area of strip}}{A} = \frac{dy}{2b}.$$

Accordingly the inertia tensor of the whole parallelogram, about G, since G is distant y**j** from P, the centroid of the strip, is

$$\mathbf{I}(G) = \sigma A \int_{-b}^{b} \frac{dy}{2b}[\tfrac{1}{3}a^2(\mathbf{U}-\mathbf{ii}) + y^2(\mathbf{U}-\mathbf{jj})]$$

$$= M[\tfrac{1}{3}a^2(\mathbf{U}-\mathbf{ii}) + \tfrac{1}{3}b^2(\mathbf{U}-\mathbf{jj})]. \qquad (2)$$

This shows that the parallelogram is equimomental with four particles at the mid points of the sides, each of mass $\frac{1}{6}$M together with a mass $\frac{1}{3}$M at the centroid: For this system has the same total mass and the same centroid as the given parallelogram, and it has the same inertia tensor about G, as follows from inspection of (2). Again (2) shows that the parallelogram is equimomental with two uniform rods through G, parallel to and terminated by the sides, each of mass M, together with a particle of mass —M at the centre.

Example. Show that a parallelogram is equimomental with four particles at the corners, each of mass $\frac{1}{12}$M, together with a particle of mass $\frac{2}{3}$M at the centre.

When the parallelogram reduces to a rectangle, **i** and **j** are perpendicular, and we may transform (2) by writing **U** = **ii** + **jj** + **kk**, **k** being a unit vector normal to the plane of the rectangle. Then

$$\mathbf{I}(G) = M[\tfrac{1}{3}b^2\mathbf{ii} + \tfrac{1}{3}a^2\mathbf{jj} + \tfrac{1}{3}(a^2+b^2)\mathbf{kk}]. \qquad (2')$$

This gives the moments of inertia about the **i**, **j**, **k** axes as $\frac{1}{3}Mb^2$, $\frac{1}{3}Ma^2$, and $\frac{1}{3}M(a^2+b^2)$. Formula (2) is, however, much more general than (2′), and more easily remembered.

If we want the moment of inertia of a *parallelogram* about the median in the **i**-direction, we work out from (2) the value of **I**(G):**ii**, obtaining

$$\tfrac{1}{3}Mb^2[1-(\mathbf{i}.\mathbf{j})^2].$$

Again, if we want the moment of inertia of a parallelogram about a diagonal, namely about the unit vector

$$\frac{a\mathbf{i}+b\mathbf{j}}{(a^2+b^2+2ab\,\mathbf{i}.\mathbf{j})^{\frac{1}{2}}}$$

through G, we find it by the same method to be

$$\frac{M}{a^2+b^2+2ab(\mathbf{i.j})}[\tfrac{1}{3}a^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{3}b^2(\mathbf{U}-\mathbf{jj})]:(a\mathbf{i}+b\mathbf{j})(a\mathbf{i}+b\mathbf{j})$$

$$=\frac{M}{a^2+b^2+2ab(\mathbf{i.j})}[\tfrac{1}{3}a^2\{(a^2+b^2+2ab(\mathbf{i.j}))-(a+b(\mathbf{i.j}))^2\}$$

$$+\tfrac{1}{3}b^2\{(a^2+b^2+2ab(\mathbf{i.j}))-(b+a(\mathbf{i.j}))^2\}]$$

$$=\frac{\tfrac{2}{3}Ma^2b^2(1-(\mathbf{i.j})^2)}{a^2+b^2+2ab(\mathbf{i.j})}.$$

This formula can be used to give also the moment of inertia of a plane triangle about its base in terms of the sides and the included angle. For this moment of inertia will be obtained by halving the above moment of inertia, putting M′ for $\frac{1}{2}$M, a′ for 2a and b′ for 2b, and then omitting primes (Fig. 80). The result is

$$\frac{\tfrac{1}{6}Ma^2b^2(1-(\mathbf{i.j})^2)}{a^2+b^2+2ab(\mathbf{i.j})}.$$

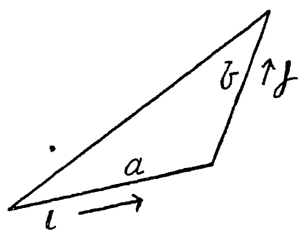

Fig. 80

334. *The inertia tensor of a solid parallelopiped.* Let G be the centroid of a uniform parallelopiped, of sides 2a, 2b, 2c in the directions of the unit vectors **i**, **j**, **k**. If ρ is the volume density, the inertia tensor of a thin elementary section of the parallelopiped, *about its centroid* P, if the section is parallel to the vectors **j** and **k**, is

$$\rho\times(\text{vol. of section})\times[\tfrac{1}{3}b^2(\mathbf{U}-\mathbf{jj})+\tfrac{1}{3}c^2(\mathbf{U}-\mathbf{kk})].$$

But if GP = x, and V = volume of parallelopiped,

$$\frac{\text{vol. of section}}{V}=\frac{dx}{2a}$$

and hence the inertia tensor of the whole parallelopiped *about its centroid* G is

$$\mathbf{I}(G)=\rho V\int_{-a}^{a}\frac{dx}{2a}[\tfrac{1}{3}b^2(\mathbf{U}-\mathbf{jj})+\tfrac{1}{3}c^2(\mathbf{U}-\mathbf{kk})+x^2(\mathbf{U}-\mathbf{ii})]$$

$$=M[\tfrac{1}{3}a^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{3}b^2(\mathbf{U}-\mathbf{jj})+\tfrac{1}{3}c^2(\mathbf{U}-\mathbf{kk})]. \qquad (3)$$

This shows that a solid parallelopiped is equimomental with six particles each of mass $\frac{1}{6}$M at the centroids of the six faces. It is also equimomental with three uniform thin rods each of mass M through the

centroid parallel to the edges and terminated by the faces, together with a mass $-2M$ at the centroid.

Example. Determine the masses of the equimomental set of particles at the eight corners and the centroid.

When the parallelopiped is rectangular, we can put $U=\mathbf{ii}+\mathbf{jj}+\mathbf{kk}$, and then

$$I(G)=M[\tfrac{1}{3}(b^2+c^2)\mathbf{ii}+\tfrac{1}{3}(c^2+a^2)\mathbf{jj}+\tfrac{1}{3}(a^2+b^2)\mathbf{kk}].$$

Example. Evaluate the moment of inertia about the diagonal, namely

$$\frac{I(G):(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})}{a^2+b^2+c^2}.$$

335. *The inertia tensor of a uniform triangular lamina about its centroid.* Let G (Fig. 81) be the centroid of a uniform triangle ABC, i.e. the intersection of the medians. Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be unit vectors along GA, GB, GC, and let 2ξ, 2η, 2ζ be the lengths of GA, GB, GC. Let $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ be unit vectors parallel to the sides BC, CA, AB; and let a, b, c be the respective lengths of these sides.

The inertia tensor of ABC about G is the sum of those of GBC, GCA, GAB. To obtain the inertia tensor of GBC, consider an elementary strip QPR parallel to BC, with mid-point P. If σ is the surface density, the inertia tensor of the strip with respect to P is

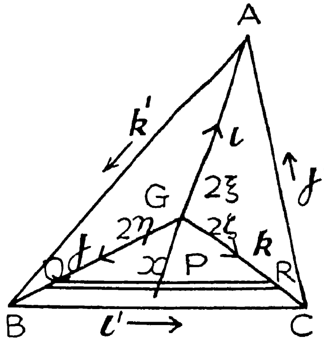

Fig. 81

$$\tfrac{1}{3}\sigma(\text{area of strip})PR^2(\mathbf{U}-\mathbf{i}'\mathbf{i}').$$

But
$$\frac{PR}{\frac{1}{2}a}=\frac{x}{\xi}, \qquad \frac{\text{area of strip}}{\frac{1}{3}\Delta}=\frac{d(x^2)}{\xi^2}=\frac{2x\,dx}{\xi^2},$$

where $GP=x$ and $\Delta=$area of ABC. Thus the inertia tensor of the elementary strip *about* P is

$$\sigma(\tfrac{1}{3}\Delta)\frac{2x\,dx}{\xi^2}\left[\tfrac{1}{3}\left(\tfrac{1}{2}a\frac{x}{\xi}\right)^2(\mathbf{U}-\mathbf{i}'\mathbf{i}')\right].$$

and hence that of the triangle GBC *about* G is

$$\sigma(\tfrac{1}{3}\Delta)\int_{x=0}^{\xi}\frac{2x\,dx}{\xi^2}\left[\tfrac{1}{3}\left(\tfrac{1}{2}a\frac{x}{\xi}\right)^2(\mathbf{U}-\mathbf{i}'\mathbf{i}')+x^2(\mathbf{U}-\mathbf{ii})\right]$$

$$=\Delta\sigma\left[\frac{a^2}{72}(\mathbf{U}-\mathbf{i}'\mathbf{i}')+\frac{\xi^2}{6}(\mathbf{U}-\mathbf{ii})\right].$$

Now $\triangle\sigma = M$, the mass of the whole lamina ABC. Hence, by addition of the inertia tensors of the three triangles GBC, GCA, GAB, we get for the whole triangle

$$\mathbf{I}(G) = \frac{M}{6}[\tfrac{1}{12}\{(\Sigma a^2)\mathbf{U} - \Sigma a^2\mathbf{i}'\mathbf{i}'\} + \{(\Sigma\xi^2)\mathbf{U} - \Sigma\xi^2\mathbf{ii}\}].$$

We proceed to simplify this. We have

$$a^2 = BC^2 = (2\zeta\mathbf{k} - 2\eta\mathbf{j})^2 = 4(\eta^2 + \zeta^2 - 2\eta\zeta(\mathbf{j}.\mathbf{k})),$$

and
$$a^2\mathbf{i}'\mathbf{i}' = 4(\zeta\mathbf{k} - \eta\mathbf{j})(\zeta\mathbf{k} - \eta\mathbf{j})$$
$$= 4(\eta^2\mathbf{jj} + \zeta^2\mathbf{kk} - \eta\zeta(\mathbf{jk} + \mathbf{kj})),$$

whence
$$\Sigma a^2 = 8\Sigma\xi^2 - 8\Sigma\eta\zeta(\mathbf{j}.\mathbf{k}),$$
$$\Sigma a^2\mathbf{i}'\mathbf{i}' = 8\Sigma\xi^2\mathbf{ii} - 4\Sigma\eta\zeta(\mathbf{jk} + \mathbf{kj}).$$

But since G is the centroid,
$$\xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k} = 0,$$

and so
$$\Sigma\xi^2 + 2\Sigma\eta\zeta(\mathbf{j}.\mathbf{k}) = 0,$$

and similarly
$$\Sigma\xi^2\mathbf{ii} + \Sigma\eta\zeta(\mathbf{jk} + \mathbf{kj}) = 0.$$

Hence
$$\Sigma a^2 = 12\Sigma\xi^2,$$
$$\Sigma a^2\mathbf{i}'\mathbf{i}' = 12\Sigma\xi^2\mathbf{ii}.$$

Thus, finally,

$$\mathbf{I}(G) = M[\tfrac{1}{3}\xi^2(\mathbf{U} - \mathbf{ii}) + \tfrac{1}{3}\eta^2(\mathbf{U} - \mathbf{jj}) + \tfrac{1}{3}\zeta^2(\mathbf{U} - \mathbf{kk})], \tag{4}$$

where, it may be recalled, ξ, η, ζ are the distances of G from the mid-points of the sides, and **i, j, k** are unit vectors parallel to the joins of G to the vertices.

This formula shows by inspection that the triangle is equimomentai with three particles at the mid-points of the sides each of mass $\frac{1}{3}$M. Likewise it is equimomental with three particles at the vertices each of mass $\frac{1}{12}$M together with a particle at the centroid of mass $\frac{3}{4}$M.

336. *The inertia tensor of a triangular prism about its centroid.* Let 2h be the length of the axis of the prism, **l** a unit vector along the axis; and let $2\xi\mathbf{i}$, $2\eta\mathbf{j}$, $2\zeta\mathbf{k}$ be the position vectors of the mid-points of the axial edges with respect to the centroid. Building up the prism by addition of elementary triangular sections parallel to the base, we have for the inertia tensor about G

$$\mathbf{I}(G) = \rho V\int_{-h}^{h}\frac{dx}{2h}[\tfrac{1}{3}\xi^2(\mathbf{U} - \mathbf{ii}) + \tfrac{1}{3}\eta^2(\mathbf{U} - \mathbf{jj}) + \tfrac{1}{3}\zeta^2(\mathbf{U} - \mathbf{kk}) + x^2(\mathbf{U} - \mathbf{ll})]$$
$$= M[\tfrac{1}{3}\xi^2(\mathbf{U} - \mathbf{ii}) + \tfrac{1}{3}\eta^2(\mathbf{U} - \mathbf{jj}) + \tfrac{1}{3}\zeta^2(\mathbf{U} - \mathbf{kk}) + \tfrac{1}{3}h^2(\mathbf{U} - \mathbf{ll})]. \tag{5}$$

337. *The inertia tensor of a triangle about a vertex.* The inertia tensor

of a triangle about a vertex, in terms of the median and base, follows from the formula for the inertia tensor of GBC about G obtained in § 335. Putting Δ' for $\frac{1}{3}\Delta$, and then omitting the prime, the inertia tensor of a triangle ABC about A (Fig. 82) is

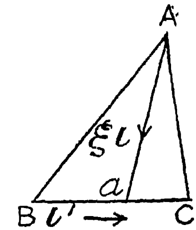

Fig. 82

$$(3\Delta)\sigma\left[\frac{a^2}{72}(\mathbf{U}-\mathbf{i}'\mathbf{i}')+\frac{\xi^2}{6}(\mathbf{U}-\mathbf{ii})\right]$$
$$=\tfrac{1}{2}M\left[\frac{a^2}{12}(\mathbf{U}-\mathbf{i}'\mathbf{i}')+\xi^2(\mathbf{U}-\mathbf{ii})\right],$$

where now ξ is the length of the median of the triangle drawn from A, $\mathbf{i}$ a unit vector along this median and $\mathbf{i}'$ a unit vector along BC.

Example. Use this formula to show that a triangle is equimomental with three particles at the mid-points of the three sides.

338. *The inertia tensor of a solid tetrahedron.* Let G (Fig. 83) be the centre of mass of a uniform solid tetrahedron ABCD, of density ρ, volume V and mass $M=\rho V$. Let A, B, C, D have position vectors $3\xi\mathbf{i}$, $3\eta\mathbf{j}$, $3\zeta\mathbf{k}$, $3\omega\mathbf{l}$ with respect to G. Let G_1 be the centre of mass of the triangle BCD and let BG_1 meet CD in B_{12}. Then the vector G_1B_{12} is equal to $\frac{1}{2}BG_1$, and $BG_1=GG_1+BG$. But G_1 is the centroid of equal particles at B, C, D. Hence

$$G_1B_{12}=\tfrac{1}{2}[GG_1+BG]$$
$$=\tfrac{1}{2}[\tfrac{1}{3}(3\eta\mathbf{j}+3\zeta\mathbf{k}+3\omega\mathbf{l})-3\eta\mathbf{j}]$$
$$=\tfrac{1}{2}[\zeta\mathbf{k}+\omega\mathbf{l}-2\eta\mathbf{j}].$$

Call this the vector $\boldsymbol{\alpha}_{12}$. Now consider an elementary section QRS of the tetrahedron GBCD parallel to the base BCD. Let the centroid J_1 of QRS be at a distance x from G. Then, since $GG_1=\xi$,

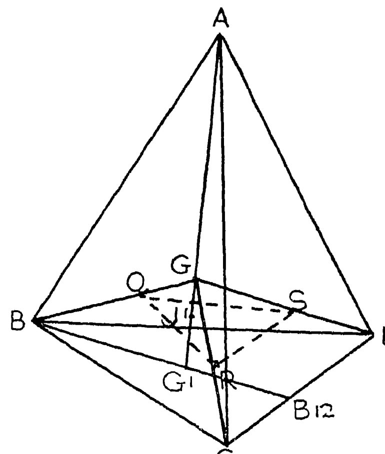

Fig. 83

$$\frac{\text{vol. of elem. section}}{\frac{1}{4}V}=\frac{d(x^3)}{\xi^3}=\frac{3x^2dx}{\xi^3}.$$

The linear dimensions of the section are to those of BCD in the ratio $x:\xi$. Hence by formula (4) for the inertia tensor of a triangle about its centroid, the inertia tensor of the elementary section about J_1 is

$$\rho(\tfrac{1}{4}V)\frac{3x^2dx}{\xi^3}\left[\frac{x^2}{\xi^2}\{\tfrac{1}{3}(\boldsymbol{\alpha}_{12}{}^2+\boldsymbol{\alpha}_{13}{}^2+\boldsymbol{\alpha}_{14}{}^2)\mathbf{U}-\tfrac{1}{3}(\boldsymbol{\alpha}_{12}\boldsymbol{\alpha}_{12}+\boldsymbol{\alpha}_{13}\boldsymbol{\alpha}_{13}+\boldsymbol{\alpha}_{14}\boldsymbol{\alpha}_{14})\}\right]$$
$$=\rho(\tfrac{1}{4}V)\frac{3x^2dx}{\xi^3}\,\frac{x^2}{2\xi^2}[\{\eta^2+\zeta^2+\omega^2-\eta\zeta(\mathbf{j.k})-\zeta\omega(\mathbf{k.l})-\omega\eta(\mathbf{l.j})\}\mathbf{U}$$
$$-(\eta^2\mathbf{jj}+\ldots+\ldots-\tfrac{1}{2}\eta\zeta(\mathbf{jk}+\mathbf{kj})-\ldots-\ldots)].$$

Hence the inertia tensor of GBCD about G is

$$\rho(\tfrac{1}{4}V)\int_{x=0}^{\xi} \frac{3x^2dx}{\xi^3}\left[\frac{x^2}{2\xi^2}\{(\eta^2+\ldots+\ldots-\eta\zeta(\mathbf{j}.\mathbf{k})-\ldots-\ldots)\mathbf{U}\right.$$

$$\left.-(\eta^2\mathbf{jj}-\ldots-\ldots-\tfrac{1}{2}\eta\zeta(\mathbf{jk}+\mathbf{kj})-\ldots-\ldots)\}+x^2(\mathbf{U}-\mathbf{ii})\right]$$

$$=\tfrac{3}{4}\rho V[\tfrac{1}{10}\{(\eta^2+\ldots+\ldots-\eta\zeta(\mathbf{j}.\mathbf{k})-\ldots-\ldots)\mathbf{U}$$

$$-(\eta^2\mathbf{jj}-\ldots-\ldots-\tfrac{1}{2}\eta\zeta(\mathbf{jk}+\mathbf{kj})-\ldots-\ldots)\}+\tfrac{1}{5}\xi^2(\mathbf{U}-\mathbf{ii})].$$

Adding four similar expressions for the four tetrahedra with vertices at G, the inertia tensor of the whole tetrahedron about G is given by

$$\mathbf{I}(G)=\tfrac{3}{4}M[\tfrac{1}{10}\{(3\Sigma\xi^2-2\Sigma\xi\eta(\mathbf{i}.\mathbf{j}))\mathbf{U}-(3\Sigma\xi^2\mathbf{ii}-\Sigma\xi\eta(\mathbf{ij}+\mathbf{ji}))\}$$
$$+\tfrac{1}{5}\{(\Sigma\xi^2)\mathbf{U}-(\Sigma\xi^2\mathbf{ii})\}].$$

But since G is the centroid,

$$\xi\mathbf{i}+\eta\mathbf{j}+\zeta\mathbf{k}+\omega\mathbf{l}=0,$$

which gives

$$\Sigma\xi^2+2\Sigma\xi\eta(\mathbf{i}.\mathbf{j})=0,$$

$$\Sigma\xi^2\mathbf{ii}+\Sigma\xi\eta(\mathbf{ij}+\mathbf{ji})=0.$$

Hence

$$\mathbf{I}(G)=\tfrac{3}{4}M[(\tfrac{2}{5}\Sigma\xi^2\mathbf{U}-\Sigma\xi^2\mathbf{ii})+\tfrac{1}{5}(\Sigma\xi^2\mathbf{U}-\Sigma\xi^2\mathbf{ii})]$$
$$=\tfrac{9}{20}M[(\Sigma\xi^2)\mathbf{U}-\Sigma\xi^2\mathbf{ii}]. \tag{6}$$

Writing this in the form

$$\mathbf{I}(G)=\tfrac{1}{20}M\left[\sum_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}}(3\xi)^2(\mathbf{U}-\mathbf{ii})\right],$$

we see that the tetrahedron is equimomental with a system of four particles at the vertices, each of mass $\frac{1}{20}M$, together with a particle of mass $\frac{4}{5}M$ at the centroid.

Example (1). Prove that the tetrahedron is equimomental with four particles at the centroids of the four faces each of mass $\frac{9}{40}M$ together with a particle of mass $-\frac{4}{5}M$ at the centroid. Hence show that the tetrahedron is equimomental with a system of four particles at the vertices of masses $\frac{-1}{40}M$ together with four particles at the centroids of the face of masses $\frac{9}{40}M$.

Example (2). Prove that the tetrahedron is equimomental with six particles at the mid-points of the edges, each of mass $\frac{1}{10}M$, together with a particle of mass $\frac{2}{5}M$ at the centroid.

339. *The inertia tensor of any plane system of particles.* Let O be an origin in the plane, $x\mathbf{i}+y\mathbf{j}$ the position vector with regard to O of a particle of mass m, where $\mathbf{i}$ and $\mathbf{j}$ are unit perpendicular vectors in the plane, and $\mathbf{k}$ a unit vector normal to the plane. Then

$$\mathbf{I}(O)=\Sigma m(x\mathbf{i}+y\mathbf{j})^2\mathbf{U}-\Sigma m(x\mathbf{i}+y\mathbf{j})(x\mathbf{i}+y\mathbf{j})$$
$$=\Sigma m(x^2+y^2)\mathbf{U}-\Sigma mx^2\mathbf{ii}-\Sigma my^2\mathbf{jj}-\Sigma mxy(\mathbf{ij}+\mathbf{ji}).$$

Putting $\quad \Sigma mx^2 = B, \quad \Sigma my^2 = A, \quad \Sigma mxy = H,$

we have
$$\mathbf{I}(O) = (A+B)(\mathbf{ii}+\mathbf{jj}+\mathbf{kk}) - B\mathbf{ii} - A\mathbf{jj} - H(\mathbf{ij}+\mathbf{ji})$$
$$= A\mathbf{ii} + B\mathbf{jj} + (A+B)\mathbf{kk} - H(\mathbf{ij}+\mathbf{ji}).$$

Thus A, B, A+B are the moments of inertia about the axes **i, j, k.** Hence the following:

Theorem: The moment of inertia of a plane distribution about an axis through a point O in the distribution perpendicular to the distribution is equal to the sum of the moments of inertia about any two perpendicular axes through O in the plane of the distribution.

Example. Show that if **i, j** are not perpendicular, then
$$\mathbf{I}(O) = (A+B+2H\mathbf{i}.\mathbf{j})\mathbf{U} - B\mathbf{ii} - A\mathbf{jj} - H(\mathbf{ij}+\mathbf{ji}),$$
where A, B, H are defined *formally* as above, i.e. with respect to the oblique axes. We have here no longer $\mathbf{U} = \mathbf{ii}+\mathbf{jj}+\mathbf{kk}$. But since
$$\mathbf{I}(O).\mathbf{i}\wedge\mathbf{j} = (A+B+2H\mathbf{i}.\mathbf{j})(\mathbf{i}\wedge\mathbf{j}),$$
$A+B+2H\mathbf{i}.\mathbf{j}$ is the principal moment of inertia of the distribution about the normal to the plane; and this normal is a principal axis.

340. *Bodies with curved boundaries.* The inertia tensors of bodies such as circles, spheres, cylinders (circular or elliptic), ellipsoids and cones can readily be built up by the following sequence of arguments. The formulæ obtained are very easy to remember or to reproduce.

341. *Inertia tensor of the circumference of a circle.* Let r be the radius of the circle. Let **i, j** be a pair of unit perpendicular vectors in the plane of the circle, **k** a unit vector normal to its plane. If G is its centre, then by the theorem of § 339, and by the symmetry of the circle, the inertia tensor of the circumference about the centre is given by
$$\mathbf{I}(G) = A\mathbf{ii} + A\mathbf{jj} + 2A\mathbf{kk},$$
where A is to be determined. Now the moment of inertia about the **k**-axis is clearly Mr^2. But this is equal to $\mathbf{I}(G):\mathbf{kk}$, which is 2A. Hence $A = \frac{1}{2}Mr^2$, or
$$\mathbf{I}(G) = \tfrac{1}{2}Mr^2(\mathbf{ii}+\mathbf{jj}+2\mathbf{kk}).$$
This can be rewritten in either of the forms
$$\mathbf{I}(G) = \tfrac{1}{2}Mr^2(\mathbf{U}+\mathbf{kk}), \tag{7}$$
$$\mathbf{I}(G) = M[\tfrac{1}{2}r^2(\mathbf{U}-\mathbf{ii}) + \tfrac{1}{2}r^2(\mathbf{U}-\mathbf{jj})]. \tag{7'}$$
The latter of these will be found to be the more significant.

342. *Circular lamina.* By adding together the inertia tensors corresponding to typical circumferential annuli, we have for the inertia tensor of a circular lamina, if σ is its surface density,
$$\mathbf{I}(G) = \int_0^r \sigma(2\pi r'dr')\tfrac{1}{2}r'^2(\mathbf{U}+\mathbf{kk})$$
$$= \tfrac{1}{4}\sigma\pi r^4(\mathbf{U}+\mathbf{kk})$$
$$= \tfrac{1}{4}Mr^2(\mathbf{U}+\mathbf{kk}) \tag{8}$$
$$= M[\tfrac{1}{4}r^2(\mathbf{U}-\mathbf{ii}) + \tfrac{1}{4}r^2(\mathbf{U}-\mathbf{jj})]. \tag{8'}$$

The moments of inertia about the **i**, **j**, **k** axes are accordingly $\frac{1}{4}Mr^2$, $\frac{1}{4}Mr^2$, and $\frac{1}{2}Mr^2$, respectively.

343. *The inertia tensor of an elliptic lamina.* Take unit vectors **i**, **j** along the axes. Then by definition,

$$\begin{aligned} \mathrm{I(G)} &= \iint [(x\mathbf{i}+y\mathbf{j})^2\mathbf{U}-(x\mathbf{i}+y\mathbf{j})(x\mathbf{i}+y\mathbf{j})]\sigma dxdy \\ &= (\mathbf{U}-\mathbf{ii})\iint x^2\sigma dxdy+(\mathbf{U}-\mathbf{jj})\iint y^2\sigma dxdy, \end{aligned}$$

the integrals $\iint \sigma xydxdy$ clearly vanishing.

Put
$$x=\frac{a}{r}x', \qquad y=\frac{b}{r}y',$$

where a, b are the semi-axes of the ellipse and r is arbitrary. Then

$$\mathrm{I(G)}=\frac{ab}{r^2}\left[\frac{a^2}{r^2}(\mathbf{U}-\mathbf{ii})\iint x'^2\sigma dx'dy'+\frac{b^2}{r^2}(\mathbf{U}-\mathbf{jj})\iint y'^2\sigma dx'dy'\right],$$

where the double integrals are now taken over the interior of the circle

$$x'^2+y'^2=r^2,$$

and are, of course, equal to one another. If we put $a=r$, $b=r$, this must reduce to the inertia tensor of a *circular* lamina. Hence

$$\iint x'^2\sigma dx'dy'=\iint y'^2\sigma dx'dy'=(\pi r^2\sigma)\tfrac{1}{4}r^2.$$

Thus

$$\begin{aligned} \mathrm{I(G)} &= \pi ab\sigma[\tfrac{1}{4}a^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{4}b^2(\mathbf{U}-\mathbf{jj})] \\ &= M[\tfrac{1}{4}a^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{4}b^2(\mathbf{U}-\mathbf{jj})]. \qquad (9') \end{aligned}$$

344. *The inertia tensor of a hollow circular cylinder.* (*Circular sectioned, open thin tube.*) Let 2h be the length of the cylinder, r its radius, **k** a unit vector along the axis. Then from the formula for the inertia tensor of the circumference of a circle, we have by building up, if σ is again the surface density,

$$\begin{aligned} \mathrm{I(G)} &= \int_{-h}^{h}(\sigma 2\pi r dx)[\tfrac{1}{2}r^2(\mathbf{U}+\mathbf{kk})+x^2(\mathbf{U}-\mathbf{kk})] \\ &= M[\tfrac{1}{2}r^2(\mathbf{U}+\mathbf{kk})+\tfrac{1}{3}h^2(\mathbf{U}-\mathbf{kk})] \qquad (10) \\ &= M[\tfrac{1}{2}r^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{2}r^2(\mathbf{U}-\mathbf{jj})+\tfrac{1}{3}h^2(\mathbf{U}-\mathbf{kk})]. \qquad (10') \end{aligned}$$

The moment of inertia about a transverse axis through G is $\mathrm{I(G)}:\mathbf{ii}$ or $M[\frac{1}{2}r^2+\frac{1}{3}h^2]$. Clearly the open-ended hollow cylinder is equimomental with an equatorial circumference of mass M, an axial rod of mass M and a particle of mass $-M$ at the centroid.

345. *Solid circular cylinder.* In the notation of the preceding section, building up the cylinder as a sum of elementary laminæ we have, if ρ is the volume density,

$$\begin{aligned} \mathrm{I(G)} &= \int_{-h}^{h}(\rho\pi r^2 dx)[\tfrac{1}{4}r^2(\mathbf{U}+\mathbf{kk})+x^2(\mathbf{U}-\mathbf{kk})] \\ &= M[\tfrac{1}{4}r^2(\mathbf{U}+\mathbf{kk})+\tfrac{1}{3}h^2(\mathbf{U}-\mathbf{kk})] \qquad (11) \\ &= M[\tfrac{1}{4}r^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{4}r^2(\mathbf{U}-\mathbf{jj})+\tfrac{1}{3}h^2(\mathbf{U}-\mathbf{kk})]. \qquad (11') \end{aligned}$$

346. *Solid elliptic cylinder.* This is evidently, by a similar proof,

$$\mathbf{I}(G)=[M\tfrac{1}{4}a^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{4}b^2(\mathbf{U}-\mathbf{jj})+\tfrac{1}{3}h^2(\mathbf{U}-\mathbf{kk})]. \tag{12}$$

347. *Hollow circular cone* (*without base*). Let h be the perpendicular height of the cone, r the radius of the open end. If $\mathbf{k}$ is a unit vector along the axis, taking a variable x along the axis measured from the vertex, we have for the inertia tensor about the vertex O

$$\mathbf{I}(O)=\int_0^h Md\left(\frac{x^2}{h^2}\right)\left[\tfrac{1}{2}\frac{x^2}{h^2}r^2(\mathbf{U}+\mathbf{kk})+x^2(\mathbf{U}-\mathbf{kk})\right]$$

$$=M[\tfrac{1}{4}r^2(\mathbf{U}+\mathbf{kk})+\tfrac{1}{2}h^2(\mathbf{U}-\mathbf{kk})] \tag{13}$$

$$=M[\tfrac{1}{4}r^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{4}r^2(\mathbf{U}-\mathbf{jj})+\tfrac{1}{2}h^2(\mathbf{U}-\mathbf{kk})]. \tag{13'}$$

348. *Solid circular cone.* The inertia tensor about the vertex is clearly given by

$$\mathbf{I}(O)=\int_0^h Md\left(\frac{x^3}{h^3}\right)\left[\tfrac{1}{4}\frac{x^2}{h^2}r^2(\mathbf{U}+\mathbf{kk})+x^2(\mathbf{U}-\mathbf{kk})\right]$$

$$=M[\tfrac{3}{20}r^2(\mathbf{U}+\mathbf{kk})+\tfrac{3}{5}h^2(\mathbf{U}-\mathbf{kk})] \tag{14}$$

$$=M[\tfrac{3}{20}r^2(\mathbf{U}-\mathbf{ii})+\tfrac{3}{20}r^2(\mathbf{U}-\mathbf{jj})+\tfrac{3}{5}h^2(\mathbf{U}-\mathbf{kk})]. \tag{14'}$$

349. *Solid cone of elliptic section.* If a, b are the semi-axes at the base, the inertia tensor about the vertex is clearly

$$\mathbf{I}(O)=M[\tfrac{3}{20}a^2(\mathbf{U}-\mathbf{ii})+\tfrac{3}{20}b^2(\mathbf{U}-\mathbf{jj})+\tfrac{3}{5}h^2(\mathbf{U}-\mathbf{kk})]. \tag{15}$$

350. *The inertia tensor of a hollow sphere.* The inertia tensor of a hollow sphere about its centre is clearly of the form

$$\mathbf{I}(G)=A\mathbf{U}.$$

But
$$\mathbf{I}(G)=\iint(\mathbf{r}^2\mathbf{U}-\mathbf{rr})\sigma dS$$

so that
$$\text{sca }\mathbf{I}(G)=2\mathbf{r}^2\iint\sigma dS=2M\mathbf{r}^2.$$

Hence
$$3A=2M\mathbf{r}^2,$$

or
$$\mathbf{I}(G)=\tfrac{2}{3}Mr^2\mathbf{U}. \tag{16}$$

The moment of inertia about any diameter is accordingly $\mathbf{I}(G):\mathbf{ii}=\tfrac{2}{3}Mr^2$.

351. *Solid sphere.* By summing for spherical shells we have

$$\mathbf{I}(G)=\int_0^r Md\left(\frac{r'^3}{r^3}\right)[\tfrac{2}{3}r'^2\mathbf{U}]$$

$$=\tfrac{2}{5}Mr^2\mathbf{U}. \tag{17}$$

352. *Solid ellipsoid.* Take unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ along the principal axes, say of lengths 2a, 2b, 2c respectively. Then by definition

$$\mathbf{I}(G)=\iiint[(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})^2\mathbf{U}-(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})]\rho dxdydz$$

$$=(\mathbf{U}-\mathbf{ii})\iiint\rho x^2dxdydz+(\mathbf{U}-\mathbf{jj})\iiint\rho y^2dxdydz+(\mathbf{U}-\mathbf{kk})\iiint\rho z^2dxdydz.$$

In this put
$$x=\frac{a}{r}x', \qquad y=\frac{b}{r}y', \qquad z=\frac{c}{r}z',$$

where r is arbitrary. Then

$$\mathbf{I}(G)=\frac{abc}{r^3}\left[\frac{a^2}{r^2}(\mathbf{U}-\mathbf{ii})+\frac{b^2}{r^2}(\mathbf{U}-\mathbf{jj})+\frac{c^2}{r^2}(\mathbf{U}-\mathbf{kk})\right]\iiint\rho x'^2dx'dy'dz',$$

where the integration is now taken over the interior of the sphere

$$x'^2+y'^2+z'^2=r^2,$$

and where we have used the equalities

$$\iiint x'^2dx'dy'dz'=\iiint y'^2dx'dy'dz'=\iiint z'^2dx'dy'dz'.$$

Using these equalities again, we have

$$\iiint x'^2dx'dy'dz'=\tfrac{1}{3}\iint(x'^2+y'^2+z'^2)dx'dy'dz'.$$

Now take the particular case $a=b=c=r$. Then $\mathbf{I}(G)$ reduces to

$$[3\mathbf{U}-(\mathbf{ii}+\mathbf{jj}+\mathbf{kk})]\tfrac{1}{3}\iiint(x'^2+y'^2+z'^2)\rho dx'dy'dz'$$
$$=\tfrac{2}{3}\mathbf{U}\rho\iiint(x'^2+y'^2+z'^2)dx'dy'dz',$$

and this must be the inertia tensor of a solid sphere of radius r and density ρ, namely

$$\tfrac{2}{5}(\tfrac{4}{3}\pi r^3\rho)r^2\mathbf{U}.$$

Hence
$$\tfrac{1}{3}\iiint(x'^2+y'^2+z'^2)\rho dx'dy'dz'=\tfrac{4}{3}\pi\frac{r^5}{5}\rho,$$

as is otherwise readily proved by elementary integral calculus. Hence

$$\mathbf{I}(G)=\tfrac{4}{3}\pi\rho abc[\tfrac{1}{5}a^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{5}b^2(\mathbf{U}-\mathbf{jj})+\tfrac{1}{5}c^2(\mathbf{U}-\mathbf{kk})]$$
$$=M[\tfrac{1}{5}a^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{5}b^2(\mathbf{U}-\mathbf{jj})+\tfrac{1}{5}c^2(\mathbf{U}-\mathbf{kk})]. \tag{18}$$

353. It will be observed that the inertia tensor of a rectangle, given by (2) with **i** and **j** perpendicular, is of the same form as that for an elliptic disc, given by (9′). These are accordingly equimomental apart from a constant multiplying factor. An elliptic disc has in fact the inertia tensor of a rectangle of equal semi-axes of $\frac{3}{4}$ of the mass.

Similarly the inertia tensor of a solid rectangular parallelopiped is of the same form as that for a solid ellipsoid. These again, therefore, are equimomental, apart from a constant multiplying factor; the ellipsoid is equimomental with a rectangular parallelopiped of equal semi-axes, of $\frac{3}{5}$ of the mass.

354. *Results in the integral calculus.* Some by-products of our formulæ for inertia tensors may be noted. The inertia tensor of the circumference of a circle has been seen to be

$$M[\tfrac{1}{2}r^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{2}r^2(\mathbf{U}-\mathbf{jj})]$$
$$=\tfrac{1}{2}(2\pi r\mu)r^2(\mathbf{U}+\mathbf{kk})$$

where μ is the line density. But this is equal, by definition of the inertia tensor, to

$$\mu\int(\mathbf{U}-\mathbf{pp})r^2 r d\theta,$$

where $\mathbf{p}$ is a variable unit vector drawn from the centre to the element $d\theta$ of the circumference. Hence

$$\int(\mathbf{U}-\mathbf{pp})d\theta=\pi(\mathbf{U}+\mathbf{kk}).$$

But since $\mathbf{U}$ is a constant tensor,

$$\int\mathbf{U}d\theta=2\pi\mathbf{U},$$

whence $$\int\mathbf{pp}d\theta=\pi(\mathbf{U}-\mathbf{kk})=\pi(\mathbf{ii}+\mathbf{jj}). \tag{1}$$

This is a very useful result. It can, of course, be verified directly by evaluating

$$\int(\mathbf{i}\cos\theta+\mathbf{j}\sin\theta)(\mathbf{i}\cos\theta+\mathbf{j}\sin\theta)d\theta.$$

It can also be evaluated by noting that on grounds of symmetry,

$$\int\mathbf{pp}d\theta=a(\mathbf{ii}+\mathbf{jj}).$$

Taking the scalar of each side we have

$$2\pi=2a,$$

or $a=\pi$, as above.

Again, since the inertia tensor of a hollow sphere is $\frac{2}{3}Mr^2\mathbf{U}$, we must have

$$\tfrac{2}{3}r^2(4\pi r^2\sigma)\mathbf{U}=\sigma\iint(\mathbf{U}-\mathbf{pp})r^2(r^2d\omega)$$

whence $$\iint(\mathbf{U}-\mathbf{pp})d\omega=\frac{8\pi}{3}\mathbf{U}.$$

But $$\iint\mathbf{U}d\omega=4\pi\mathbf{U}.$$

Hence $$\iint\mathbf{pp}d\omega=\frac{4\pi}{3}\mathbf{U}.$$

This may also be verified trigonometrically. It is, of course, æsthetically satisfactory to be able to make the evaluation of such integrals a process of pure vector algebra, without the use of trigonometric integrals.

With the aid of the above results, many other integrals may be found. For example, it is sometimes required to evaluate the mean value of

$$(\mathbf{p}.\mathbf{A})(\mathbf{p}\wedge\mathbf{A}),$$

where $\mathbf{p}$ is a unit vector describing uniformly a circle, centre O, normal to a unit vector $\mathbf{z}$, and $\mathbf{A}$ is a given constant vector (Fig. 84). This may be written

$$\frac{1}{2\pi}\int(\mathbf{p}.\mathbf{A})(\mathbf{p}\wedge\mathbf{A})d\theta$$

$$=\left[\left(\frac{1}{2\pi}\int\mathbf{pp}d\theta\right).\mathbf{A}\right]\wedge\mathbf{A}$$

$$=\frac{1}{2\pi}\pi[(\mathbf{U}-\mathbf{zz}).\mathbf{A}]\wedge\mathbf{A}$$

$$=-\tfrac{1}{2}(\mathbf{z}.\mathbf{A})(\mathbf{z}\wedge\mathbf{A}).$$

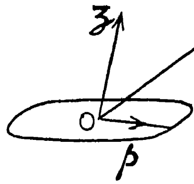

Fig. 84

355. The inertia tensors of arcs, sectors, etc., may be found by obvious extensions of the foregoing methods if they are required. Some examples follow.

356. *Inertia tensor of an arc of a circle.* Let the arc, of radius r, subtend an angle 2α at the centre O (Fig. 85). If μ is its line density,

$$\mathbf{I}(O)=\mu\int_{\theta=-\alpha}^{\theta=\alpha}(\mathbf{U}-\mathbf{pp})r^2rd\theta,$$

where $\mathbf{p}$ is a unit vector in the direction from O to the element $d\theta$ of the arc. Let the unit vector $\mathbf{i}$ bisect the angle subtended by the arc at O, and let $\mathbf{j}$ be a unit vector perpendicular to it, in the plane of the arc. Then

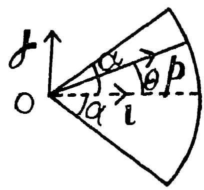

Fig. 85

$$\mathbf{p}=\mathbf{i}\cos\theta+\mathbf{j}\sin\theta,$$

and $$\mathbf{I}(O)=\mu r^3\left[2\alpha\mathbf{U}-\int_{-\alpha}^{\alpha}(\mathbf{i}\cos\theta+\mathbf{j}\sin\theta)(\mathbf{i}\cos\theta+\mathbf{j}\sin\theta)d\theta\right].$$

The terms in $\mathbf{ij}$ and $\mathbf{ji}$ vanish. We are left with

$$\mathbf{I}(O)=Mr^2\left[\mathbf{U}-\tfrac{1}{2}(\mathbf{ii}+\mathbf{jj})-\frac{\sin\alpha\cos\alpha}{2\alpha}(\mathbf{ii}-\mathbf{jj})\right]$$

$$=\tfrac{1}{2}Mr^2\left[\mathbf{U}+\mathbf{kk}-\frac{\sin\alpha\cos\alpha}{\alpha}(\mathbf{ii}-\mathbf{jj})\right]. \qquad (19)$$

357. *Plane sectorial lamina.* Replacing M in the foregoing by $\sigma(2\alpha r dr)$, where σ is the surface density of the lamina, of angle 2α, and integrating with respect to r, we have for a sector

$$\mathbf{I}(O)=\left[\mathbf{U}+\mathbf{kk}-\frac{\sin\alpha\cos\alpha}{\alpha}(\mathbf{ii}-\mathbf{jj})\right]\int_0^r\tfrac{1}{2}r^2\sigma(2\alpha r dr)$$

$$=\tfrac{1}{4}Mr^2\left[\mathbf{U}+\mathbf{kk}-\frac{\sin\alpha\cos\alpha}{\alpha}(\mathbf{ii}-\mathbf{jj})\right] \qquad (20)$$

since here $M=\sigma\tfrac{1}{2}(2\alpha r^2)=\sigma\alpha r^2$.

In both these formulæ, (19) and (20), the term in $\sin\alpha\cos\alpha$ vanishes when $\alpha=\frac{1}{2}\pi$, and so the inertia tensor of a semi-circular arc and of a semi-circular lamina are of the same form as for a complete circular arc or circular disc, (7) and (8).

358. *Inertia tensor of the cap of a sector of a sphere, about the centre of the sphere.* Let α be the semi-vertical angle of the cone subtended by the cap of the sphere at the centre O of the sphere. Then we have

$$\mathbf{I}(\mathrm{O})=\iint(\mathbf{U}-\mathbf{pp})\mathrm{r}^2\sigma\mathrm{r}^2d\omega,$$

where $\mathbf{p}$ is a unit vector from O towards the element (θ,φ) of the cap. Taking a unit vector $\mathbf{i}$ along the axis of the cap, we have

$$\mathbf{p}=\mathbf{i}\cos\theta+\mathbf{j}\sin\theta\cos\varphi+\mathbf{k}\sin\theta\sin\varphi,$$

and, the integrals being taken over the cap

$$\iint\cos^2\theta d\omega=\frac{2\pi}{3}(1-\cos^3\alpha),$$

$$\iint\sin^2\theta\cos^2\varphi d\omega=\iint\sin^2\theta\sin^2\varphi d\omega=2\pi[\tfrac{1}{2}(1-\cos\alpha)-\tfrac{1}{6}(1-\cos^3\alpha)].$$

Hence $$\mathbf{I}(\mathrm{O})=\sigma\mathrm{r}^4[2\pi(1-\cos\alpha)\mathbf{U}-2\pi\{\tfrac{1}{3}(1-\cos^3\alpha)\mathbf{ii} \\ +(\tfrac{1}{2}(1-\cos\alpha)-\tfrac{1}{6}(1-\cos^3\alpha))(\mathbf{jj}+\mathbf{kk})\}].$$

But here, since the area of the cap is $2\pi\mathrm{r}^2(1-\cos\alpha)$, we have

$$\mathrm{M}=2\pi\mathrm{r}^2\sigma(1-\cos\alpha).$$

Hence

$$\mathbf{I}(\mathrm{O})=\mathrm{Mr}^2[\mathbf{U}-\tfrac{1}{2}(\mathbf{jj}+\mathbf{kk})-\tfrac{1}{6}(2\mathbf{ii}-\mathbf{jj}-\mathbf{kk})(1+\cos\alpha+\cos^2\alpha)]. \quad (21)$$

To check this, take $\alpha=\pi$. We get them for the surface of a complete sphere

$$\mathbf{I}(\mathrm{O})=\mathrm{Mr}^2[\mathbf{U}-\tfrac{1}{2}(\mathbf{jj}+\mathbf{kk})-\tfrac{1}{6}(2\mathbf{ii}-\mathbf{jj}-\mathbf{kk})] \\ =\tfrac{2}{3}\mathrm{Mr}^2\mathbf{U},$$

as it should be.

359. *Inertia tensor of a solid sector of a sphere about its vertex.* In (21) we replace M by $\rho 2\pi(1-\cos\alpha)\mathrm{r}^2d\mathrm{r}$, integrate with respect to r and then write M for $\frac{2}{3}\pi\rho(1-\cos\alpha)\mathrm{r}^3$. We find

$$\mathbf{I}(\mathrm{O})=\tfrac{3}{5}\mathrm{Mr}^2[\mathbf{U}-\tfrac{1}{2}(\mathbf{jj}+\mathbf{kk})-\tfrac{1}{6}(2\mathbf{ii}-\mathbf{jj}-\mathbf{kk})(1+\cos\alpha+\cos^2\alpha)]. \quad (22)$$

If in (21) or (22) we put $\alpha=\frac{1}{2}\pi$, the square bracket reduces to $\frac{2}{3}\mathbf{U}$, just as when $\alpha=\pi$. Thus the momental ellipsoid of a uniform hemisphere or hemispherical shell, about its centre, is a sphere.

360. *List of inertia tensors.*

M = mass. $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, $\mathbf{l}$ unit vectors. G, centre of mass.

(1) *Uniform rod* of length 2a, in direction of $\mathbf{i}$:

$$\mathbf{I}(\mathrm{G})=\tfrac{1}{3}\mathrm{Ma}^2(\mathbf{U}-\mathbf{ii}).$$

(2) *Uniform parallelogram* (lamina), sides 2a, 2b, in directions of **i**, **j** respectively :

$$\mathbf{I}(G)=M[\tfrac{1}{3}a^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{3}b^2(\mathbf{U}-\mathbf{jj})].$$

(3) *Uniform solid parallelopiped*, sides 2a, 2b, 2c, in directions **i**, **j**, **k** respectively :

$$\mathbf{I}(G)=M[\tfrac{1}{3}a^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{3}b^2(\mathbf{U}-\mathbf{jj})+\tfrac{1}{3}c^2(\mathbf{U}-\mathbf{kk})].$$

(4) *Uniform triangular lamina*, vertices distant from G by lengths 2ξ, 2η, 2ζ in directions **i**, **j**, **k** ($\xi\mathbf{i}+\eta\mathbf{j}+\zeta\mathbf{k}=0$) :

$$\mathbf{I}(G)=M[\tfrac{1}{3}\xi^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{3}\eta^2(\mathbf{U}-\mathbf{jj})+\tfrac{1}{3}\zeta^2(\mathbf{U}-\mathbf{kk})].$$

(5) *Uniform triangular prism*, section as in (4), of height 2h in direction **l** :

$$\mathbf{I}(G)=M[\tfrac{1}{3}\xi^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{3}\eta^2(\mathbf{U}-\mathbf{jj})+\tfrac{1}{3}\zeta^2(\mathbf{U}-\mathbf{kk})+\tfrac{1}{3}h^2(\mathbf{U}-\mathbf{ll})].$$

(6) *Uniform tetrahedron*, vertices distant 3ξ, 3η, 3ζ, 3ω from G in directions **i**, **j**, **k**, **l** :

$$\mathbf{I}(G)=\tfrac{9}{20}M[\xi^2(\mathbf{U}-\mathbf{ii})+\eta^2(\mathbf{U}-\mathbf{jj})+\zeta^2(\mathbf{U}-\mathbf{kk})+\omega^2(\mathbf{U}-\mathbf{ll})].$$

(7) *Circumference of a circle*, of radius r, the normal to its plane in direction **k**, and **i**, **j** any two perpendicular unit vectors in its plane :

$$\mathbf{I}(G)=M[\tfrac{1}{2}r^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{2}r^2(\mathbf{U}-\mathbf{jj})]=\tfrac{1}{2}Mr^2[\mathbf{U}+\mathbf{kk}].$$

(8) *Uniform circular lamina*, of radius r, vectors **i**, **j**, **k** as in (7) :

$$\mathbf{I}(G)=M[\tfrac{1}{4}r^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{4}r^2(\mathbf{U}-\mathbf{jj})]=\tfrac{1}{4}Mr^2[\mathbf{U}+\mathbf{kk}].$$

(9) *Uniform elliptic lamina*, of semi-axes a, b in directions **i**, **j**:

$$\mathbf{I}(G)=M[\tfrac{1}{4}a^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{4}b^2(\mathbf{U}-\mathbf{jj})].$$

(10) *Hollow circular cylinder*, of radius r and height 2h, axis in direction **k**, and **i**, **j** any two unit perpendicular vectors normal to **k** :

$$\mathbf{I}(G)=M[\tfrac{1}{2}r^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{2}r^2(\mathbf{U}-\mathbf{jj})+\tfrac{1}{3}h^2(\mathbf{U}-\mathbf{kk})]$$
$$=M[\tfrac{1}{2}r^2(\mathbf{U}+\mathbf{kk})+\tfrac{1}{3}h^2(\mathbf{U}-\mathbf{kk})].$$

(11) *Solid circular cylinder*, definitions as in (10) :

$$\mathbf{I}(G)=M[\tfrac{1}{4}r^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{4}r^2(\mathbf{U}-\mathbf{jj})+\tfrac{1}{3}h^2(\mathbf{U}-\mathbf{kk})]$$
$$=M[\tfrac{1}{4}r^2(\mathbf{U}+\mathbf{kk})+\tfrac{1}{3}h^2(\mathbf{U}-\mathbf{kk})].$$

(12) *Solid elliptic cylinder*, of semi-axes a, b in directions **i**, **j**, and axis 2h in direction **k** :

$$\mathbf{I}(G)=M[\tfrac{1}{4}a^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{4}b^2(\mathbf{U}-\mathbf{jj})+\tfrac{1}{3}h^2(\mathbf{U}-\mathbf{kk})].$$

(13) *Hollow circular cone*, of base-radius r and axis of length h, in direction **k**, O being the vertex :

$$\mathbf{I}(O)=M[\tfrac{1}{4}r^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{4}r^2(\mathbf{U}-\mathbf{jj})+\tfrac{1}{2}h^2(\mathbf{U}-\mathbf{kk})]$$
$$=M[\tfrac{1}{4}r^2(\mathbf{U}+\mathbf{kk})+\tfrac{1}{2}h^2(\mathbf{U}-\mathbf{kk})].$$

(14) *Solid circular cone*, definition as in (13):

$$\mathbf{I}(O)=M[\tfrac{3}{20}r^2(\mathbf{U}-\mathbf{ii})+\tfrac{3}{20}r^2(\mathbf{U}-\mathbf{jj})+\tfrac{3}{5}h^2(\mathbf{U}-\mathbf{kk})]$$
$$=M[\tfrac{3}{20}r^2(\mathbf{U}+\mathbf{kk})+\tfrac{3}{5}h^2(\mathbf{U}-\mathbf{kk})].$$

(15) *Solid cone of elliptic section*, of semi-axes of base a, b in directions $\mathbf{i}, \mathbf{j}$:

$$\mathbf{I}(O)=M[\tfrac{3}{20}a^2(\mathbf{U}-\mathbf{ii})+\tfrac{3}{20}b^2(\mathbf{U}-\mathbf{jj})+\tfrac{3}{5}h^2(\mathbf{U}-\mathbf{kk})].$$

(16) *Hollow sphere*, of radius r:

$$\mathbf{I}(G)=\tfrac{2}{3}Mr^2\mathbf{U}.$$

(17) *Solid sphere*, of radius r:

$$\mathbf{I}(G)=\tfrac{2}{5}Mr^2\mathbf{U}.$$

(18) *Solid ellipsoid*, of semi-axes a, b, c in directions $\mathbf{i}, \mathbf{j}, \mathbf{k}$:

$$\mathbf{I}(G)=M[\tfrac{1}{5}a^2(\mathbf{U}-\mathbf{ii})+\tfrac{1}{5}b^2(\mathbf{U}-\mathbf{jj})+\tfrac{1}{5}c^2(\mathbf{U}-\mathbf{kk})].$$

The inertia tensors of all the solid bodies possessing three axes of symmetry $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are given by

$$\mathbf{I}(G)=M\left[\frac{1}{n}a^2(\mathbf{U}-\mathbf{ii})+\frac{1}{n'}b^2(\mathbf{U}-\mathbf{jj})+\frac{1}{n''}c^2(\mathbf{U}-\mathbf{kk})\right],$$

where the typical denominators n, n′, n″ are equal either to 3 (for rods and rod-like dimensions), to 4 for elliptic-sectioned dimensions, or to 5 for ellipsoidal dimensions.

361. *Some formulæ involving the inertia tensor.* This seems a convenient place to derive some commonly-met-with formulæ involving the inertia tensor. They are usually derived in their proper physical contexts, but it seems desirable to give them here whilst the inertia-tensor technique is fresh in the reader's mind.

362. *The gravitational potential of a body at a distant point.* Let G (Fig. 86) be the centre of mass of a given body, P a fixed distant point

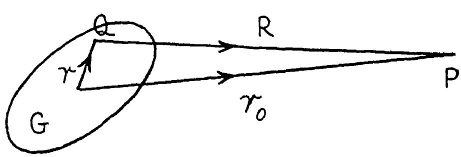

Fig. 86

outside the body, Q a variable point in the body. Let GQ $=\mathbf{r}$, GP $=\mathbf{r}_0$, QP $=\mathbf{R}$, so that

$$\mathbf{R}=\mathbf{r}_0-\mathbf{r}.$$

Then the gravitational potential at P due to the body, say V_P, is given by

$$V_P=\gamma\iiint\frac{\rho d\tau}{|\mathbf{R}|},$$

where γ is the constant of gravitation, ρ the volume-density and $d\tau$ the volume element. Now

$$\frac{1}{|\mathbf{R}|}=\frac{1}{(\mathbf{R}^2)^{\frac{1}{2}}}=\frac{1}{(\mathbf{r}_0{}^2-2\mathbf{r}_0.\mathbf{r}+\mathbf{r}^2)^{\frac{1}{2}}}$$
$$=\frac{1}{|\mathbf{r}_0|}\left[1-\tfrac{1}{2}\left(\frac{-2\mathbf{r}_0.\mathbf{r}+\mathbf{r}^2}{\mathbf{r}_0{}^2}\right)+\frac{-\frac{1}{2}.-\frac{3}{2}}{1.2}\left(\frac{-2\mathbf{r}.\mathbf{r}_0+\mathbf{r}^2}{\mathbf{r}_0{}^2}\right)^2+\dots\right]$$
$$=\frac{1}{|\mathbf{r}_0|}\left[1+\frac{\mathbf{r}_0.\mathbf{r}}{\mathbf{r}_0{}^2}+\frac{3(\mathbf{r}.\mathbf{r}_0)^2-\mathbf{r}_0{}^2\mathbf{r}^2}{2\mathbf{r}_0{}^4}+\dots\right],$$

and so $$V_P=\gamma\iiint\left[\frac{1}{|\mathbf{r}_0|}+\frac{\mathbf{r}_0.\mathbf{r}}{|\mathbf{r}_0|^3}+\frac{3(\mathbf{r}.\mathbf{r}_0)^2-\mathbf{r}_0{}^2\mathbf{r}^2}{2|\mathbf{r}_0|^5}+\dots\right]\rho d\tau.$$

But since G is the centre of mass,

$$\int\rho\mathbf{r}d\tau=0.$$

Further, since $$\mathbf{I}(G)=\iiint\rho(\mathbf{r}^2\mathbf{U}-\mathbf{r}\mathbf{r})d\tau,$$

we have $$\mathbf{I}(G):\frac{\mathbf{r}_0\mathbf{r}_0}{|\mathbf{r}_0|^2}=\frac{1}{|\mathbf{r}_0|^2}\iiint[\mathbf{r}^2\mathbf{r}_0{}^2-(\mathbf{r}.\mathbf{r}_0)^2]\rho d\tau,$$

and $$\text{sca }\mathbf{I}(G)=2\iiint\rho\mathbf{r}^2d\tau.$$

Hence $$\iiint\frac{3(\mathbf{r}.\mathbf{r}_0)^2-\mathbf{r}_0{}^2\mathbf{r}^2}{2|\mathbf{r}_0|^5}\rho d\tau=\frac{1}{2|\mathbf{r}_0|^3}\left[\text{sca }\mathbf{I}(G)-3\mathbf{I}(G):\frac{\mathbf{r}_0\mathbf{r}_0}{\mathbf{r}_0{}^2}\right].$$

Since $\mathbf{r}_0/|\mathbf{r}_0|$ is a unit vector along GP, $\mathbf{I}(G):\frac{\mathbf{r}_0\mathbf{r}_0}{|\mathbf{r}_0|^2}$ is the moment of inertia of the body about GP, say $\mu(GP)$. Also sca $\mathbf{I}(G)=A+B+C$, where A, B, C are the principal moments of inertia at G. Hence

$$V_P=\gamma\left[\frac{M}{|\mathbf{r}_0|}+\frac{A+B+C-3\mu(GP)}{2|\mathbf{r}_0|^3}+\dots\right].$$

363. The mutual potential energy of two distant bodies in one another's presence can now be readily found. This is given by

$$-\iiint V_P\rho d\tau,$$

the negative arising from the sign convention used in defining the gravitational potential (Fig. 87). The volume integral is to be taken over the domain of the second body. Using the approximation of the previous section, we get

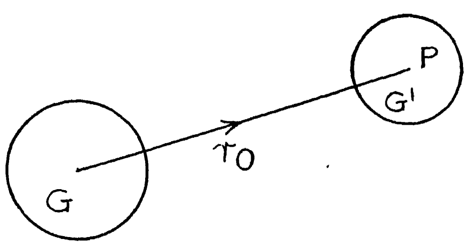

Fig. 87

$$-\gamma\iiint\left[\frac{M}{|\mathbf{r}_0|}+\frac{\text{sca }\mathbf{I}(G)-3\mathbf{I}(G):\frac{\mathbf{r}_0\mathbf{r}_0}{|\mathbf{r}_0|^2}}{2|\mathbf{r}_0|^3}\right]\rho d\tau.$$

To a sufficient order, the second term in the square bracket can be treated as constant during the integration. The reciprocal $1/|\mathbf{r}_0|$ in the first term can be expanded as in § 362. The result is

$$-\left[\gamma\frac{MM'}{|\bar{\mathbf{r}}|}+\gamma M\frac{\operatorname{sca}\mathbf{I}'(G')-3\mathbf{I}'(G'):\frac{\bar{\mathbf{r}}\bar{\mathbf{r}}}{|\bar{\mathbf{r}}|^2}}{2|\bar{\mathbf{r}}|^3}+\gamma M'\frac{\operatorname{sca}\mathbf{I}(G)-3\mathbf{I}(G):\frac{\bar{\mathbf{r}}\bar{\mathbf{r}}}{|\bar{\mathbf{r}}|^2}}{2|\bar{\mathbf{r}}|^3}\right].$$

Here r is the position vector of the centre of mass of either body with regard to the centre of mass of the other, and primed symbols refer to the second (distant) body.

364. *Couple due to a distant body.* Let $\mathbf{\Gamma}$ be the couple exerted by the distant body of mass M' on the body of mass M whose centre of mass G is taken as origin. Let this body at the origin undergo a small rotation $\boldsymbol{\epsilon}$ about the origin. Then the work done is $\mathbf{\Gamma}.\boldsymbol{\epsilon}$, and hence the mutual energy of the two bodies decreases by this amount. It follows that

$$-\mathbf{\Gamma}.\boldsymbol{\epsilon}=-\gamma M'\delta\frac{\operatorname{sca}\mathbf{I}(G)-3\mathbf{I}(G):\frac{\bar{\mathbf{r}}\bar{\mathbf{r}}}{\bar{\mathbf{r}}^2}}{2|\bar{\mathbf{r}}|^3}.$$

But sca $\mathbf{I}(G)$ is unaltered by a rigid displacement of the body concerned. And by § 211, since the tensor $\mathbf{I}(G)$ is unaltered *in its own frame*, the change of $\mathbf{I}(G)$ with respect to a fixed frame is given by

$$\delta\mathbf{I}(G)=\boldsymbol{\epsilon}\wedge\mathbf{I}(G)-\mathbf{I}(G)\wedge\boldsymbol{\epsilon}$$

(where as usual the notation denotes cross-products of tensors and vectors). Making free use of the theorems relating to such cross-products, we have

$$(\boldsymbol{\epsilon}\wedge\mathbf{I}):\bar{\mathbf{r}}\bar{\mathbf{r}}=[(\boldsymbol{\epsilon}\wedge\mathbf{I}).\bar{\mathbf{r}}].\bar{\mathbf{r}}=[\boldsymbol{\epsilon}\wedge(\mathbf{I}.\bar{\mathbf{r}})].\bar{\mathbf{r}}=[(\mathbf{I}.\bar{\mathbf{r}})\wedge\bar{\mathbf{r}}].\boldsymbol{\epsilon},$$

and $$(\mathbf{I}\wedge\boldsymbol{\epsilon}):\bar{\mathbf{r}}\bar{\mathbf{r}}=[(\mathbf{I}\wedge\boldsymbol{\epsilon}).\bar{\mathbf{r}}].\bar{\mathbf{r}}=[\mathbf{I}.(\boldsymbol{\epsilon}\wedge\bar{\mathbf{r}})].\bar{\mathbf{r}},$$

or, since $\mathbf{I}$ is self-conjugate, the last expression is equal to

$$(\mathbf{I}.\bar{\mathbf{r}}).(\boldsymbol{\epsilon}\wedge\bar{\mathbf{r}})=-[(\mathbf{I}.\bar{\mathbf{r}})\wedge\bar{\mathbf{r}}].\boldsymbol{\epsilon}.$$

Thus, since $\mathbf{r}$ is constant under the operator δ,

$$-\mathbf{\Gamma}.\boldsymbol{\epsilon}=+\frac{3\gamma M'}{|\bar{\mathbf{r}}|^5}[(\mathbf{I}.\bar{\mathbf{r}})\wedge\bar{\mathbf{r}}].\boldsymbol{\epsilon}.$$

This is to be true for all arbitrary small displacements $\boldsymbol{\epsilon}$. Hence

$$\mathbf{\Gamma}=-\frac{3\gamma M'}{|\bar{\mathbf{r}}|^5}(\mathbf{I}.\bar{\mathbf{r}})\wedge\bar{\mathbf{r}}.$$

If $\bar{\mathbf{r}}=\bar{x}\mathbf{i}+\bar{y}\mathbf{j}+\bar{z}\mathbf{k}$, and if $\mathbf{I}(G)=A\mathbf{ii}+B\mathbf{jj}+C\mathbf{kk}$, when referred to its principal axes, then the last expression reduces to

$$\mathbf{\Gamma}=\frac{3}{|\bar{\mathbf{r}}|^5}[(C-B)\bar{y}\bar{z}\mathbf{i}+(A-C)\bar{z}\bar{x}\mathbf{j}+(B-A)\bar{x}\bar{y}\mathbf{k}].$$

365. The formula just derived by using general theory is readily obtained from first principles. Let P be a distant particle, of mass M′. If Q is a particle of the given body, and if GP$=\mathbf{r}_0$, GQ$=\mathbf{r}$ (Fig. 88), the force at Q on the element $\rho d\tau$ at Q is given by

$$\gamma M' \frac{\mathbf{r}_0-\mathbf{r}}{|\mathbf{r}_0-\mathbf{r}|^3}\rho d\tau,$$

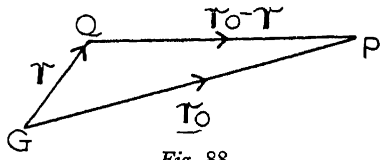

Fig. 88

and the element of couple about G due to this is

$$\gamma M'\mathbf{r}\wedge \frac{\mathbf{r}_0-\mathbf{r}}{|\mathbf{r}_0-\mathbf{r}|^3}\rho d\tau$$

or

$$\gamma M' \frac{\mathbf{r}\wedge\mathbf{r}_0}{|\mathbf{r}_0-\mathbf{r}|^3}\rho d\tau.$$

The total couple is accordingly

$$\mathbf{\Gamma}=\gamma M'\iiint\frac{\mathbf{r}\wedge\mathbf{r}_0}{|\mathbf{r}_0-\mathbf{r}|^3}\rho d\tau$$

$$=\gamma M'\iiint\frac{\mathbf{r}\wedge\mathbf{r}_0}{|\mathbf{r}_0|^3}\left[1+3\frac{\mathbf{r}.\mathbf{r}_0}{\mathbf{r}_0^{\,2}}+\ldots\right]\rho d\tau.$$

Now, since G is the centre of mass of the given body,

$$\iiint\mathbf{r}\rho d\tau=0,$$

and the first term in the integral vanishes. The second term gives

$$\mathbf{\Gamma}=\frac{3\gamma M'}{|\mathbf{r}_0|^5}\iiint(\mathbf{r}\wedge\mathbf{r}_0)(\mathbf{r}.\mathbf{r}_0)\rho d\tau$$

$$=\frac{3\gamma M'}{|\mathbf{r}_0|^5}\left[\left(\iiint\mathbf{r}\mathbf{r}\rho d\tau\right).\mathbf{r}_0\right]\wedge\mathbf{r}_0.$$

But since $$\mathbf{I}(G)=\iiint(\mathbf{r}^2\mathbf{U}-\mathbf{r}\mathbf{r})\rho d\tau,$$

and since $$(\mathbf{U}.\mathbf{r}_0)\wedge\mathbf{r}_0=0,$$

we have $$\left[\left(\iiint\mathbf{r}\mathbf{r}\rho d\tau\right).\mathbf{r}_0\right]\wedge\mathbf{r}_0=-[\mathbf{I}(G).\mathbf{r}_0]\wedge\mathbf{r}_0.$$

Hence $$\mathbf{\Gamma}=-\frac{3\gamma M'}{|\mathbf{r}_0|^5}[\mathbf{I}.\mathbf{r}_0]\wedge\mathbf{r}_0$$

in accordance with § 364.

CHAPTER XV

THE DYNAMICS OF RIGID BODIES

366. We have now considered the dynamics of a system of *particles*. We shall next consider the dynamics of a system of particles constituting a *rigid body*.

367. In the case of a system of particles, we saw (§ 296) that since the actions between any two particles, members of the system, are equal and opposite, the equations of motion take the form

$$\frac{d\mathbf{L}}{dt}=\mathbf{R}, \qquad \frac{d\mathbf{H}(O)}{dt}=\mathbf{\Gamma}(O),$$

where $\mathbf{L}$ is the linear momentum, $\mathbf{H}(O)$ the angular momentum about a fixed point O, and $(\mathbf{R}, \mathbf{\Gamma}(O))$ is the system of external forces reduced to O as base point. If it is legitimate to regard a rigid body as a system of particles, these equations apply as they stand.

368. *D'Alembert's principle.* There are, however, logical difficulties in the way of regarding the rigid body as a system of particles, for in the limit the number of small elements into which it can be considered as divided are infinite in number, and none of them are particles. Moreover it is not clear that the total force system acting on a given element of the rigid body can be divided up into actions between it and some or all of the other elements. It is better to make some specific assumption, to be justified by its success.

The equation of motion of a particle acted on by a force $\mathbf{P}$ is

$$m\frac{d^2\mathbf{r}}{dt^2}=\mathbf{P}.$$

Hence
$$\mathbf{P}-m\frac{d^2\mathbf{r}}{dt^2}=0.$$

This states that if we introduce a fictitious force (called the *effective force*) equal to m times the acceleration of the particle, then this force taken negatively together with the applied force $\mathbf{P}$ forms a system of forces equivalent to zero. Now consider the system of line vectors $(\mathbf{P})$ and the system of line vectors $(+m\ddot{\mathbf{r}})$, obtained by aggregating respectively the various applied forces and the various effective forces. Then these two systems of line vectors are equivalent. Let us now make the assumption that the system of forces acting on the totality of small elements of a *rigid body* is equivalent to the *external* system of forces. This, of course,

follows from Newton's third law when the *internal* forces may be divided into pairs of equal and opposite forces. The assumption is, however, wider than this, for it includes the possibility of the internal actions taking the form of *couples* or other force systems, and it avoids explicit mention of the necessity to associate the several constituents of the action on a given element A of the rigid body with other specified elements B, C, D, ... Making, then, this assumption, we assert that *the system of effective forces is equivalent to the system of external forces.* This is called d'Alembert's principle.

369. *Equations of motion.* The theory of systems of line vectors, as given in Chapter VI, then provides the analytical conditions of equivalence in the form

$$\Sigma \mathrm{m}\frac{d^2\mathbf{r}}{dt^2}=\mathbf{R}, \qquad \Sigma\mathbf{r}\wedge \mathrm{m}\frac{d^2\mathbf{r}}{dt^2}=\mathbf{\Gamma}(O),$$

where $\mathbf{r}$ is the position vector of a typical element of the body, of mass m, relative to a fixed point O. These equations are the embodiment of d'Alembert's principle.

370. D'Alembert's principle may equally be embodied in any other set of equations expressing the equivalence of two systems of line vectors. For example, if $\delta\mathbf{r}$ denotes an arbitrary rigid-body displacement associated with the element at $\mathbf{r}$, then the conditions of equivalence may be expressed by the statement that the work of the effective forces in the rigid-body displacement is equal to the work of the externally applied forces in the same rigid-body displacement. Thus

$$\Sigma \mathrm{m}\frac{d^2\mathbf{r}}{dt^2}.\delta\mathbf{r}=\Sigma\mathbf{P}.\delta\mathbf{r}.$$

371. *Equations of motion in terms of momentum.* The foregoing conditions of equivalence may now be transformed as for a system of particles. Denoting the linear momentum by $\mathbf{L}$, the angular momentum about O by $\mathbf{H}(O)$, so that

$$\mathbf{L}=\Sigma \mathrm{m}\frac{d\mathbf{r}}{dt}, \qquad \mathbf{H}(O)=\Sigma\mathbf{r}\wedge \mathrm{m}\frac{d\mathbf{r}}{dt},$$

we have

$$\frac{d\mathbf{L}}{dt}=\Sigma \mathrm{m}\frac{d^2\mathbf{r}}{dt^2}, \qquad \frac{d\mathbf{H}(O)}{dt}=\Sigma\mathbf{r}\wedge \mathrm{m}\frac{d^2\mathbf{r}}{dt^2},$$

whence

$$\frac{d\mathbf{L}}{dt}=\mathbf{R}, \qquad \frac{d\mathbf{H}(O)}{dt}=\mathbf{\Gamma}(O).$$

It is particularly to be noted that in the differentiation of $\mathbf{H}(O)$ with respect to t, O is to be treated as a fixed point; otherwise we should not have the vanishing of the term $\frac{d\mathbf{r}}{dt}\wedge \mathrm{m}\frac{d\mathbf{r}}{dt}$, for the two $\frac{d\mathbf{r}}{dt}$'s would mean different things.

372. Again, these equations may be deduced from the principle of virtual work. If the small rigid-body displacements $\delta\mathbf{r}$ arise from a displacement $\mathbf{u}$ at O, and a rotation $\boldsymbol{\epsilon}$, then by § 370

$$\Sigma m\frac{d^2\mathbf{r}}{dt^2}.(\mathbf{u}+\boldsymbol{\epsilon}\wedge\mathbf{r})=\Sigma\mathbf{P}.(\mathbf{u}+\boldsymbol{\epsilon}\wedge\mathbf{r}),$$

or
$$\left(\Sigma m\frac{d^2\mathbf{r}}{dt^2}\right).\mathbf{u}+\boldsymbol{\epsilon}.\left(\Sigma\mathbf{r}\wedge m\frac{d^2\mathbf{r}}{dt^2}\right)=\mathbf{u}.\Sigma\mathbf{P}+\boldsymbol{\epsilon}.\Sigma\mathbf{r}\wedge\mathbf{P}.$$

Taking $\boldsymbol{\epsilon}=0$, since the equality must hold for all $\mathbf{u}$ we have

$$\Sigma m\frac{d^2\mathbf{r}}{dt^2}=\Sigma\mathbf{P}=\mathbf{R},$$

and taking $\mathbf{u}=0$, since the equality must hold for all $\boldsymbol{\epsilon}$ we have

$$\Sigma\mathbf{r}\wedge m\frac{d^2\mathbf{r}}{dt^2}=\Sigma\mathbf{r}\wedge\mathbf{P}=\boldsymbol{\Gamma}(O).$$

In the right-hand side of the last two equations, the summations are to be extended solely to the points of application of the external forces. These yield as before

$$\frac{d\mathbf{L}}{dt}=\mathbf{R},\qquad \frac{d\mathbf{H}(O)}{dt}=\boldsymbol{\Gamma}(O).$$

373. The rate of change of angular momentum about the fixed point O can now be transformed into the rate of change about a moving point instantaneously coinciding with O, by any of the formulæ of § 308. For example, it is sometimes convenient to choose O as a point on the line of action of an unknown reaction. The student will find it, however, safer and more convenient to choose for O either the centre of mass of the body or a point of the body which is fixed. Older treatises on dynamics were at much pains to solve problems involving the calculation of moments of forces and rates of change of angular momenta about moving points. But it will be found in almost all cases more expeditious and less burdensome on the memory to introduce unknown reactions as specifically mentioned vectors, and then to eliminate them. The following examples of this method will be worked out in full.

EXAMPLES OF THE MOTION OF RIGID BODIES

374. *Motion of a sphere rolling on a rough horizontal plane.* Let M be the mass, CU the inertia tensor * of a spherical body of radius a, in motion without slipping on a rough horizontal plane. Let A (Fig. 89) be the

* About G.

point of contact, **R** the reaction at A, **Ω** the angular velocity of the body at any instant, **r** the position vector of the centre G of the sphere, **z** a unit vector vertically upwards. Without loss of generality we can take **r** as perpendicular to **z**, so that $\mathbf{r}.\mathbf{z}=0$. The angular momentum **H**(G) about G is $C\mathbf{U}.\mathbf{\Omega}=C\mathbf{\Omega}$. The equation for the rate of change of linear momentum is

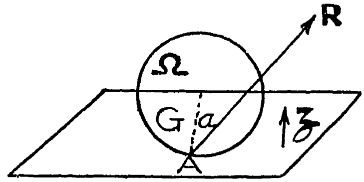

Fig. 89

$$M\frac{d^2\mathbf{r}}{dt^2}=\mathbf{R}-Mg\mathbf{z}, \tag{1}$$

and the equation for the rate of change of angular momentum about G is

$$C\frac{d\mathbf{\Omega}}{dt}=-a\mathbf{z}\wedge\mathbf{R}, \tag{2}$$

the right-hand side of the latter equation being the moment of the external force system about G. The equation of rolling contact, expressing that the particle of the sphere in contact with the plane is at rest, is

$$\frac{d\mathbf{r}}{dt}+\mathbf{\Omega}\wedge(-a\mathbf{z})=0. \tag{3}$$

The first step is to eliminate **R** between (1) and (2). This gives

$$C\frac{d\mathbf{\Omega}}{dt}=-a\mathbf{z}\wedge M\frac{d^2\mathbf{r}}{dt^2}. \tag{4}$$

Differentiating (3) with regard to t and eliminating $\frac{d\mathbf{\Omega}}{dt}$ between the result of this and relation (4), we get

$$\frac{d^2\mathbf{r}}{dt^2}+\frac{Ma^2}{C}\left(\mathbf{z}\wedge\frac{d^2\mathbf{r}}{dt^2}\right)\wedge\mathbf{z}=0,$$

or, expanding the continued vector product and using $\mathbf{z}.\frac{d^2\mathbf{r}}{dt^2}=0$,

$$\left(1+\frac{Ma^2}{C}\right)\frac{d^2\mathbf{r}}{dt^2}=0.$$

Integrating this, $$\frac{d\mathbf{r}}{dt}=\text{const.}=\mathbf{V},$$

say. Relation (4) then gives

$$\mathbf{\Omega}=\text{const.}$$

To relate the constant value of **Ω** with **V**, we multiply (3) vectorially by **z**. We get

$$\mathbf{V}\wedge\mathbf{z}-a[-\mathbf{\Omega}+\mathbf{z}(\mathbf{z}.\mathbf{\Omega})]=0,$$

or $$\mathbf{\Omega}=-\frac{\mathbf{V}\wedge\mathbf{z}}{a}+\mathbf{z}(\mathbf{z}.\mathbf{\Omega}).$$

Since $\mathbf{\Omega}$ and $\mathbf{z}$ are constants, $\mathbf{z}.\mathbf{\Omega}$ is constant. (This can be seen directly by multiplying (4) scalarly by $\mathbf{z}$, when we get $\mathbf{z}.d\mathbf{\Omega}/dt=0$.) Clearly (3) is impotent to determine the $\mathbf{z}$-component of $\mathbf{\Omega}$, and so $\mathbf{\Omega}.\mathbf{z}$ must be an arbitrary constant, the spin of the sphere about the normal. Denote this by n. Then

$$\mathbf{\Omega}=-\frac{\mathbf{V}\wedge\mathbf{z}}{a}+n\mathbf{z}.$$

This is the desired relation. The reaction $\mathbf{R}$, by (1), is just $Mg\mathbf{z}$.

375. *Motion of a sphere on a rough inclined plane.* With the same specification of the sphere as in § 374, let $\mathbf{i}$ be a unit vector normal to the inclined plane, in the upward direction, and $\mathbf{z}$ a unit vertically upward vector (Fig. 90). Let α be the inclination of the plane to the horizontal.

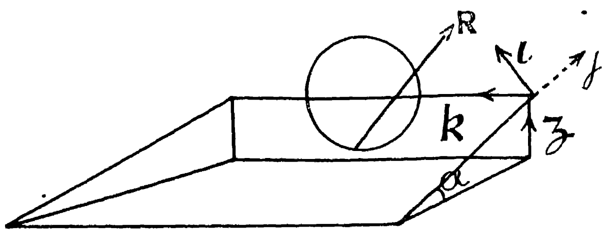

Fig. 90

Take an origin O in a plane through the centre of the sphere parallel to the inclined plane, and let $\mathbf{r}$ be the position vector of the centre of the sphere with respect to O.

Let $\mathbf{\Omega}$ be the angular velocity of the sphere at any instant, $\mathbf{R}$ the reaction at the point of contact. The equations of motion are

$$M\frac{d^2\mathbf{r}}{dt^2}=\mathbf{R}-Mg\mathbf{z}, \tag{1}$$

$$C\frac{d\mathbf{\Omega}}{dt}=-a\mathbf{i}\wedge\mathbf{R}, \tag{2}$$

and the relation expressing rolling contact is

$$\frac{d\mathbf{r}}{dt}+\mathbf{\Omega}\wedge(-a\mathbf{i})=0. \tag{3}$$

Eliminating $\mathbf{R}$ between (1) and (2), we have

$$C\frac{d\mathbf{\Omega}}{dt}=-Ma\mathbf{i}\wedge\left[\frac{d^2\mathbf{r}}{dt^2}+g\mathbf{z}\right]. \tag{4}$$

This integrates as it stands in the form

$$C\mathbf{\Omega}=-Ma\mathbf{i}\wedge\left[\frac{d\mathbf{r}}{dt}+g\mathbf{z}t\right]-Ma\mathbf{A},$$

where $\mathbf{A}$ is a vector constant. Eliminating $\mathbf{\Omega}$ we have now, from (3),

$$\frac{d\mathbf{r}}{dt}+\frac{Ma^2}{C}\left[\mathbf{i}\wedge\left(\frac{d\mathbf{r}}{dt}+g\mathbf{z}t\right)+\mathbf{A}\right]\wedge\mathbf{i}=0. \tag{5}$$

Now expand the continued vector products. Since $\mathbf{i}.\mathbf{r}=0$, we have

$$\left(\mathbf{i}\wedge\frac{d\mathbf{r}}{dt}\right)\wedge\mathbf{i}=\frac{d\mathbf{r}}{dt};$$

and if we take a unit vector $\mathbf{j}$ up the plane along a line of greatest slope, and $\mathbf{k}=\mathbf{i}\wedge\mathbf{j}$, then since

$$\mathbf{z}=\mathbf{i}\cos\alpha+\mathbf{j}\sin\alpha,$$

we have
$$\mathbf{i}\wedge\mathbf{z}=\mathbf{k}\sin\alpha$$

and
$$(\mathbf{i}\wedge\mathbf{z})\wedge\mathbf{i}=\mathbf{j}\sin\alpha.$$

Hence, from (5),

$$\frac{d\mathbf{r}}{dt}\left[1+\frac{Ma^2}{C}\right]+\frac{Ma^2}{C}gt\mathbf{j}\sin\alpha+\text{const.}=0,$$

or, say
$$\frac{d\mathbf{r}}{dt}=-\frac{1}{1+C/Ma^2}gt\mathbf{j}\sin\alpha+\mathbf{V}_0,$$

where $\mathbf{V}_0$ is evidently the initial velocity of the centre of the sphere, and is parallel to the plane. The angular velocity $\mathbf{\Omega}$ of the sphere follows on vectorial multiplication of (3) by $\mathbf{i}$; we get

$$\mathbf{\Omega}-(\mathbf{\Omega}.\mathbf{i})\mathbf{i}=\frac{1}{a}\left(\mathbf{i}\wedge\frac{d\mathbf{r}}{dt}\right),$$

or
$$\mathbf{\Omega}=n\mathbf{i}-\frac{g\sin\alpha}{a}\left(1+\frac{C}{Ma^2}\right)^{-1}t\mathbf{k}+\frac{\mathbf{i}\wedge\mathbf{V}_0}{a},$$

where n is the spin of the sphere about the normal to the plane. From (4),

$$\frac{d\mathbf{\Omega}}{dt}.\mathbf{i}=0$$

and so
$$\mathbf{\Omega}.\mathbf{i}=\text{const.},$$

or
$$n=\text{const.}$$

The equation for $d\mathbf{r}/dt$ integrates at once in the form

$$\mathbf{r}=-\left(1+\frac{C}{Ma^2}\right)^{-1}\tfrac{1}{2}gt^2\mathbf{j}\sin\alpha+\mathbf{V}_0t,$$

and the path of the centre is clearly a parabola.

376. *Motion of a sphere on a rough horizontal plane which is compelled to rotate with constant angular velocity* ω *about an axis normal to itself.* Let $\mathbf{z}$ be a unit vector normal to the plane, vertically upwards, and let $\mathbf{r}$

be the vector distance of the centre of the sphere from the axis of rotation of the plane, so that $\mathbf{r}.\mathbf{z}=0$. The angular velocity of the plane is $\omega\mathbf{z}$. The angular velocity of the sphere is, say, $\mathbf{\Omega}$, and the equations of motion of the sphere are, as before (§ 374),

$$M\frac{d^2\mathbf{r}}{dt^2}=\mathbf{R}-Mg\mathbf{z}, \qquad C\frac{d\mathbf{\Omega}}{dt}=-a\mathbf{z}\wedge\mathbf{R}, \qquad (1), (2)$$

but the condition of rolling contact is now

$$\omega\mathbf{z}\wedge(\mathbf{r}-a\mathbf{z})=\frac{d\mathbf{r}}{dt}+\mathbf{\Omega}\wedge(-a\mathbf{z}),$$

which simplifies to

$$\omega\mathbf{z}\wedge\mathbf{r}=\frac{d\mathbf{r}}{dt}-a\mathbf{\Omega}\wedge\mathbf{z}. \qquad (3)$$

Eliminating $\mathbf{R}$ between (1) and (2),

$$-a\mathbf{z}\wedge M\left(\frac{d^2\mathbf{r}}{dt^2}+g\mathbf{z}\right)=C\frac{d\mathbf{\Omega}}{dt}$$

or

$$\frac{d\mathbf{\Omega}}{dt}=\frac{Ma}{C}\frac{d^2\mathbf{r}}{dt^2}\wedge\mathbf{z}. \qquad (4)$$

Differentiating (3) with respect to t and substituting for $\frac{d\mathbf{\Omega}}{dt}$ from (4), we get

$$\omega\mathbf{z}\wedge\frac{d\mathbf{r}}{dt}=\left[1+\frac{Ma^2}{C}\right]\frac{d^2\mathbf{r}}{dt^2}.$$

This integrates in the form

$$\frac{d\mathbf{r}}{dt}=\frac{\omega}{1+Ma^2/C}\mathbf{z}\wedge(\mathbf{r}-\mathbf{r}_0),$$

where $\mathbf{r}_0$ is an arbitrary vector constant. This is the complete kinematic solution of the problem, for it states that the path of the centre of the sphere is a circle of arbitrary centre and radius, described with angular speed $\omega(1+Ma^2/C)^{-1}$.

From (4),

$$\frac{d\mathbf{\Omega}}{dt}.\mathbf{z}=0, \qquad \text{or} \qquad \mathbf{\Omega}.\mathbf{z}=\text{const.}=n,$$

say. Solving (3) for $\mathbf{\Omega}$ by vectorial multiplication by $\mathbf{z}$, we have

$$\mathbf{\Omega}=n\mathbf{z}+\frac{\omega}{a}\mathbf{r}-\frac{1}{a}\frac{d\mathbf{r}}{dt}\wedge\mathbf{z}=n\mathbf{z}+\frac{\omega}{a}\mathbf{r}-\frac{\omega}{a}\frac{C}{C+Ma^2}(\mathbf{r}-\mathbf{r}_0).$$

This determines $\mathbf{\Omega}$ in any position $\mathbf{r}$.

The centre $\mathbf{r}_0$ of the circle described by the centre of the sphere can be found if the initial velocity $\mathbf{V}_1$ and position $\mathbf{r}_1$ are given ; for then

$$\mathbf{V}_1=\frac{\omega}{1+\mathrm{Ma}^2/\mathrm{C}}\mathbf{z}\wedge(\mathbf{r}_1-\mathbf{r}_0),$$

or

$$\mathbf{r}_0=\mathbf{r}_1-\frac{1+\mathrm{Ma}^2/\mathrm{C}}{\omega}(\mathbf{V}_1\wedge\mathbf{z}).$$

377. *Motion of a sphere on a rough inclined plane compelled to rotate with constant angular velocity ω about an axis normal to itself.* In the notation of §§ 375, 376, the equations of motion are

$$\mathrm{M}\frac{d^2\mathbf{r}}{dt^2}=\mathbf{R}-\mathrm{Mg}\mathbf{z}, \qquad \mathrm{C}\frac{d\mathbf{\Omega}}{dt}=-\mathrm{a}\mathbf{i}\wedge\mathbf{R}, \qquad (1), (2)$$

and the equation of rolling contact is

$$\omega\mathbf{i}\wedge(\mathbf{r}-\mathrm{a}\mathbf{i})=\frac{d\mathbf{r}}{dt}+\mathbf{\Omega}\wedge(-\mathrm{a}\mathbf{i})$$

or

$$\omega\mathbf{i}\wedge\mathbf{r}=\frac{d\mathbf{r}}{dt}-\mathrm{a}\mathbf{\Omega}\wedge\mathbf{i}. \qquad (3)$$

Eliminating **R** between (1) and (2),

$$\frac{d\mathbf{\Omega}}{dt}=\frac{\mathrm{aM}}{\mathrm{C}}\left(\frac{d^2\mathbf{r}}{dt^2}+\mathrm{g}\mathbf{z}\right)\wedge\mathbf{i}. \qquad (4)$$

Hence

$$\frac{d\mathbf{\Omega}}{dt}.\mathbf{i}=0$$

or

$$\mathbf{\Omega}.\mathbf{i}=\text{const.}=n,$$

say. Integrating (4),

$$\mathbf{\Omega}+\text{const.}=\frac{\mathrm{Ma}}{\mathrm{C}}\left[\frac{d\mathbf{r}}{dt}+\mathrm{gt}\mathbf{z}\right]\wedge\mathbf{i}.$$

Hence, by (3),

$$\omega\mathbf{i}\wedge\mathbf{r}=\frac{d\mathbf{r}}{dt}\left(1+\frac{\mathrm{Ma}^2}{\mathrm{C}}\right)+\frac{\mathrm{Ma}^2}{\mathrm{C}}\mathrm{gt}\mathbf{j}\sin\alpha+\text{const.}$$

This can be written in the form

$$\frac{d\mathbf{r}}{dt}=\frac{\omega}{1+\mathrm{Ma}^2/\mathrm{C}}\mathbf{i}\wedge\left[\mathbf{r}-\mathbf{r}_0+\frac{\mathrm{Ma}^2/\mathrm{C}}{\omega}\mathrm{gt}\mathbf{k}\sin\alpha\right]. \qquad (5)$$

The motion is therefore one of uniform rotation about a centre which moves uniformly in the direction of **k**, i.e. along a horizontal line of the plane, with the constant speed

$$\frac{\mathrm{Ma}^2/\mathrm{C}}{\omega}\mathrm{g}\sin\alpha.$$

This is clearly the time average of $d\mathbf{r}/dt$, so that on the whole the sphere does not travel *down* the inclined plane but progresses epicycloidally in a horizontal direction.

378. This problem has a remarkable formal similarity to the problem of the motion of a charged particle in a combined magnetic and gravitational field. The latter problem was first discussed in relation to the motion of ions in the sun's magnetic field, but the result is a general one.*

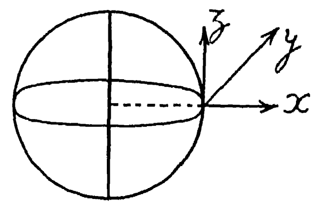

Fig. 91

Consider a particle of charge e in motion with velocity $d\mathbf{r}/dt$ at any moment. Let the intensity of the magnetic field be $H\mathbf{z}$ (Fig. 91), where $\mathbf{z}$ is a unit vector, and let the gravitational field be $-g\mathbf{x}$, where $\mathbf{x}$ is a perpendicular unit vector. (This corresponds to the case of a point on the sun's equator.) Then the equation of motion of the particle is

$$m\frac{d^2\mathbf{r}}{dt^2}=-mg\mathbf{x}+\frac{eH}{c}\left(\frac{d\mathbf{r}}{dt}\wedge\mathbf{z}\right).$$

This integrates in the form

$$\frac{d\mathbf{r}}{dt}=-gt\mathbf{x}+\frac{eH}{mc}(\mathbf{r}\wedge\mathbf{z})+\text{const.},$$

which can be rewritten as

$$\frac{d\mathbf{r}}{dt}=-\frac{eH}{mc}\mathbf{z}\wedge\left[(\mathbf{r}-\mathbf{r}_0)-\frac{mcg}{eH}t\mathbf{y}\right]+\mathbf{V}_0,$$

where $\mathbf{y}=\mathbf{z}\wedge\mathbf{x}$, $\mathbf{V}_0$ is parallel to $\mathbf{z}$ and $\mathbf{r}_0$ is a constant. This equation is fully analogous to (5) of § 377. It states that the velocity at any instant *perpendicular to* $\mathbf{z}$ is one of rotation about a parallel to the $\mathbf{z}$-axis with angular velocity eH/mc, the centre of rotation progressing in the $\mathbf{y}$-direction with linear speed mcg/eH. The motion *parallel to* the $\mathbf{z}$-axis is unchanged by either the magnetic or gravitational pull.

The remarkable aspect of the motion is that though gravity exerts a steady downward pull, in the direction $-\mathbf{x}$, yet there is no systematic tendency of the particle to move in the $\mathbf{x}$-direction. The particle systematically progresses in the $\mathbf{y}$-direction, across the lines of both magnetic and gravitational force. The physical reason is that the tending to fall downwards under gravity results in the acquisition of so much velocity perpendicular to the magnetic field that this in turn results in a deflecting force, perpendicular to the field and to the velocity, large enough to swing the particle systematically east or west. Apart from the northerly component of velocity the path is an epicycloid. Similarly, in the case of the inclined rotating plane, the velocity of descent down the plane ultimately carries the sphere so far from the axis of rotation of the plane that the velocity imparted by the rotation is sufficient to shift the

* S. Chapman, *Monthly Notices, R. A. S.*, 89, 61, 1928.

particle up the plane again and systematically in a horizontal direction. This is seen by writing (5) of § 377 in the alternative form

$$\frac{d\mathbf{r}}{dt}=-\frac{\omega}{1+Ma^2/C}\frac{Ma^2}{C\omega}g\sin\alpha t\mathbf{j}+\frac{\omega}{1+Ma^2/C}\mathbf{i}\wedge(\mathbf{r}-\mathbf{r}_0),$$

wherein the first term exhibits the velocity *down* the plane.

379. *Example* (Routh). Two equal rods, whose mass centres are at their mid-points, are freely jointed together, the other ends being free. The system falls freely under gravity. Obtain the equations of motion.

Let $\mathbf{r}$ be the position vector of the joint, $\mathbf{i}_1$ and $\mathbf{i}_2$ unit vectors along the rods, 2a the length of a rod, C the moment of inertia of a rod about an axis through its mass centre perpendicular to it (Fig. 92). Let $\mathbf{R}$, $-\mathbf{R}$ be the reactions at the hinge on the two rods, respectively. Take a unit vector $\mathbf{z}$ vertically downwards. The equations of linear momentum are

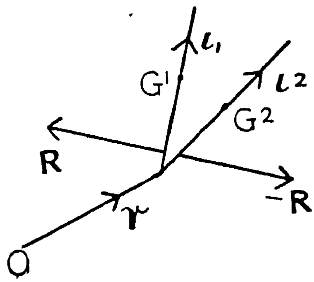

Fig. 92

$$Mg\mathbf{z}+\mathbf{R}=M\frac{d^2}{dt^2}(\mathbf{r}+a\mathbf{i}_1), \tag{1}$$

$$Mg\mathbf{z}-\mathbf{R}=M\frac{d^2}{dt^2}(\mathbf{r}+a\mathbf{i}_2), \tag{2}$$

and the equations of angular momentum about the mass centres are

$$(-a\mathbf{i}_1)\wedge\mathbf{R}=\frac{d}{dt}\left[C\mathbf{i}_1\wedge\frac{d\mathbf{i}_1}{dt}\right], \tag{3}$$

$$(-a\mathbf{i}_2)\wedge(-\mathbf{R})=\frac{d}{dt}\left[C\mathbf{i}_2\wedge\frac{d\mathbf{i}_2}{dt}\right], \tag{4}$$

where we have used a result of § 231, namely that the angular velocity of a rod in direction $\mathbf{i}$ is of the form $\mathbf{i}\wedge d\mathbf{i}/dt$.

Equations (1), ... (4) are equivalent to four vector equations for the four unknown vectors $\mathbf{R}$, $\mathbf{r}$, $\mathbf{i}_1$, $\mathbf{i}_2$. Subtracting (1) and (2) we have

$$2\mathbf{R}=Ma\frac{d^2}{dt^2}(\mathbf{i}_1-\mathbf{i}_2),$$

whence, substituting for $\mathbf{R}$ in (3) and (4),

$$\left(1+\frac{C}{\frac{1}{2}Ma^2}\right)\mathbf{i}_1\wedge\frac{d^2\mathbf{i}_1}{dt^2}-\mathbf{i}_1\wedge\frac{d^2\mathbf{i}_2}{dt^2}=0,$$

$$\left(1+\frac{C}{\frac{1}{2}Ma^2}\right)\mathbf{i}_2\wedge\frac{d^2\mathbf{i}_2}{dt^2}-\mathbf{i}_2\wedge\frac{d^2\mathbf{i}_1}{dt^2}=0.$$

If we take $C = \frac{1}{3}Ma^2$, then the first of these equations shows that $\mathbf{i}_1$ is parallel to

$$\frac{5}{3}\frac{d^2\mathbf{i}_1}{dt^2} - \frac{d^2\mathbf{i}_2}{dt^2},$$

which is Routh's result. In principle the last two equations in $\mathbf{i}_1$ and $\mathbf{i}_2$ can be solved (they are easily seen to be equivalent to four scalar equations), whence (1) and (2) determine $\mathbf{r}$ and $\mathbf{R}$.

CHAPTER XVI

THE MOTION OF A RIGID BODY ABOUT ITS CENTRE OF MASS

380. *Introduction.* We are now in a position to consider the dynamics of a single rigid body given (*a*) its inertia tensor about its centre of mass G, (*b*) the system of external forces acting on it. The system of external forces may be reduced to a force **R** at G and a couple $\mathbf{\Gamma}$(G). If M is the mass of the body, the motion of the centre of mass has been shown to be that of a particle of mass M under the force **R**. And the motion relative to the centre of mass has been shown to depend only on the moment of the applied forces about the centre of mass. We have in fact established the equations

$$\frac{d}{dt}\mathbf{H}(G)=\mathbf{\Gamma}(G),$$

where
$$\mathbf{H}(G)=\mathbf{I}(G).\mathbf{\Omega},$$

H(G) being the angular momentum about G and **I**(G) the inertia tensor about G. In particular, these equations describe the motion of a body whose centre of mass is fixed.

381. *Euler's equations of motion.* As the rigid body moves, the inertia tensor **I**(G) changes when reckoned in any fixed frame of reference. But it remains constant in any frame moving with the rigid body. For brevity of notation we shall for the time being omit the symbol G from **I**(G), $\mathbf{\Gamma}$(G), **H**(G), and write them shortly as **I**, $\mathbf{\Gamma}$, **H**, though these vectors must always be thought of as relating to G.

Since, now, the apparent rate of change $\partial\mathbf{I}/\partial t$ of the inertia tensor **I** relative to a frame moving with the body is zero, we have by the result of § 211,

$$\frac{d\mathbf{I}}{dt}=\mathbf{\Omega}\wedge\mathbf{I}-\mathbf{I}\wedge\mathbf{\Omega},$$

using the notation for cross-products of tensors and vectors. Hence the equation of motion of § 380 may be written

$$\mathbf{\Gamma}=\frac{d}{dt}(\mathbf{I}.\mathbf{\Omega})=(\mathbf{\Omega}\wedge\mathbf{I}-\mathbf{I}\wedge\mathbf{\Omega}).\mathbf{\Omega}+\mathbf{I}.\frac{d\mathbf{\Omega}}{dt}.$$

By theorems of § 68,

$$(\mathbf{\Omega}\wedge\mathbf{I}).\mathbf{\Omega}=\mathbf{\Omega}\wedge(\mathbf{I}.\mathbf{\Omega})$$

and $$(\mathbf{I}\wedge\mathbf{\Omega}).\mathbf{\Omega}=\mathbf{I}.(\mathbf{\Omega}\wedge\mathbf{\Omega})=0.$$

Hence $$\mathbf{\Gamma}=\mathbf{I}.\frac{d\mathbf{\Omega}}{dt}+\mathbf{\Omega}\wedge(\mathbf{I}.\mathbf{\Omega}). \tag{1}$$

This is the desired form of the equation of motion.

This equation may be derived more simply without using the theory of cross-products for tensors, as follows. In a frame moving with the rigid body, the apparent rate of change of $\mathbf{H}$, $\partial\mathbf{H}/\partial t$, is connected with the actual rate of change $d\mathbf{H}/dt$ by the relation (§ 207)

$$\frac{d\mathbf{H}}{dt}=\frac{\partial\mathbf{H}}{\partial t}+\mathbf{\Omega}\wedge\mathbf{H}.$$

In this moving frame, $\mathbf{I}$ is a constant tensor, and so

$$\frac{\partial\mathbf{H}}{\partial t}=\frac{\partial}{\partial t}(\mathbf{I}.\mathbf{\Omega})=\mathbf{I}.\frac{\partial\mathbf{\Omega}}{\partial t}=\mathbf{I}.\frac{d\mathbf{\Omega}}{dt},$$

the last equality following from § 209. Hence

$$\mathbf{\Gamma}=\frac{d\mathbf{H}}{dt}=\mathbf{I}.\frac{d\mathbf{\Omega}}{dt}+\mathbf{\Omega}\wedge(\mathbf{I}.\mathbf{\Omega}),$$

as before.

382. If $\mathbf{I}$ is specified with respect to its principal axes, $\mathbf{i}, \mathbf{j}, \mathbf{k}$, say

$$\mathbf{I}=A\mathbf{ii}+B\mathbf{jj}+C\mathbf{kk},$$

then the $\mathbf{i}$-component of $\mathbf{I}.d\mathbf{\Omega}/dt$ is $A\,d\omega_1/dt$, where $(\omega_1, \omega_2, \omega_3)$ are the components of $\mathbf{\Omega}$ along the principal axes $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Also

$$\mathbf{\Omega}\wedge(\mathbf{I}.\mathbf{\Omega})=(\omega_1\mathbf{i}+\omega_2\mathbf{j}+\omega_3\mathbf{k})\wedge(A\omega_1\mathbf{i}+B\omega_2\mathbf{j}+C\omega_3\mathbf{k}),$$

and the right-hand side of this has for its $\mathbf{i}$-component

$$-\omega_2\omega_3(B-C).$$

Hence the equation of motion (1) gives the three scalar equations

$$\left.\begin{aligned}\Gamma_1&=A\frac{d\omega_1}{dt}-(B-C)\,\omega_2\omega_3,\\ \Gamma_2&=B\frac{d\omega_2}{dt}-(C-A)\,\omega_3\omega_1,\\ \Gamma_3&=C\frac{d\omega_3}{dt}-(A-B)\,\omega_1\omega_2.\end{aligned}\right\} \tag{2}$$

More generally, if $\mathbf{I}$ has for its components in some triad

$$\begin{matrix} A, & -H, & -G \\ -H, & B, & -F \\ -G, & -F, & C, \end{matrix}$$

then the equation of motion (1) yields

$$\begin{aligned}\Gamma_1 = {} & A\dot{\omega}_1 - H\dot{\omega}_2 - G\dot{\omega}_3 + \omega_2(-G\omega_1 - F\omega_2 + C\omega_3) \\ & - \omega_3(-H\omega_1 + B\omega_2 - F\omega_3) \\ = {} & A\dot{\omega}_1 - H\dot{\omega}_2 - G\dot{\omega}_3 - (B-C)\omega_2\omega_3 - F(\omega_2{}^2 - \omega_3{}^2) \\ & - G\omega_1\omega_2 + H\omega_1\omega_3, \end{aligned} \tag{3}$$

and two similar equations.

Equations (2) are known as Euler's equations of motion. Equations (3) are the general form of Euler's equations in Cartesian components, and equation (1) is the general vector form. It is seldom necessary to use Euler's equations in their scalar form (2) in an investigation of genuine dynamical interest. The student should always attempt to solve problems directly from the vector form (1), which is usually handled without difficulty. It is best to obtain the inertia tensor **I**(G) first, and then evaluate **H**(G). Differentiation of this with respect to the time and the equating of the result to the external couple at once produce an equation of motion whose form suggests the next step in the treatment of the problem.

383. The physical meaning of the term $\boldsymbol{\Omega}\wedge(\mathbf{I}.\boldsymbol{\Omega})$ or $\boldsymbol{\Omega}\wedge\mathbf{H}$ is of some interest. Let P be any particle $\mathbf{r}$ of the rigid body, N the foot of the perpendicular from P on to the instantaneous axis $\boldsymbol{\Omega}$ (Fig. 93). Then, if $\omega = |\boldsymbol{\Omega}|$,

$$PN = -\mathbf{r} + \frac{(\mathbf{r}.\boldsymbol{\Omega})}{|\boldsymbol{\Omega}|}\frac{\boldsymbol{\Omega}}{|\boldsymbol{\Omega}|},$$

and so $\qquad \mathbf{r}\wedge\omega^2 PN = (\mathbf{r}.\boldsymbol{\Omega})(\mathbf{r}\wedge\boldsymbol{\Omega}).$

But $\qquad \mathbf{H} = \Sigma\mathbf{r}\wedge m(\boldsymbol{\Omega}\wedge\mathbf{r})$

$$= \Sigma m[-\mathbf{r}(\boldsymbol{\Omega}.\mathbf{r}) + \boldsymbol{\Omega}\mathbf{r}^2],$$

so that $\qquad \boldsymbol{\Omega}\wedge\mathbf{H} = \Sigma m(\boldsymbol{\Omega}.\mathbf{r})(\mathbf{r}\wedge\boldsymbol{\Omega}).$

Thus $\qquad \boldsymbol{\Omega}\wedge\mathbf{H} = \Sigma\mathbf{r}\wedge m\omega^2 PN.$

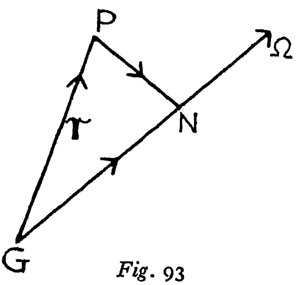

Fig. 93

In words, the term $\boldsymbol{\Omega}\wedge\mathbf{H}$ is equal to the moment about G of the *effective forces* of the particles of the rigid body as arising from their motion of rotation about the instantaneous axis.

MOTION OF A RIGID BODY UNDER NO FORCES

384. *Integrals of the motion.* When the external couple $\boldsymbol{\Gamma}$ vanishes, the equation of motion is just

$$\frac{d}{dt}(\mathbf{I}.\boldsymbol{\Omega}) = 0$$

or

$$\mathbf{I}.\frac{\partial\boldsymbol{\Omega}}{\partial t} + \boldsymbol{\Omega}\wedge(\mathbf{I}.\boldsymbol{\Omega}) = 0.$$

Multiplying the latter form scalarly by $\mathbf{\Omega}$ we have,

$$\mathbf{I}:\mathbf{\Omega}\frac{\partial\mathbf{\Omega}}{\partial t}=0.$$

Since $\mathbf{I}$ is self-conjugate, this integrates in the form

$$\mathbf{I}:\mathbf{\Omega}\mathbf{\Omega}=\text{const.}=2\mathrm{T}, \tag{1}$$

say. We have also from the first form of the equation of motion, on integration,

$$\mathbf{H}=\mathbf{I}.\mathbf{\Omega}=\text{const.} \tag{2}$$

Thus the kinetic energy T and vector angular momentum $\mathbf{H}$ are constants of the motion.

385. *Poinsot's geometrical construction for the motion.* The momental ellipsoid is given by

$$\mathbf{I}:\mathbf{rr}=\text{const.}=\mathrm{K}, \tag{3}$$

and the normal to this ellipsoid at the point where it is met by the vector $\mathbf{r}$, from G is known to be parallel to $\mathbf{I.r}$. Hence the normal to the momental ellipsoid at the point where it is met by the instantaneous axis $\mathbf{\Omega}$ is parallel to $\mathbf{I}.\mathbf{\Omega}$, i.e. to $\mathbf{H}$. It follows, since $\mathbf{H}$ is a constant vector, that this normal is in a fixed direction. Let P be the point in which the instantaneous axis meets the momental ellipsoid, N the foot of the perpendicular from G to the tangent plane at P; and set GP $=\mathbf{r}$ (Fig. 94). Then

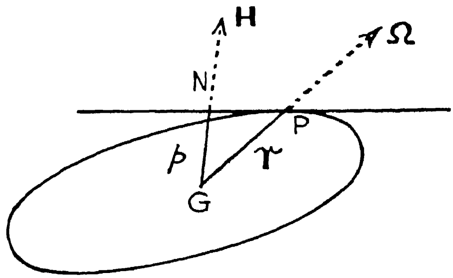

Fig. 94

$$\frac{\mathbf{\Omega}^2}{\mathbf{r}^2}=\frac{\mathbf{I}:\mathbf{\Omega\Omega}}{\mathbf{I}:\mathbf{rr}}=\frac{2\mathrm{T}}{\mathrm{K}}, \tag{4}$$

and so $|\mathbf{r}|\propto|\mathbf{\Omega}|$ during the motion. Again, if p is the length of the perpendicular GN,

$$\begin{aligned}
\mathrm{p}&=\text{projection of } \mathbf{r} \text{ on normal at } \mathbf{r}\\
&=\mathbf{r}.\frac{\mathbf{I.r}}{|\mathbf{I.r}|}=\frac{\mathbf{I}:\mathbf{rr}}{|\mathbf{I.r}|}=\frac{\mathrm{K}}{|\mathbf{I}.\mathbf{\Omega}|}\,\frac{|\mathbf{\Omega}|}{|\mathbf{r}|}\\
&=\frac{\mathrm{K}}{|\mathbf{H}|}\left(\frac{2\mathrm{T}}{\mathrm{K}}\right)^{\frac{1}{2}}=\frac{(2\mathrm{KT})^{\frac{1}{2}}}{|\mathbf{H}|}.
\end{aligned} \tag{5}$$

and so p is constant, in magnitude as well as in direction. It follows that the tangent plane at P to the momental ellipsoid is a fixed plane. Further, the points of the body in GP, being on the instantaneous axis are at rest. Hence if the momental ellipsoid is considered as a rigid body,

rigidly attached in its proper position to the given rigid body, the motion of this ellipsoid must be one of rolling contact with a fixed plane. For the point of contact P is a particle of the rigid body instantaneously at rest. Further, the rate of turning of the ellipsoid about GP, namely $|\mathbf{\Omega}|$, is proportional to GP, being $(2T/K)^{\frac{1}{2}}$ times GP, and so the instantaneous motion of the ellipsoid is determinate.

It follows that the motion of the rigid body with constants of integration $2T$ and $\mathbf{H}$ can be reconstructed by rolling the momental ellipsoid $\mathbf{I}:\mathbf{rr}=K$ in contact with a fixed plane (whose normal is parallel to $\mathbf{H}$ and whose distance from G is $(2KT)^{\frac{1}{2}}/|\mathbf{H}|$) at a rate equal to $(2T/K)^{\frac{1}{2}}$ times the radius vector to the point of contact. This is Poinsot's construction for the motion.

386. *Locus of the instantaneous axis in the body.* The locus of the instantaneous axis in a frame fixed in the rigid body is readily found, since the point $\mathbf{r}$ satisfies the relation

$$\frac{(\mathbf{I}.\mathbf{r})^2}{\mathbf{I}:\mathbf{rr}}=\frac{(\mathbf{I}.\mathbf{\Omega})^2}{\mathbf{I}:\mathbf{\Omega\Omega}}=\frac{\mathbf{H}^2}{2T}, \tag{6}$$

and since $\mathbf{I}$ has fixed components relative to the body, the equality of the first and last members yields the equation of a cone which is the body-locus of the instantaneous axis. Referred to its principal axes, which are the principal axes of the inertia tensor at G, its Cartesian equation is

$$(Ax)^2+(By)^2+(Cz)^2=\frac{H^2}{2T}(Ax^2+By^2+Cz^2),$$

or

$$\Sigma Ax^2\left(A-\frac{H^2}{2T}\right)=0.$$

This equation can also be written in the form

$$\left[\mathbf{I}.\mathbf{I}-\frac{\mathbf{H}^2}{2T}\mathbf{I}\right]:\mathbf{rr}=0. \tag{7}$$

The cone is the *polhode* cone of the motion.

387. The body-locus of the invariable direction GN (in the direction of H) is also readily found. We now put

$$\mathbf{r}'=\mathbf{I}.\mathbf{\Omega}=\mathbf{H},$$

and then from the previous relation

$$\frac{(\mathbf{I}.\mathbf{\Omega})^2}{\mathbf{I}:\mathbf{\Omega\Omega}}=\frac{\mathbf{H}^2}{2T}$$

and from

$$\mathbf{I}:\mathbf{\Omega\Omega}=(\mathbf{I}.\mathbf{\Omega}).\mathbf{\Omega}=\mathbf{r}'.(\mathbf{I}^{-1}.\mathbf{r}'),$$

we find

$$\mathbf{r}'^2=\frac{\mathbf{H}^2}{2T}(\mathbf{I}^{-1}.\mathbf{r}').\mathbf{r}'$$

or

$$\left(\mathbf{U}-\frac{\mathbf{H}^2}{2T}\mathbf{I}^{-1}\right):\mathbf{r}'\mathbf{r}'=0. \tag{8}$$

This is the *herpolhode* cone of the motion. It has for its Cartesian equation, referred to the principal dynamical axes,

$$\Sigma\left(1-\frac{\mathbf{H}^2}{2AT}\right)x^2=0. \tag{8'}$$

388. *Steady rotation.* A motion of steady rotation is possible when $\mathbf{\Omega}=$const. is a solution of the equation of motion (1) with $\mathbf{\Gamma}=0$. This requires $d\mathbf{\Omega}/dt=0$, i.e. $\partial\mathbf{\Omega}/\partial t=0$, i.e.

$$\mathbf{\Omega}\wedge(\mathbf{I}.\mathbf{\Omega})=0,$$

that is, $\mathbf{I}.\mathbf{\Omega}$ must be parallel to $\mathbf{\Omega}$. But this is the condition that $\mathbf{\Omega}$ shall lie along a principal axis of $\mathbf{I}$. It follows that the only possible motions of steady rotation arise when the axis of rotation coincides with a principal dynamical axis. The speed of rotation is then arbitrary.

389. *Stability of motion of steady rotation.* Consider a rigid body in motion with constant angular speed ω_1 about the principal dynamical axis $\mathbf{i}$, corresponding to the moment of inertia A. Let the body be slightly disturbed. After the disturbance, let the angular velocity be $\omega_1\mathbf{i}+\boldsymbol{\epsilon}$. Then in the disturbed motion,

$$2T=\mathbf{I}:(\omega_1\mathbf{i}+\boldsymbol{\epsilon})(\omega_1\mathbf{i}+\boldsymbol{\epsilon})$$
$$=\omega_1{}^2\mathbf{I}:\mathbf{ii}+2\omega_1\mathbf{I}:\boldsymbol{\epsilon}\mathbf{i}+\mathbf{I}:\boldsymbol{\epsilon\epsilon}.$$

or, since $\mathbf{i}$ is a principal axis,

$$2T=A\omega_1{}^2+2A\omega_1(\mathbf{i}.\boldsymbol{\epsilon})+\mathbf{I}:\boldsymbol{\epsilon\epsilon}.$$

Further
$$\mathbf{H}^2=|\mathbf{I}.(\omega_1\mathbf{i}+\boldsymbol{\epsilon})|^2$$
$$=\omega_1{}^2(\mathbf{I}.\mathbf{i})^2+2\omega_1(\mathbf{I}.\mathbf{i}).(\mathbf{I}.\boldsymbol{\epsilon})+(\mathbf{I}.\boldsymbol{\epsilon})^2$$

or, since $\mathbf{i}$ is a principal axis, and so $\mathbf{I}.\mathbf{i}=A\mathbf{i}$,

$$\mathbf{H}^2=A^2\omega_1{}^2+2\omega_1A\mathbf{i}.(\mathbf{I}.\boldsymbol{\epsilon})+(\mathbf{I}.\boldsymbol{\epsilon})^2.$$

But
$$\mathbf{i}.(\mathbf{I}.\boldsymbol{\epsilon})=(\mathbf{i}.\mathbf{I}).\boldsymbol{\epsilon}=A\mathbf{i}.\boldsymbol{\epsilon}.$$

Hence
$$\mathbf{H}^2=A^2\omega_1{}^2+2A^2\omega_1(\mathbf{i}.\boldsymbol{\epsilon})+(\mathbf{I}.\boldsymbol{\epsilon})^2.$$

We now determine the herpolhode cone, the locus in the body of a line fixed in space, by (8′). We have, for $\boldsymbol{\epsilon}$ small,

$$1-\frac{\mathbf{H}^2}{2AT}\sim\frac{A\mathbf{I}:\boldsymbol{\epsilon\epsilon}-(\mathbf{I}.\boldsymbol{\epsilon})^2}{2AT},$$

$$1-\frac{\mathbf{H}^2}{2BT}\sim 1-\frac{A}{B},$$

$$1-\frac{\mathbf{H}^2}{2CT}\sim 1-\frac{A}{C}.$$

Further, if we put

$$\boldsymbol{\epsilon}=\varepsilon_1\mathbf{i}+\varepsilon_2\mathbf{j}+\varepsilon_3\mathbf{k},$$

with
$$\mathbf{I}=A\mathbf{ii}+B\mathbf{jj}+C\mathbf{kk},$$

then
$$\frac{A\mathbf{I}:\boldsymbol{\epsilon\epsilon}-(\mathbf{I}.\boldsymbol{\epsilon})^2}{2AT}=\frac{B(A-B)\varepsilon_2{}^2+C(A-C)\varepsilon_3{}^2}{2AT}.$$

Hence so long as ϵ is small, the body-locus of the invariable line is approximately the cone

$$\frac{B(A-B)\varepsilon_2{}^2+C(A-C)\varepsilon_3{}^2}{2AT}x^2+\left(1-\frac{A}{B}\right)y^2+\left(1-\frac{A}{C}\right)z^2=0.$$

Though ε_2 and ε_3 are not separately constant, the coefficient of x^2 is constant. It follows that if $A>B$ and $A>C$, the expression

$$B(A-B)\varepsilon_2{}^2+C(A-C)\varepsilon_3{}^2$$

is not only constant, but positive, and accordingly, if ε_2 and ε_3 are initially small they remain small. The cone is thus an elliptic cone of small solid angle, and the motion is therefore *stable*. Similarly it is stable if $A<B$ and $A<C$. But if A lies between B and C, ε_2 and ε_3 are not necessarily always small, the cone is a hyperbolic cone and the motion is unstable in the sense that Ω does not remain in the vicinity of its original value.

To consider the motion more fully in this case, it is convenient to suppose $A>B>C$ and to examine the behaviour of ω_2. We have

$$B\frac{d\omega_2}{dt}=(C-A)\omega_1\omega_3,$$

where
$$2AT-H^2=B(A-B)\omega_2{}^2+C(A-C)\omega_3{}^2$$
$$2CT-H^2=A(C-A)\omega_1{}^2+B(C-B)\omega_2{}^2.$$

Hence

$$(AC)^{\frac{1}{2}}B\frac{d\omega_2}{dt}=-[(2AT-H^2)-B(A-B)\omega_2{}^2]^{\frac{1}{2}}[(H^2-2CT)-B(B-C)\omega_2{}^2]^{\frac{1}{2}}.$$

During this motion, $\omega_2{}^2$ cannot exceed the smaller, say $(\omega_2)^2{}_0$, of the two values

$$\frac{2AT-H^2}{B(A-B)},\qquad \frac{H^2-2CT}{B(B-C)},$$

and accordingly ω_2 executes a periodic variation between the limits $\pm(\omega_2)_0$. If, when $\omega_2=+(\omega_2)_0$, ω_1 and ω_3 are very small, then they are also small when $\omega_2=-(\omega_2)_0$, and the instantaneous axis practically turns end for end during the half-period. If $\omega_1=\omega_3=0$ when $\omega_2=(\omega_2)_0$, so that $H^2=2BT$, then the instability may be represented as the complete passage from the position in which Ω lies along the B-axis in one sense to the position in which Ω lies along the B-axis in the opposite sense. The approach is, however, asymptotically slow with respect to either limiting position. The motion when ω_1 and ω_3 are not exactly 0 for $\omega_2=(\omega_2)_0$ is along a cone fitting in between the pairs of planes

$$\left(1-\frac{B}{A}\right)^{\frac{1}{2}}x=\pm\left(-1+\frac{B}{C}\right)^{\frac{1}{2}}z.$$

It is possible to investigate the time relations of the motion in great detail, but the investigations are tedious and disclose no further points of dynamical interest.

390. *Examples.*

Example (1). A body is in motion about a fixed point under forces whose moment about the instantaneous axis is always zero. Show that the angular velocity at any instant is proportional to the radius vector of the momental ellipsoid drawn in the direction of the instantaneous axis.

By hypothesis, the moment of the external couple $\mathbf{\Gamma}$ about the instantaneous axis being zero, we have

$$\mathbf{\Gamma}.\mathbf{\Omega}=0.$$

But by the equation of motion,

$$\mathbf{\Gamma}=\mathbf{I}.\frac{\partial\mathbf{\Omega}}{\partial t}+\mathbf{\Omega}\wedge(\mathbf{I}.\mathbf{\Omega}).$$

Hence $$\mathbf{I}.\frac{\partial\mathbf{\Omega}}{\partial t}.\mathbf{\Omega}=0,$$

or, integrating, $$\mathbf{I}:\mathbf{\Omega\Omega}=\text{const.}$$

But the momental ellipsoid is given by

$$\mathbf{I}:\mathbf{rr}=\text{const.}$$

Hence, when $\mathbf{r}$ is parallel to $\mathbf{\Omega}$,

$$|\mathbf{r}|\propto|\mathbf{\Omega}|.$$

Example (2). Show that the component of the angular velocity of a body moving under no forces, about the direction of constant angular momentum, is constant and equal to $2T/|\mathbf{H}|$.

Example (3). A rigid body contains a rotating flywheel, whose axis is fixed relative to the rigid body. Obtain the modified form of Euler's equations (Lamb, *H.M.*).

Let $\mathbf{\Omega}$ be the angular velocity of the rigid body, $\mathbf{i}$ a unit vector in the axis of the flywheel. Let $\mathbf{I}$ be the inertia tensor of the rigid body and the flywheel together, C' the moment of inertia of the *flywheel* about its axis $\mathbf{i}$, A' the moment of inertia of the *flywheel* about a transverse axis through its centre of mass. Let n be the spin of the flywheel about its own axis.

The angular momentum of the flywheel about its centre of mass is (§ 330)

$$C'n\mathbf{i}+A'\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}.$$

If $\mathbf{\Gamma}'$ is the couple acting on the flywheel, then

$$\mathbf{\Gamma}'=C'\frac{d}{dt}(n\mathbf{i})+A'\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}.$$

But since the only forces acting on the flywheel pass through its axis,

$$\mathbf{\Gamma}'.\mathbf{i}=0,$$

and accordingly $$C'\left[\mathbf{i}\frac{dn}{dt}+n\frac{d\mathbf{i}}{dt}\right].\mathbf{i}=0,$$

or, since $\mathbf{i}.d\mathbf{i}/dt=0$,
$$\frac{dn}{dt}=0$$
i.e.
$$n=\text{const.}$$

Assuming the centres of mass of the flywheel and the body to coincide, the total angular momentum of the complete system about its centre of mass is seen to be
$$\mathbf{I}.\boldsymbol{\Omega}+C'(n-\boldsymbol{\Omega}.\mathbf{i})\mathbf{i}.$$
For we reproduce the angular momentum of the complete system by first calculating it as if the flywheel were frozen in, and then adding the angular momentum due to the excess axial spin of the flywheel. This may be written
$$(\mathbf{I}-C'\mathbf{ii}).\boldsymbol{\Omega}+C'n\mathbf{i}.$$
Now let $\boldsymbol{\Gamma}$ be the couple applied externally to the complete system. Then
$$\boldsymbol{\Gamma}-\frac{d}{dt}[(\mathbf{I}-C'\mathbf{ii}).\boldsymbol{\Omega}+C'n\mathbf{i}].$$
But
$$\frac{d}{dt}=\frac{\partial}{\partial t}+\boldsymbol{\Omega}\wedge\quad,$$
and the tensor $\mathbf{I}$ and the vector $\mathbf{i}$, being fixed relatively to the body, are constant under the operator $\partial/\partial t$. Thus
$$\frac{\partial\mathbf{I}}{\partial t}=0,\qquad \frac{\partial\mathbf{i}}{\partial t}=0,$$
whence
$$\boldsymbol{\Gamma}=(\mathbf{I}-C'\mathbf{ii}).\frac{\partial\boldsymbol{\Omega}}{\partial t}+\boldsymbol{\Omega}\wedge[(\mathbf{I}-C'\mathbf{ii}).\boldsymbol{\Omega}+C'n\mathbf{i}],$$
or
$$(\mathbf{I}-C'\mathbf{ii}).\frac{\partial\boldsymbol{\Omega}}{\partial t}-[(\mathbf{I}-C'\mathbf{ii}).\boldsymbol{\Omega}]\wedge\boldsymbol{\Omega}=\boldsymbol{\Gamma}-C'n\boldsymbol{\Omega}\wedge\mathbf{i}.$$

In the solution of this vector equation, $\mathbf{i}$ is to be treated as a constant vector. In comparison with Euler's standard equations, the effect of the flywheel is to replace the tensor $\mathbf{I}$ by $\mathbf{I}-C'\mathbf{ii}$, and to correct the applied couple by $-C'n\boldsymbol{\Omega}\wedge\mathbf{i}$. The axis of this correcting couple is parallel to the motion of $\mathbf{i}$, for $d\mathbf{i}/dt=\boldsymbol{\Omega}\wedge\mathbf{i}$. The tensor $\mathbf{I}-C'\mathbf{ii}$ is, of course, the inertia tensor less that part which is due to the rotational freedom of the flywheel (but including, of course, the transverse inertia of the flywheel).

It should be noted that this equation has its simplest Cartesian equivalents when the triad of reference chosen is that formed by the principal axes of the tensor $\mathbf{I}-C'\mathbf{ii}$, not by those of $\mathbf{I}$.

If the centres of mass of the body and flywheel do not coincide, exactly the same analysis holds, provided that $\mathbf{I}$ denotes the inertia tensor of the whole system about the centre of mass of the whole system. For to calculate the angular momentum of the whole system we have to add

to $\mathbf{I.\Omega}$ the angular momentum, about the resultant mass centre G, of the flywheel mass moving with the additional angular velocity $n-(\mathbf{\Omega.i})$ about its axis $\mathbf{i}$. This is simply $C'(n-\mathbf{\Omega.i})\mathbf{i}$, and no correction is required for the linear momentum of the flywheel, since this is all properly counted in the term $\mathbf{I.\Omega}$. The expression for the total angular momentum about G is thus unaltered.

If $\mathbf{\Gamma}=0$, the equation of motion is immediately integrable on scalar multiplication by $\mathbf{\Omega}$. For this gives on integration

$$(\mathbf{I}-C'\mathbf{ii}):\mathbf{\Omega\Omega}=\text{const.}$$

This replaces the energy integral. Again, if $\mathbf{\Gamma}=0$ the angular momentum is constant, and so

$$(\mathbf{I}-C'\mathbf{ii}).\mathbf{\Omega}+C'n\mathbf{i}=\mathbf{H}=\text{const.}$$

It follows that
$$(\mathbf{H}-C'n\mathbf{i}).\mathbf{\Omega}=\text{const.},$$

but in this equation it must be remembered that $\mathbf{i}$ is variable, satisfying

$$d\mathbf{i}/dt=\mathbf{\Omega}\wedge\mathbf{i}.$$

Example (4). If a body having kinetic symmetry about the axis of greatest moment be subject to a retarding couple about the instantaneous axis, whose magnitude varies as the angular velocity, then the instantaneous axis will approach asymptotically the axis of symmetry (Lamb, *H.M.*).

Since the body has kinetic symmetry about an axis, say the $\mathbf{i}$-axis, the inertia tensor is of the form

$$\mathbf{I}=A\mathbf{ii}+C(\mathbf{jj}+\mathbf{kk}), \qquad (A>C)$$

and the equation of motion is of the form

$$\mathbf{I}.\frac{\partial\mathbf{\Omega}}{\partial t}+\mathbf{\Omega}\wedge(\mathbf{I.\Omega})=-\lambda\mathbf{\Omega}. \qquad (\lambda>0)$$

This gives three Cartesian equations

$$\begin{aligned} A\dot{\omega}_1 \qquad\qquad\qquad &= -\lambda\omega_1, \\ C\dot{\omega}_2+(A-C)\omega_3\omega_1 &= -\lambda\omega_2, \\ C\dot{\omega}_3+(C-A)\omega_1\omega_2 &= -\lambda\omega_3. \end{aligned}$$

The first equation shows that ω_1 tends to zero as $e^{-\lambda t/A}$. The second and third equations, multiplied respectively by ω_2 and ω_3 and added, give

$$C\frac{d}{dt}[\tfrac{1}{2}(\omega_2{}^2+\omega_3{}^2)]=-2\lambda(\tfrac{1}{2}\omega_2{}^2+\tfrac{1}{2}\omega_3{}^2)$$

or
$$\tfrac{1}{2}(\omega_2{}^2+\omega_3{}^2)=\text{const. } e^{-2\lambda t/C}.$$

Hence $(\omega_2{}^2+\omega_3{}^2)^{\frac{1}{2}}$ tends to zero as $e^{-\lambda t/C}$, and so, if $A>C$, $(\omega_2{}^2+\omega_3{}^2)^{\frac{1}{2}}$ tends to zero faster than ω_1, and accordingly the direction of the instantaneous axis tends to coincidence with the axis $\mathbf{i}$.

Example (5). A body is in motion about its centre of mass under no forces. Determine the rate of change of the component of angular momentum about a line OP fixed in the body and moving with it. [If $\mathbf{i}$ is a unit vector along OP, the required quantity is $\mathbf{H}.(\mathbf{\Omega}\wedge\mathbf{i})$ (cf. Lamb, *H.M.*).]

Example (6). A body is compelled to rotate about its centre of mass with uniform angular velocity $\omega\mathbf{z}$ about an axis along a unit vector $\mathbf{z}$ not coinciding with a principal axis of inertia. Determine the components of the requisite applied couple about the principal axes of inertia.

We have

$$\mathbf{H}=\omega\mathbf{z}.\mathbf{I}=\omega(A\mathbf{ii}+B\mathbf{jj}+C\mathbf{kk}).\mathbf{z}.$$

Here $$\frac{d\mathbf{i}}{dt}=\omega\mathbf{z}\wedge\mathbf{i},\qquad \frac{d\mathbf{j}}{dt}=\omega\mathbf{z}\wedge\mathbf{j},\qquad \frac{d\mathbf{k}}{dt}=\omega\mathbf{z}\wedge\mathbf{k},$$

whence $$\mathbf{\Gamma}=\frac{d\mathbf{H}}{dt}=\omega^2[\Sigma A(\mathbf{z}\wedge\mathbf{i})\mathbf{i}+\Sigma A\mathbf{i}(\mathbf{z}\wedge\mathbf{i})].\mathbf{z}=\omega^2\Sigma A(\mathbf{z}\wedge\mathbf{i})(\mathbf{i}.\mathbf{z}).$$

Hence a typical component of the applied couple is given by

$$\begin{aligned}\mathbf{\Gamma}.\mathbf{i}&=\omega^2[B(\mathbf{z}\wedge\mathbf{j})(\mathbf{j}.\mathbf{z}).\mathbf{i}+C(\mathbf{z}\wedge\mathbf{k})(\mathbf{k}.\mathbf{z}).\mathbf{i}]\\&=\omega^2(\mathbf{z}.\mathbf{j})(\mathbf{z}.\mathbf{k})(C-B).\end{aligned}$$

This is, of course, equivalent in these circumstances to

$$\mathbf{\Gamma}.\mathbf{i}=-(B-C)\,\omega_2\omega_3,$$

and as such is equivalent to a particular case of Euler's equations. But the foregoing derivation from first principles is of some interest.

391. *Uniform precession.* It is of interest to determine the couple necessary to compel a body possessing dynamical symmetry about an axis to move in such a way that the axis of symmetry precesses uniformly about a fixed direction.

Let A, A, C be the principal moments of inertia at the mass centre. Let $\mathbf{k}$ be a unit vector along the axis of symmetry (corresponding to C). Then the angular momentum is given by

$$\mathbf{H}=Cn\mathbf{k}+A\mathbf{k}\wedge\frac{d\mathbf{k}}{dt},$$

where n defines the spin about $\mathbf{k}$; and the angular velocity $\mathbf{\Omega}$ is given by

$$\mathbf{\Omega}=n\mathbf{k}+\mathbf{k}\wedge\frac{d\mathbf{k}}{dt}.$$

The necessary couple $\mathbf{\Gamma}$ is given by

$$\mathbf{\Gamma}=\frac{d\mathbf{H}}{dt}=Cn\frac{d\mathbf{k}}{dt}+C\frac{dn}{dt}\mathbf{k}+A\mathbf{k}\wedge\frac{d^2\mathbf{k}}{dt^2}.$$

Now a motion of steady precession, in which n remains constant and $\mathbf{k}$ rotates uniformly about a fixed direction $\mathbf{z}$, will be possible provided that in the last equation

$$\frac{d\mathbf{k}}{dt} = \omega \mathbf{z} \wedge \mathbf{k}, \qquad n = \text{const.},$$

where ω is the precessional angular velocity. This gives

$$\begin{aligned} \mathbf{\Gamma} &= Cn\omega(\mathbf{z}\wedge\mathbf{k}) + A\omega^2\mathbf{k}\wedge[\mathbf{z}\wedge(\mathbf{z}\wedge\mathbf{k})] \\ &= Cn\omega(\mathbf{z}\wedge\mathbf{k}) - A\omega^2(\mathbf{z}\wedge\mathbf{k})(\mathbf{z}.\mathbf{k}). \end{aligned}$$

We can choose the sense of $\mathbf{z}$ so that necessarily $\omega > 0$. Then the last equation determines $\mathbf{\Gamma}$ as parallel to $\mathbf{z}\wedge\mathbf{k}$, i.e. the *plane* of the couple lies in the plane defined by the axis of precession and the axis of symmetry.

To express $\mathbf{\Gamma}$ in terms of $\mathbf{\Omega}$, we notice that by the above expressions for $\mathbf{\Omega}$ and $d\mathbf{k}/dt$,

$$\mathbf{\Omega}.\mathbf{k} = n, \qquad \mathbf{\Omega}\wedge\mathbf{k} = [\mathbf{k}\wedge(\omega\mathbf{z}\wedge\mathbf{k})]\wedge\mathbf{k} = \omega\mathbf{z}\wedge\mathbf{k}.$$

Hence, since $$\mathbf{\Gamma} = \omega(\mathbf{z}\wedge\mathbf{k})[Cn - A\omega(\mathbf{z}.\mathbf{k})],$$

we have $$\mathbf{\Gamma} = (\mathbf{\Omega}\wedge\mathbf{k})\left[C(\mathbf{\Omega}.\mathbf{k}) - A(\mathbf{z}.\mathbf{k})\frac{|\mathbf{\Omega}\wedge\mathbf{k}|}{|\mathbf{z}\wedge\mathbf{k}|}\right],$$

where since $\omega > 0$ we have put

$$\omega = |\mathbf{\Omega}\wedge\mathbf{k}|/|\mathbf{z}\wedge\mathbf{k}|.$$

If α is the angle between the axis of symmetry ($\mathbf{k}$) and the axis of precession ($\mathbf{z}$), β the angle between the instantaneous axis ($\mathbf{\Omega}$) and the axis of precession, then

$$\mathbf{\Omega}.\mathbf{k} = |\mathbf{\Omega}| \cos\beta, \qquad \mathbf{z}.\mathbf{k} = \cos\alpha,$$

$$\omega = \frac{|\mathbf{\Omega}| \sin\beta}{\sin\alpha}, \qquad |\mathbf{z}\wedge\mathbf{k}| = \sin\alpha,$$

and so $$\mathbf{\Gamma} = \frac{\Omega^2(\mathbf{z}\wedge\mathbf{k})\sin\beta}{\sin\alpha}\left[C\cos\beta - A\frac{\cos\alpha\sin\beta}{\sin\alpha}\right]$$

whence $$|\mathbf{\Gamma}| = \Omega^2 \sin\beta\left[C\cos\beta - A\frac{\cos\alpha\sin\beta}{\sin\alpha}\right],$$

as given in Lamb's *Higher Mechanics*.

392. *Further example.* A rigid body, of mass M, is in motion under the action of a force which acts always at a given point of the body. If $\boldsymbol{\rho}$ is the position vector of this point with regard to the centre of mass of the body, $\mathbf{r}$ the position vector of the centre of mass of the body with

regard to a fixed origin, **I** the inertia tensor of the body about its centre of mass, show that

$$M\boldsymbol{\rho}\wedge\frac{d^2\mathbf{r}}{dt^2}-\mathbf{I}.\frac{d\boldsymbol{\Omega}}{dt}-\boldsymbol{\Omega}\wedge(\mathbf{I}.\boldsymbol{\Omega})=0.$$

Let **R** be the force. The equations of motion of the body are

$$M\frac{d^2\mathbf{r}}{dt^2}=\mathbf{R}, \qquad \frac{d}{dt}(\mathbf{I}.\boldsymbol{\Omega})=\boldsymbol{\rho}\wedge\mathbf{R}.$$

Hence

$$M\boldsymbol{\rho}\wedge\frac{d^2\mathbf{r}}{dt^2}=\frac{d}{dt}(\mathbf{I}.\boldsymbol{\Omega})=\frac{\partial}{\partial t}(\mathbf{I}.\boldsymbol{\Omega})+\boldsymbol{\Omega}\wedge(\mathbf{I}.\boldsymbol{\Omega})$$

$$=\mathbf{I}.\frac{\partial\boldsymbol{\Omega}}{\partial t}+\boldsymbol{\Omega}\wedge(\mathbf{I}.\boldsymbol{\Omega})$$

$$=\mathbf{I}.\frac{d\boldsymbol{\Omega}}{dt}+\boldsymbol{\Omega}\wedge(\mathbf{I}.\boldsymbol{\Omega}).$$

CHAPTER XVII

GYROSTATIC PROBLEMS

393. *Rigid bodies possessing spin.* In this chapter it is proposed to discuss a class of problems of considerable interest. These are problems in which one of the constituent objects is a spinning rigid body. We shall consider the cases of tops, spheres, cylinders, hoops and other bodies possessing some feature of symmetry, spinning and possibly rolling on other spheres, cones or cylinders. We shall obtain the general equation of motion of some vector fixing the configuration of the system, and we shall obtain the conditions of steady precession where steady precession is a possible motion. We shall also examine the stability of the steady motion in certain cases.

In these contexts vector methods reach their greatest fruition. The vector methods of obtaining, for example, the condition of steady precession of a top, or the condition of stability of a nearly vertical top, are incomparably more direct, more *dynamical* so to say, easier to handle and more insight-giving than the customary methods by means of Eulerian angles, Lagrangian equations or integrals of energy and of angular momentum about the vertical. The latter integrals play very little part in our development and in their place other integrals make their appearance, integrals analogous to the constancy of axial spin of a top, but having no simple Lagrangian setting in more complicated problems. Throughout, we write down equations of motion from first principles, and the technique required is of the simplest. Angular momentum and angular velocity are handled as vectors throughout.

The vector method is no mere shorthand equivalent of the usual scalar procedures ; it deals with the dynamical situation actually arising, and gives a picture of the motions occurring. There are, it is true, occasionally certain problems, such as the determination of the period of oscillation near a state of steady motion, where the vector method is more cumbrous. But even here it provides more insight into the nature of the motion. In all cases where we seek not so much quantitative relations between scalars artificially introduced (such as Eulerian angles) but rather a description of the motion as it would appear to actual view, the vector method has the advantage.

THE MOTION OF A TOP

394. *The equation of motion of a spinning top.* By a top we mean dynamically a rigid system possessing an axis of symmetry about which

it is capable of rotating. The body thus possesses two equal moments of inertia about any two axes perpendicular to the axis of symmetry and to one another. The top may have one particle fixed or one particle constrained to move in a particular way: the particle in question is sometimes the centre of mass, sometimes elsewhere on the axis of symmetry.

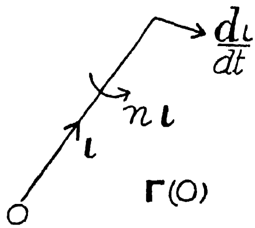

Fig. 95

We consider first the case in which one particle O on the axis is fixed (Fig. 95). (This is the case of the spinning top as ordinarily understood, for which the lowest particle of the axis is in contact with the ground.) Let $\mathbf{i}$ be a unit vector along the axis, C the moment of inertia about $\mathbf{i}$, A the moment of inertia about any axis through O perpendicular to $\mathbf{i}$. Then the angular velocity $\mathbf{\Omega}$ of the top is of the form (§ 231).

$$\mathbf{\Omega}=\mathrm{n}\mathbf{i}+\mathbf{i}\wedge\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}.$$

It is of interest to re-derive this formula in the present context. The motion of the extremity of the unit vector $\mathbf{i}$ along the axis is given by

$$\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}=\mathbf{\Omega}\wedge\mathbf{i}.$$

Hence
$$\mathbf{i}\wedge\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}=\mathbf{i}\wedge(\mathbf{\Omega}\wedge\mathbf{i})=\mathbf{\Omega}-(\mathbf{\Omega}.\mathbf{i})\mathbf{i}.$$

But $\mathbf{\Omega}.\mathbf{i}$ is the spin n about the axis. The formula for $\mathbf{\Omega}$ follows.

The inertia tensor about O is of the form (§ 330)

$$\mathbf{I}(\mathrm{O})=\mathrm{A}\mathbf{U}+(\mathrm{C}-\mathrm{A})\mathbf{ii},$$

and the angular momentum about O is of the form (§ 330)

$$\mathbf{H}(\mathrm{O})=\mathrm{Cn}\mathbf{i}+\mathrm{A}\mathbf{i}\wedge\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}.$$

The latter formula is also obvious from first principles, since $\mathrm{n}\mathbf{i}$ and $\mathbf{i}\wedge\mathrm{d}\mathbf{i}/\mathrm{dt}$ are the components of angular velocity along and perpendicular to the axis of symmetry.

If $\mathbf{\Gamma}(\mathrm{O})$ is the external couple about O, since O is a particle at rest, the equation of motion is

$$\frac{\mathrm{d}\mathbf{H}(\mathrm{O})}{\mathrm{dt}}=\mathbf{\Gamma}(\mathrm{O})$$

or
$$\mathrm{C}\frac{\mathrm{dn}}{\mathrm{dt}}\mathbf{i}+\mathrm{Cn}\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}+\mathrm{A}\mathbf{i}\wedge\frac{\mathrm{d}^2\mathbf{i}}{\mathrm{dt}^2}=\mathbf{\Gamma},$$

where we have written $\mathbf{\Gamma}$ for $\mathbf{\Gamma}(O)$. Multiplying the last equation scalarly by $\mathbf{i}$, we have

$$C\frac{dn}{dt}=\mathbf{\Gamma}.\mathbf{i}.$$

If, as is usually the case, the external forces have zero moment about the axis of symmetry $\mathbf{i}$, then $\mathbf{\Gamma}.\mathbf{i}=0$, and hence

$$n=\text{constant}. \tag{1}$$

Accordingly the equation of motion reduces to

$$Cn\frac{d\mathbf{i}}{dt}+A\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}=\mathbf{\Gamma}. \tag{2}$$

395. This equation of motion may be written in the form

$$A\frac{d}{dt}\left(\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}\right)=\mathbf{\Gamma}-Cn\frac{d\mathbf{i}}{dt}.$$

This is the equation of motion of a particle P of mass A at the position vector $\mathbf{i}$ from O, under a couple of moment $\mathbf{\Gamma}-Cnd\mathbf{i}/dt$. The effect of the spin n is thus to give rise to a couple

$$-Cn\frac{d\mathbf{i}}{dt},$$

whose axis is opposite in direction (for $n>0$) to the velocity $d\mathbf{i}/dt$. Since

$$-Cn\frac{d\mathbf{i}}{dt}=\mathbf{i}\wedge\left(Cn\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}\right)$$

the couple arising from the spin may be regarded as caused by a force

$$Cn\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}$$

acting at P. This 'force due to the spin' is at right angles to $\mathbf{i}$ and to the velocity $d\mathbf{i}/dt$, and, for $n>0$, makes with these a positive triad.

The physical origin of this couple is evident from the analysis by which it was deduced. When $\mathbf{i}$ changes to $\mathbf{i}+d\mathbf{i}$, the component of angular momentum about the axis of symmetry changes from $Cn\mathbf{i}$ to $Cn(\mathbf{i}+d\mathbf{i})$. The rate of change of this component is $Cnd\mathbf{i}/dt$, and thus the contribution $Cnd\mathbf{i}/dt$ is required from the applied couple on account of the spin n. The remainder $\mathbf{\Gamma}-Cnd\mathbf{i}/dt$ is available for giving an acceleration to $\mathbf{i}$.

Equations (1) and (2) above contain the complete dynamics of the motion of a top.

396. *Case of a top under gravity.* Let the centre of mass G of the top be at a point in the axis distant h from the fixed particle O (Fig. 96). If $\mathbf{z}$ is a unit vector vertically upwards, M the mass, then the moment of the weight about O is given by

$$\mathbf{\Gamma}=h\mathbf{i}\wedge(-Mg\mathbf{z}),$$

and the equation of motion is therefore

$$Cn\frac{d\mathbf{i}}{dt}+A\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}=-Mgh\mathbf{i}\wedge\mathbf{z}. \tag{3}$$

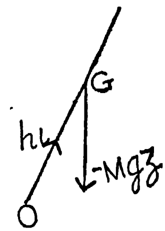

Fig. 96

397. *Condition for steady precession.* Steady precession will be possible if (3) possesses a solution in which $\mathbf{i}$ rotates round $\mathbf{z}$ at a constant rate. This will be so if we can find a number ω such that the motion

$$\frac{d\mathbf{i}}{dt}=\omega\mathbf{z}\wedge\mathbf{i} \tag{4}$$

is a solution of (3). From (4),

$$\frac{d\mathbf{i}}{dt}.\mathbf{z}=0$$

or

$$\mathbf{i}.\mathbf{z}=\text{const.}=\cos\alpha,$$

say. Further, from (4) again,

$$\frac{d^2\mathbf{i}}{dt^2}=\omega\mathbf{z}\wedge(\omega\mathbf{z}\wedge\mathbf{i})=\omega^2[-\mathbf{i}+\mathbf{z}\cos\alpha]$$

and so

$$\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}=\omega^2\cos\alpha\mathbf{i}\wedge\mathbf{z}.$$

Substituting in (3) we require ω to satisfy

$$Cn\omega(\mathbf{z}\wedge\mathbf{i})+A\omega^2\cos\alpha(\mathbf{i}\wedge\mathbf{z})=-Mgh(\mathbf{i}\wedge\mathbf{z}).$$

This will be an identity if ω satisfies the quadratic equation

$$A\omega^2\cos\alpha-Cn\omega+Mgh=0. \tag{5}$$

Relation (5) is the condition for steady precession at inclination α at the rate ω. For any given value of α for which

$$C^2n^2>4AMgh\cos\alpha,$$

there are two possible values of ω, given by

$$\omega_1,\ \omega_2=\frac{Cn\pm[C^2n^2-4AMgh\cos\alpha]^{\frac{1}{2}}}{2A\cos\alpha}.$$

If n is large compared with $(4AMgh\cos\alpha)^{\frac{1}{2}}/C$, the two solutions are approximately

$$\omega_1=\frac{Cn}{A\cos\alpha},\qquad \omega_2=\frac{Mgh}{Cn}.$$

The one is 'fast,' the other 'slow.' The latter is the one often approximately realized in a spinning top.

If the axis of the top is *hanging*, i.e. if the centre of mass is below the point of support, we may take either h or cos α as negative, in which case states of steady precession are always possible. The slow precession is then retrograde.

398. *Integrals of angular momentum and energy for a top.* The integrals of constancy of energy, and of constancy of angular momentum about the vertical, can readily be obtained from (3) of § 396, if required. Multiply (3) scalarly by $\mathbf{z}$ and integrate. We find

$$\mathrm{Cn}\mathbf{i}.\mathbf{z}+\mathrm{A}\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}.\mathbf{z}=\text{const.} \tag{1}$$

This expresses that the component of angular momentum about the vertical is constant—a result which is due to the circumstance that all the forces acting either pass through O or are parallel to the vertical through O. Again, multiply (3) scalarly by $\mathbf{i}\wedge d\mathbf{i}/dt$. We get

$$\mathrm{A}\left(\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}\right).\left(\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}\right)=\mathrm{Mgh}(\mathbf{z}\wedge\mathbf{i}).\left(\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}\right)$$

$$=-\mathrm{Mgh}\mathbf{z}.\frac{d\mathbf{i}}{dt}.$$

This integrates as it stands in the form

$$\tfrac{1}{2}\mathrm{A}\left(\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}\right)^2+\mathrm{Mgh}\mathbf{z}.\mathbf{i}=\text{const.} \tag{2}$$

Since the contribution of the spin n to the kinetic energy is a constant, the last equation expresses the constancy of the sum of kinetic and potential energy.

Integrals (1) and (2) are scalar relations. They are usually expressed in terms of the polar co-ordinates (θ, φ) defining the position of $\mathbf{i}$. To translate them into the θ, φ notation, we note first that

$$\mathbf{z}.\mathbf{i}=\cos\theta,$$

secondly, that the motion of $\mathbf{i}$ is given by

$$\frac{d\mathbf{i}}{dt}=\mathbf{\Omega}'\wedge\mathbf{i},$$

where $\mathbf{\Omega}'$ is a vector expressing the angular velocity of $\mathbf{i}$. Clearly in terms of $\dot\theta$ and $\dot\varphi$ we can take $\mathbf{\Omega}'$ to be given by

$$\mathbf{\Omega}'=\dot\theta\frac{\mathbf{z}\wedge\mathbf{i}}{\sin\theta}+\dot\varphi\mathbf{z},$$

and so, in turn, $$\frac{d\mathbf{i}}{dt}=\frac{\dot\theta}{\sin\theta}(-\mathbf{z}+\mathbf{i}\cos\theta)+\dot\varphi(\mathbf{z}\wedge\mathbf{i}),$$

$$\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}=\frac{\dot\theta}{\sin\theta}(\mathbf{z}\wedge\mathbf{i})+\dot\varphi(\mathbf{z}-\mathbf{i}\cos\theta)$$

$$\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}.\mathbf{z}=\dot\varphi\sin^2\theta.$$

The integrals (1) and (2) accordingly become

$$\mathrm{Cn}\cos\theta+\mathrm{A}\dot\varphi\sin^2\theta=\text{const.},\tag{1'}$$

$$\tfrac{1}{2}\mathrm{A}(\dot\theta^2+\dot\varphi^2\sin^2\theta)+\mathrm{Mgh}\cos\theta=\text{const.}\tag{2'}$$

Elimination of $\dot\varphi$ between these equations gives an equation for $\dot\theta$ which determines what is called the nutation, i.e. the variation of the inclination of the vector $\mathbf{i}$ to the vertical. We shall not investigate this motion in general,* but content ourselves with giving later an investigation of the nutations consequent on a small disturbance of a state of steady precession.

399. *Examples on steady precession.* (1) Find the condition of steady precession of a top whose vertex is in contact with a rough horizontal plane which is compelled to rotate with an angular velocity ω about the vertical.

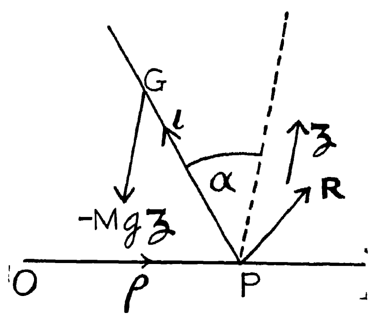

Fig. 97

Let O (Fig. 97) be the point in the horizontal plane about which it is rotating, P the vertex of the top, and put $\mathrm{OP}=\boldsymbol{\rho}$. If G is the centre of mass, with our previous notation for a top, the position vector of G with respect to O is

$$\boldsymbol{\rho}+\mathrm{h}\mathbf{i},$$

in which $$\frac{d\boldsymbol{\rho}}{dt}=\omega\mathbf{z}\wedge\boldsymbol{\rho},\qquad \frac{d^2\boldsymbol{\rho}}{dt^2}=-\omega^2\boldsymbol{\rho}.\tag{1}$$

If C, A are the inertia constants *for axes through* G, then

$$\mathbf{H}(\mathrm{G})=\mathrm{Cn}\mathbf{i}+\mathrm{A}\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}.$$

If $\mathbf{R}$ is the reaction at P, the equations of rate of change of linear and angular momentum are

$$\mathrm{M}\frac{d^2}{dt^2}(\boldsymbol{\rho}+\mathrm{h}\mathbf{i})=\mathbf{R}-\mathrm{Mg}\mathbf{z},\tag{2}$$

$$\mathrm{C}\frac{d}{dt}(\mathrm{n}\mathbf{i})+\mathrm{A}\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}=(-\mathrm{h}\mathbf{i})\wedge\mathbf{R}.\tag{3}$$

* We have nothing to add to the standard methods usually given in books on dynamics.

The last equation, on scalar multiplication by $\mathbf{i}$, gives

$$\frac{dn}{dt}=0, \qquad \text{or} \qquad n=\text{const.}^* \tag{4}$$

Eliminating $\mathbf{R}$ between (2) and (3) we get

$$Cn\frac{d\mathbf{i}}{dt}+A\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}=-h\mathbf{i}\wedge\left[Mg\mathbf{z}+M\frac{d^2}{dt^2}(\boldsymbol{\rho}+h\mathbf{i})\right],$$

or, using (1),

$$Cn\frac{d\mathbf{i}}{dt}+A\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}=-Mh\mathbf{i}\wedge\left[g\mathbf{z}-\omega^2\boldsymbol{\rho}+h\frac{d^2\mathbf{i}}{dt^2}\right]. \tag{5}$$

This is the equation determining the motion of $\mathbf{i}$.

In a state of steady precession, $\mathbf{i}$ will rotate about the vertical at the same speed as $\boldsymbol{\rho}$. We therefore seek a solution of (5) in which

$$\frac{d\mathbf{i}}{dt}=\omega\mathbf{z}\wedge\mathbf{i}$$

so that
$$\frac{d^2\mathbf{i}}{dt^2}=\omega^2\mathbf{z}\wedge(\mathbf{z}\wedge\mathbf{i})=\omega^2[-\mathbf{i}+\mathbf{z}\cos\alpha],$$

where we have put
$$\mathbf{z}.\mathbf{i}=\cos\alpha.$$

Insertion in (5) now gives

$$Cn\omega(\mathbf{z}\wedge\mathbf{i})+(A+Mh^2)\omega^2\cos\alpha(\mathbf{i}\wedge\mathbf{z})=-Mgh(\mathbf{i}\wedge\mathbf{z})+Mh\omega^2(\mathbf{i}\wedge\boldsymbol{\rho}).$$

But, since
$$\mathbf{i}=\mathbf{z}\cos\alpha-\frac{\boldsymbol{\rho}}{|\boldsymbol{\rho}|}\sin\alpha,$$

we have
$$\mathbf{i}\wedge\mathbf{z}=\mathbf{z}\wedge\frac{\boldsymbol{\rho}}{|\boldsymbol{\rho}|}\sin\alpha,$$
$$\mathbf{i}\wedge\boldsymbol{\rho}=(\mathbf{z}\wedge\boldsymbol{\rho})\cos\alpha.$$

We hence have an identity, all the vectors being parallel to $\mathbf{z}\wedge\boldsymbol{\rho}$, provided that

$$[(A+Mh^2)\omega^2\cos\alpha-Cn\omega+Mgh]\sin\alpha=Mh\omega^2|\boldsymbol{\rho}|\cos\alpha.$$

This is accordingly the condition of steady precession.

Example (2). Three mutually perpendicular light rods, Ox, Oy, Oz, are rigidly connected together at O. Three equal gyrostats of mass M and moment of inertia C about the axis are mounted on the rods with their centres of mass at equal distances *a* from O. The system is suspended from O and is free to rotate about O under gravity. Obtain the equations of motion and show that the conditions of steady precession with an angular speed ω are

$$\frac{rm-qn}{m-n}=\frac{pn-rl}{n-l}=\frac{ql-pm}{l-m}=-\frac{Mga}{C\omega},$$

* A similar integral should always be determined as a preliminary step in any case of steady precession.

where p, q, r are the spins of the gyrostats and l, m, n are the cosines of the inclinations of the rods to the vertical (*M. T.* 1911 ; Besant and Ramsey).

Take three unit vectors **i, j, k** along the gyrostats' axes. Since each flywheel (gyrostat) is acted on only by forces intersecting its axis, by the usual argument the spins p, q, r are constants. If A is the moment of inertia of a gyrostat about a transverse axis *at* O, then the angular momentum of the first gyrostat about O is

$$C p\mathbf{i} + A\mathbf{i} \wedge \frac{d\mathbf{i}}{dt}.$$

Now take **z** to be a unit vector vertically downwards. Since the total moment about O of the external forces (gravity) is $\Sigma(a\mathbf{i} \wedge Mg\mathbf{z})$, the equation of rate of change of angular momentum about O for the complete system is

$$\frac{d}{dt}\left[\Sigma C p\mathbf{i} + \Sigma A\mathbf{i} \wedge \frac{d\mathbf{i}}{dt}\right] = Mga(\mathbf{i}+\mathbf{j}+\mathbf{k}) \wedge \mathbf{z}.$$

But since the axes of the gyrostats are rigidly connected together, there exists a single vector **Ω**, the angular velocity of the system of axes, with the property that

$$\frac{d\mathbf{i}}{dt} = \mathbf{\Omega} \wedge \mathbf{i}, \qquad \frac{d\mathbf{j}}{dt} = \mathbf{\Omega} \wedge \mathbf{j}, \qquad \frac{d\mathbf{k}}{dt} = \mathbf{\Omega} \wedge \mathbf{k}. \qquad (1), (2), (3)$$

Hence $$\mathbf{i} \wedge \frac{d\mathbf{i}}{dt} = \mathbf{i} \wedge (\mathbf{\Omega} \wedge \mathbf{i}) = \mathbf{\Omega} - \mathbf{i}(\mathbf{\Omega}.\mathbf{i}),$$

$$\Sigma\mathbf{i} \wedge \frac{d\mathbf{i}}{dt} = 3\mathbf{\Omega} - \Sigma\mathbf{i}(\mathbf{\Omega}.\mathbf{i}) = 3\mathbf{\Omega} - \mathbf{\Omega} = 2\mathbf{\Omega}.$$

Hence $$C\mathbf{\Omega} \wedge (p\mathbf{i} + q\mathbf{j} + r\mathbf{k}) + 2A\frac{d\mathbf{\Omega}}{dt} = Mga(\mathbf{i}+\mathbf{j}+\mathbf{k}) \wedge \mathbf{z}. \qquad (4)$$

Equations (1), (2), (3), (4) determine the four vectors **Ω, i, j, k**.

If the motion is one of steady precession, **Ω** is a constant, say $\omega\mathbf{z}$. If $l = \mathbf{z}.\mathbf{i}$, $m = \mathbf{z}.\mathbf{j}$, $n = \mathbf{z}.\mathbf{k}$, then in steady precession l, m, n are constants, and

$$\mathbf{\Omega} = \omega(l\mathbf{i} + m\mathbf{j} + n\mathbf{k}).$$

Hence (4) requires

$$C\omega(l\mathbf{i} + m\mathbf{j} + n\mathbf{k}) \wedge (p\mathbf{i} + q\mathbf{j} + r\mathbf{k}) = Mga(\mathbf{i}+\mathbf{j}+\mathbf{k}) \wedge (l\mathbf{i} + m\mathbf{j} + n\mathbf{k}).$$

Equating coefficients of the linearly independent vectors $\mathbf{j} \wedge \mathbf{k}$, $\mathbf{k} \wedge \mathbf{i}$, $\mathbf{i} \wedge \mathbf{j}$ we get

$$C\omega(mr - nq) = Mga(n - m),$$

etc., which is the desired result.

Example (3). Four equal gyrostats are mounted symmetrically on four equal light rods forming a rhombus ODEF, of side 2a, hung from O. The masses of the gyrostats are each M, and their moments of inertia about axes through their centres of mass are C, A, in the usual notation. A mass M′ is hung from E. Obtain the condition of steady precession about the vertical.

Let **R** be the reaction at the joint D (Fig. 98), i.e. let **R** at D be the force on rod ED, so that $-\mathbf{R}$ at D is the force on rod OD. Let **i, j** be unit vectors in ED, DO, **z** a unit vector in EO. The equation of angular momentum about the fixed point O for the rod OD is

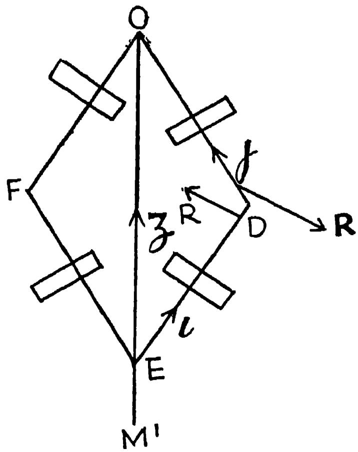

Fig. 98

$$\frac{d}{dt}\left[Cn\mathbf{j}+(A+Ma^2)\mathbf{j}\wedge\frac{d\mathbf{j}}{dt}\right]=-a\mathbf{j}\wedge Mg(-\mathbf{z})+(-2a\mathbf{j})\wedge(-\mathbf{R}),$$

where n is the spin of a gyrostat. In steady precession, E is also a fixed point, and the equation of moments about E for the rod ED is similarly

$$\frac{d}{dt}\left[Cn\mathbf{i}+(A+Ma^2)\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}\right]=a\mathbf{i}\wedge Mg(-\mathbf{z})+2a\mathbf{i}\wedge\mathbf{R}.$$

Further, in steady precession, the equation of linear momentum for the pair of rods DEF is

$$(M'+2M)g=2\mathbf{R}.\mathbf{z}.$$

Also $$\mathbf{i}+\mathbf{j}=2\mathbf{z}\cos\alpha,\qquad \mathbf{i}-\mathbf{j}=2\mathbf{x}\sin\alpha,$$

where **x** is along FD and the angle EOD $=\alpha$. Further, in steady precession,

$$\frac{d\mathbf{i}}{dt}=\omega\mathbf{z}\wedge\mathbf{i},\qquad \frac{d^2\mathbf{i}}{dt^2}=\omega^2[-\mathbf{i}+\mathbf{z}\cos\alpha],$$

$$\frac{d\mathbf{j}}{dt}=\omega\mathbf{z}\wedge\mathbf{j},\qquad \frac{d^2\mathbf{j}}{dt^2}=\omega^2[-\mathbf{j}+\mathbf{z}\cos\alpha].$$

Subtracting the two equations of angular momentum, we have in steady precession

$$[Cn\omega-(A+Ma^2)\omega^2\cos\alpha]\mathbf{z}\wedge(\mathbf{i}-\mathbf{j})=-Mga(\mathbf{i}+\mathbf{j})\wedge\mathbf{z}+2a(\mathbf{i}-\mathbf{j})\wedge\mathbf{R}$$

or $$[Cn\omega-(A+Ma^2)\omega^2\cos\alpha]2\sin\alpha\,(\mathbf{z}\wedge\mathbf{x})=4a\sin\alpha\,\mathbf{x}\wedge\mathbf{R}$$

on using previous relations. Since the component of angular momentum of FED about the vertical EO is constant, the force **R** at D and its analogue

at F can have no resultant moment about OE, whence by symmetry **R** has no component perpendicular to the plane of the rhombus. Hence

$$\mathbf{x}\wedge\mathbf{R}=\mathbf{x}\wedge[(\mathbf{R.z})\mathbf{z}+(\mathbf{R.x})\mathbf{x}]=(\mathbf{R.z})(\mathbf{x}\wedge\mathbf{z})=\tfrac{1}{2}(M'+2M)g(\mathbf{x}\wedge\mathbf{z}).$$

Hence the desired condition is

$$Cn\omega-(A+Ma^2)\omega^2\cos\alpha=-(M'+2M)ga.$$

400. *The nearly vertical top. Stability.* The equation of motion of the top under gravity has been shown to be of the form (§ 396 (3))

$$Cn\frac{d\mathbf{i}}{dt}+A\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}=-Mgh(\mathbf{i}\wedge\mathbf{z}). \tag{1}$$

This has a solution

$$\mathbf{i}=\text{const.}=\mathbf{z}, \qquad n=\text{const.},$$

which describes a top spinning with its axis vertical. To discuss the stability * of this particular motion, put

$$\mathbf{i}=\mathbf{z}+\boldsymbol{\rho}$$

where $\boldsymbol{\rho}$ is a small vector. Then to a sufficient accuracy, the equation of motion (1) reduces to

$$Cn\frac{d\boldsymbol{\rho}}{dt}+A\mathbf{z}\wedge\frac{d^2\boldsymbol{\rho}}{dt^2}=-Mgh\boldsymbol{\rho}\wedge\mathbf{z}. \tag{2}$$

This equation is akin to types discussed earlier under particle dynamics. It is solved by putting

$$\frac{d\boldsymbol{\rho}}{dt}=\omega\mathbf{z}\wedge\boldsymbol{\rho}, \tag{3}$$

where ω is a constant. This gives

$$\frac{d^2\boldsymbol{\rho}}{dt^2}=\omega^2\mathbf{z}\wedge(\mathbf{z}\wedge\boldsymbol{\rho})=-\omega^2\boldsymbol{\rho},$$

since $\mathbf{z}.\boldsymbol{\rho}=0$ approximately. Hence, inserting in (2), ω must satisfy

$$Cn\omega(\mathbf{z}\wedge\boldsymbol{\rho})-A\omega^2(\mathbf{z}\wedge\boldsymbol{\rho})=-Mgh(\boldsymbol{\rho}\wedge\mathbf{z})$$

or

$$A\omega^2-Cn\omega+Mgh=0.$$

This has real roots provided

$$C^2n^2>4AMgh.$$

If this condition is satisfied, and the (real) roots are ω_1, ω_2, the motion is given by

$$\boldsymbol{\rho}=\boldsymbol{\rho}_1+\boldsymbol{\rho}_2, \tag{4}$$

* This problem is often discussed by a very roundabout method. The equations of motion of the upper end of the axis are obtained in scalars x, y, then combined to form a complex number $x+iy$ (equivalent to a vector) and then the real and imaginary parts of the solution sorted out. It is logically preferable, since the laws of motion are enunciated physically in terms of vectors initially, to carry through the solution in these same vectors throughout.

where $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$ are rotating vectors of constant modulus and arbitrary phase, rotating with angular velocities ω_1 and ω_2 according to the relations

$$\frac{d\boldsymbol{\rho}_1}{dt}=\omega_1\mathbf{z}\wedge\boldsymbol{\rho}_1, \qquad \frac{d\boldsymbol{\rho}_2}{dt}=\omega_2\mathbf{z}\wedge\boldsymbol{\rho}_2. \tag{5}$$

This solution clearly contains four scalar constants, and so is the complete solution of (2), this being a second-order differential equation for the two-dimensional vector $\boldsymbol{\rho}$. The motion near the vertical is accordingly *stable.*

401. *Unstable motion.* If $C^2n^2<4AMgh$, ω_1 and ω_2 are complex. We should anticipate that in this case the motion near the vertical would be unstable, but to establish the nature of the motion we seek a solution of a type different from (3) above. To proceed quite generally, let us seek a solution of (2) of the form

$$\frac{d\boldsymbol{\rho}}{dt}=\omega\mathbf{z}\wedge\boldsymbol{\rho}-\gamma\boldsymbol{\rho} \tag{6}$$

where ω and γ are real constants. Then

$$\frac{d^2\boldsymbol{\rho}}{dt^2}=\omega\mathbf{z}\wedge[\omega\mathbf{z}\wedge\boldsymbol{\rho}-\gamma\boldsymbol{\rho}]-\gamma[\omega\mathbf{z}\wedge\boldsymbol{\rho}-\gamma\boldsymbol{\rho}]$$

$$=-2\omega\gamma(\mathbf{z}\wedge\boldsymbol{\rho})+\boldsymbol{\rho}(\gamma^2-\omega^2).$$

Insertion in (2) gives

$$Cn[\omega\mathbf{z}\wedge\boldsymbol{\rho}-\gamma\boldsymbol{\rho}]+2A\omega\gamma\boldsymbol{\rho}+A(\gamma^2-\omega^2)\mathbf{z}\wedge\boldsymbol{\rho}+Mgh\boldsymbol{\rho}\wedge\mathbf{z}=0.$$

In this the coefficients of $\boldsymbol{\rho}$ and $\mathbf{z}\wedge\boldsymbol{\rho}$ must vanish. Hence

$$-Cn\gamma+2A\omega\gamma=0, \tag{7}$$

$$Cn\omega+A(\gamma^2-\omega^2)-Mgh=0. \tag{8}$$

Relation (7) gives either $\gamma=0$ or $\omega=Cn/2A$. Taking $\gamma=0$, we have from (8)

$$A\omega^2-Cn\omega+Mgh=0,$$

and we recover the solution of § 400, valid for $C^2n^2>4AMgh$. Taking $\omega=Cn/2A$, and inserting in (8), we get

$$\gamma^2=\frac{1}{4A^2}[4AMgh-C^2n^2]\;;$$

γ is therefore real and non-zero if $C^2n^2<4AMgh$, in which case $\omega=Cn/2A$. Hence, if we write

$$\gamma_1,-\gamma_1=\pm\frac{[4AMgh-C^2n^2]^{\frac{1}{2}}}{2A},$$

the solution for $C^2n^2<4AMgh$ is

$$\boldsymbol{\rho}=\boldsymbol{\rho}_1+\boldsymbol{\rho}_2,$$

where by (6)

$$\frac{d(e^{\gamma_1 t}\boldsymbol{\rho}_1)}{dt}=\frac{Cn}{2A}\mathbf{z}\wedge(\boldsymbol{\rho}_1 e^{\gamma_1 t}), \tag{9}$$

$$\frac{d(e^{-\gamma_1 t}\boldsymbol{\rho}_2)}{dt}=\frac{Cn}{2A}\mathbf{z}\wedge(\boldsymbol{\rho}_2 e^{-\gamma_1 t}). \tag{10}$$

It is clear that in this solution $\boldsymbol{\rho}_1 e^{\gamma_1 t}$ is a uniformly rotating vector of constant modulus, so that $\boldsymbol{\rho}_1$ shrinks exponentially to zero. Similarly, $\boldsymbol{\rho}_2 e^{-\gamma_1 t}$ is a uniformly rotating vector of constant modulus, so that $\boldsymbol{\rho}_2$ exponentially increases with t. This is the general solution for $C^2n^2<4AMgh$. Since one of the components of $\boldsymbol{\rho}$ increases indefinitely with t, the motion for $C^2n^2<4AMgh$ is in general *unstable*. It will be *apparently* stable only if the term in $\boldsymbol{\rho}_2$ is absent.

Suppose $\boldsymbol{\rho}_0$ and $(d\boldsymbol{\rho}/dt)_0$ are the initial displacement and velocity of the axis, say at $t=t_0$. Then when $C^2n^2<4AMgh$, we have

$$\boldsymbol{\rho}_0=(\boldsymbol{\rho}_1)_0+(\boldsymbol{\rho}_2)_0,$$

$$\left(\frac{d\boldsymbol{\rho}}{dt}\right)_0=\frac{Cn}{2A}\mathbf{z}\wedge[(\boldsymbol{\rho}_1)_0+(\boldsymbol{\rho}_2)_0]-\gamma_1[(\boldsymbol{\rho}_1)_0-(\boldsymbol{\rho}_2)_0].$$

For the disturbed motion not to increase indefinitely, we need to have $(\boldsymbol{\rho}_2)_0=0$, which requires

$$\left(\frac{d\boldsymbol{\rho}}{dt}\right)_0=\frac{Cn}{2A}\mathbf{z}\wedge\boldsymbol{\rho}_0-\gamma_1\boldsymbol{\rho}_0.$$

Only if this condition is satisfied will the essentially unstable top possess a motion of its axis decaying exponentially to zero.

402. *Alternative procedure.* An alternative procedure is to use a rotating frame of reference. In (2) introduce

$$\frac{d\boldsymbol{\rho}}{dt}=\frac{\partial\boldsymbol{\rho}}{\partial t}+\omega\mathbf{z}\wedge\boldsymbol{\rho},$$

where ω is a constant to be determined. Then since

$$\frac{d^2\boldsymbol{\rho}}{dt^2}=\frac{\partial^2\boldsymbol{\rho}}{\partial t^2}+2\omega\mathbf{z}\wedge\frac{\partial\boldsymbol{\rho}}{\partial t}-\omega^2\boldsymbol{\rho},$$

(2) gives

$$Cn\left(\frac{\partial\boldsymbol{\rho}}{\partial t}+\omega\mathbf{z}\wedge\boldsymbol{\rho}\right)+A\mathbf{z}\wedge\left(\frac{\partial^2\boldsymbol{\rho}}{\partial t^2}-\omega^2\boldsymbol{\rho}\right)-2\omega A\frac{\partial\boldsymbol{\rho}}{\partial t}+Mgh\boldsymbol{\rho}\wedge\mathbf{z}=0.$$

Since ω is at our disposal, we may choose it so that the coefficient of $\partial\boldsymbol{\rho}/\partial t$ vanishes. This requires $Cn=2A\omega$, and we then get

$$\mathbf{z}\wedge\left[A\frac{\partial^2\boldsymbol{\rho}}{\partial t^2}+\boldsymbol{\rho}(-Mgh-A\omega^2+Cn\omega)\right]=0.$$

Inserting $\omega=Cn/2A$, we get

$$\frac{\partial^2\boldsymbol{\rho}}{\partial t^2}+\boldsymbol{\rho}\frac{C^2n^2-4AMgh}{4A^2}=0.$$

If $C^2n^2 < 4AMgh$, we can put

$$\gamma^2 = \frac{4AMgh - C^2n^2}{4A^2},$$

and then the equation $$\frac{\partial^2 \rho}{\partial t^2} - \gamma^2 \rho = 0$$

implies $$\rho = \rho_1 + \rho_2$$

where $$\rho_1 = (\rho_1)_0 e^{-\gamma t}, \qquad \rho_2 = (\rho_2)_0 e^{+\gamma t}.$$

Thus, in the frame rotating with angular speed $Cn/2A$, ρ is the sum of two vectors of constant (apparent) direction, the one exponentially shrinking, the other exponentially growing, provided γ is real. This is the case of instability.

If γ^2 is negative, put

$$\gamma'^2 = -\gamma^2 = \frac{C^2n^2 - 4AMgh}{4A^2}.$$

Then the equation $$\frac{\partial^2 \rho}{\partial t^2} + \gamma'^2 \rho = 0$$

has for its general solution $$\rho = \rho_1 + \rho_2,$$

where $$\frac{\partial \rho_1}{\partial t} = \gamma' \mathbf{z} \wedge \rho_1, \qquad \frac{\partial \rho_2}{\partial t} = -\gamma' \mathbf{z} \wedge \rho_2.$$

The motion in the frame rotating with angular speed $\omega = Cn/2A$ is thus the sum of two oppositely rotating vectors, of arbitrary amplitude and phase, the speed of *relative* rotation being $\pm\gamma'$. It will be seen that this is the *stable* motion found in § 400.

The above methods of discussing vector differential equations are capable of application in other contexts.

403. *Oscillations near steady precession.* We proceed to a direct discussion of the disturbed motion of a top near a state of steady precession. In the general equation of motion of the axis,

$$Cn\frac{d\mathbf{i}}{dt} + A\mathbf{i} \wedge \frac{d^2\mathbf{i}}{dt^2} = Mgh\mathbf{z} \wedge \mathbf{i}, \tag{1}$$

put $$\mathbf{i} = \mathbf{i}_0 + \mathbf{p},$$

where $\mathbf{i}_0$ is the unit vector describing the axis of the top in the state of precession in question, and $|\mathbf{p}|$ is small compared with unity. Since $\mathbf{i}$ and $\mathbf{i}_0$ are unit vectors, to a sufficient order $\mathbf{i}_0 . \mathbf{p} = 0$. The vector $\mathbf{i}_0$ possesses, as we have seen, the motion

$$\frac{d\mathbf{i}_0}{dt} = \omega \mathbf{z} \wedge \mathbf{i}_0, \qquad \mathbf{i}_0 . \mathbf{z} = \text{const.} \tag{2}$$

where ω is a root of $$A\omega^2(\mathbf{i}_0 . \mathbf{z}) - Cn\omega + Mgh = 0, \tag{3}$$

(§ 397). In this case $$\frac{d^2\mathbf{i}_0}{dt^2} = \omega^2[\mathbf{z}(\mathbf{i}_0 . \mathbf{z}) - \mathbf{i}_0] \tag{2'}$$

and $\mathbf{i}_0$ satisfies
$$\mathrm{Cn}\frac{d\mathbf{i}_0}{dt}+\mathrm{A}\mathbf{i}_0\wedge\frac{d^2\mathbf{i}_0}{dt^2}=\mathrm{Mgh}(\mathbf{z}\wedge\mathbf{i}_0). \tag{4}$$

Using these relations, (1) reduces to
$$\mathrm{Cn}\frac{d\mathbf{p}}{dt}+\mathrm{A}\left[\mathbf{i}_0\wedge\frac{d^2\mathbf{p}}{dt^2}+\omega^2(\mathbf{i}_0.\mathbf{z})(\mathbf{p}\wedge\mathbf{z})-\omega^2(\mathbf{p}\wedge\mathbf{i}_0)\right]=\mathrm{Mgh}(\mathbf{z}\wedge\mathbf{p}). \tag{5}$$

Eliminating Mgh between (3) and (5) we get
$$\mathrm{A}\mathbf{i}_0\wedge\left(\frac{d^2\mathbf{p}}{dt^2}+\omega^2\mathbf{p}\right)=-\mathrm{Cn}\left(\frac{d\mathbf{p}}{dt}-\omega\mathbf{z}\wedge\mathbf{p}\right). \tag{6}$$

Equation (6) determines the behaviour of the small vector $\mathbf{p}$ as a function of t. Its form at once suggests that we should adopt as frame of reference a frame rotating with angular velocity $\omega\mathbf{z}$. This is the frame rigidly attached to $\mathbf{z}$ and $\mathbf{i}_0$. The motion of $\mathbf{p}$ relative to this frame will give us a picture of the disturbed motion. We are interested less in the motion of $\mathbf{p}$ in space than in its motion relative to $\mathbf{i}_0$. We write therefore
$$\frac{d\mathbf{p}}{dt}=\frac{\partial\mathbf{p}}{\partial t}+\omega\mathbf{z}\wedge\mathbf{p},$$
$$\frac{d^2\mathbf{p}}{dt^2}=\frac{\partial^2\mathbf{p}}{\partial t^2}+2\omega\mathbf{z}\wedge\frac{\partial\mathbf{p}}{\partial t}+\omega^2[-\mathbf{p}+\mathbf{z}(\mathbf{p}.\mathbf{z})],$$
where $\mathbf{i}_0$ is to remain constant under the operator $\partial/\partial t$. Then (6) becomes
$$\mathrm{A}\mathbf{i}_0\wedge\frac{\partial^2\mathbf{p}}{\partial t^2}+[\mathrm{Cn}-2\mathrm{A}\omega(\mathbf{i}_0.\mathbf{z})]\frac{\partial\mathbf{p}}{\partial t}+\mathrm{A}\omega^2(\mathbf{p}.\mathbf{z})(\mathbf{i}_0\wedge\mathbf{z})=0.$$

Multiplying this vectorially by $\mathbf{i}_0$ and remembering that $\mathbf{i}_0.\partial^2\mathbf{p}/\partial t^2=0$, we get
$$\frac{\partial^2\mathbf{p}}{\partial t^2}+\lambda\frac{\partial\mathbf{p}}{\partial t}\wedge\mathbf{i}_0+\omega^2(\mathbf{p}.\mathbf{z})(\mathbf{z}-\mathbf{i}_0\cos\alpha)=0, \tag{7}$$
where
$$\lambda=\frac{\mathrm{Cn}-2\mathrm{A}\omega\cos\alpha}{\mathrm{A}}, \qquad (\cos\alpha=\mathbf{i}_0.\mathbf{z}) \tag{8}$$

In (7), $\mathbf{i}_0$ is to be treated as a constant. The vector $(\mathbf{z}-\mathbf{i}_0\cos\alpha)$ is perpendicular to $\mathbf{i}_0$, and we therefore take an orthogonal triad of unit vectors $\mathbf{i}_0$, $\mathbf{j}$, $\mathbf{k}$, $\mathbf{k}$ being in the plane of $\mathbf{z}$ and $\mathbf{i}_0$, in such a way (Fig. 99) that
$$\mathbf{z}=\mathbf{i}_0\cos\alpha+\mathbf{k}\sin\alpha,$$
$$\mathbf{p}.\mathbf{z}=(\mathbf{p}.\mathbf{i}_0)\cos\alpha+(\mathbf{p}.\mathbf{k})\sin\alpha=(\mathbf{p}.\mathbf{k})\sin\alpha.$$
Then (7) becomes
$$\frac{\partial^2\mathbf{p}}{\partial t^2}+\lambda\frac{\partial\mathbf{p}}{\partial t}\wedge\mathbf{i}_0+\omega^2\sin^2\alpha(\mathbf{p}.\mathbf{k})\mathbf{k}=0. \tag{9}$$

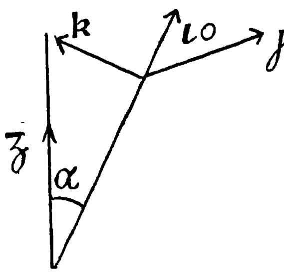

Fig. 99

This is precisely the equation we have already discussed, § 262, equation (4), in connexion with the disturbed motion of the spherical pendulum near steady precession. Equation (9) reduces, in fact, to the pendulum

equation when we put $n=0$. We can therefore use the solution previously found. The solution $\mathbf{p}$ of (9) is given by

$$\mathbf{p}=(\rho.\mathbf{k})\mathbf{k}+\mu(\rho.\mathbf{j})\mathbf{j}, \tag{10}$$

where ρ is an arbitrary vector rotating round $\mathbf{i}_0$ in the plane of $\mathbf{j}$ and $\mathbf{k}$ with relative angular velocity $\omega'\mathbf{i}_0$, so that

$$\frac{\partial\rho}{\partial t}=\omega'\mathbf{i}_0\wedge\rho.$$

Instead of quoting the values of ω' and μ, it is better to note that relation (10) gives

$$\frac{\partial\mathbf{p}}{\partial t}=\omega'[(\mathbf{j}.\rho)\mathbf{k}-\mu(\mathbf{k}.\rho)\mathbf{j}],$$

$$\frac{\partial^2\mathbf{p}}{\partial t^2}=-\omega'^2[(\rho.\mathbf{k})\mathbf{k}+\mu(\rho.\mathbf{j})\mathbf{j}],$$

whence, inserting in (9) and equating to zero the coefficients of $\mathbf{j}$ and $\mathbf{k}$ separately, we find

$$-\omega'^2+\mu\lambda\omega'+\omega^2\sin^2\alpha=0,$$
$$-\mu\omega'^2+\lambda\omega'=0.$$

Hence $\qquad \lambda=\mu\omega', \qquad \omega'^2=\lambda^2+\omega^2\sin^2\alpha.$

Thus $\qquad \omega'=+(\lambda^2+\omega^2\sin^2\alpha)^{\frac{1}{2}}, \qquad \mu=\dfrac{\lambda}{(\lambda^2+\omega^2\sin^2\alpha)^{\frac{1}{2}}}.$

If $\qquad x=\mathbf{p}.\mathbf{j}=\mu(\rho.\mathbf{j}), \qquad y=\mathbf{p}.\mathbf{k}=\rho.\mathbf{k},$

then $\qquad \dfrac{x^2}{\mu^2}+y^2=\rho^2=\text{const.}$

The motion of $\mathbf{p}$ in the precessing frame consists in the describing of an ellipse, period $2\pi/\omega'$. If $\lambda>0$, then $0<\mu<1$, and the ellipse has its shorter axis horizontal, and is described in the positive sense about $\mathbf{i}_0$. If $\lambda<0$, then $0>\mu>-1$, and though ω' is positive the ellipse is described in the negative sense about $\mathbf{i}_0$. Taking the other sign before the surd for ω' gives no new solution, for μ is also reversed in sign.

For the 'rapid' precession $\omega=\omega_1\sim Cn/A\cos\alpha$, we have $\lambda\sim-Cn/A$; for the 'slow' precession $\omega=\omega_2\sim Mgh/Cn$, we have $\lambda\sim+Cn/A$. Since ω' is essentially real, the precessional motion is essentially stable.

404. *Top on smooth horizontal table.* Let $R\mathbf{z}$ be the reaction, where $\mathbf{z}$ is a unit vector vertically upwards (Fig. 100). If $\mathbf{i}$ is a unit vector in the axis of the top, the equations of rate of change of linear momentum and of rate of change of angular momentum *about* G are

$$R-Mg=M\frac{d^2}{dt^2}h(\mathbf{i}.\mathbf{z}),$$

$$-h\mathbf{i}\wedge R\mathbf{z}=\frac{d}{dt}\left[Cn\mathbf{i}+A\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}\right],$$

where, in the second equation, A denotes the moment of inertia about a transverse axis through G. Scalar multiplication of the second equation by $\mathbf{i}$ gives as usual n=const.; and it is clear that G moves vertically up and down in a straight line, since the only forces acting are vertical. Eliminating R we have

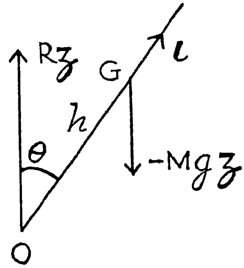

Fig. 100

$$\mathrm{Cn}\frac{d\mathbf{i}}{dt}+\mathrm{A}\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}=\mathrm{Mh}\left[g+h\frac{d^2}{dt^2}(\mathbf{i}.\mathbf{z})\right](\mathbf{z}\wedge\mathbf{i}).$$

This is the equation determining $\mathbf{i}$, and so the position of G. Multiplying scalarly by $\mathbf{z}$, we have the integral of angular momentum about the vertical in the form

$$\left[\mathrm{Cn}\mathbf{i}+\mathrm{A}\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}\right].\mathbf{z}=\text{const.}$$

Multiplying scalarly by $\mathbf{i}\wedge d\mathbf{i}/dt$, we get, since

$$(\mathbf{z}\wedge\mathbf{i}).\left(\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}\right)=-\mathbf{z}.\frac{d\mathbf{i}}{dt},$$

the relation

$$\mathrm{A}\left(\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}\right).\left(\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}\right)=-\mathrm{Mh}\left[g+h\left(\mathbf{z}.\frac{d^2\mathbf{i}}{dt^2}\right)\right]\mathbf{z}.\frac{d\mathbf{i}}{dt}$$

which integrates as it stands in the form

$$\tfrac{1}{2}\mathrm{A}\left(\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}\right)^2=-\mathrm{Mgh}(\mathbf{z}.\mathbf{i})-\tfrac{1}{2}\mathrm{Mh}^2\left(\mathbf{z}.\frac{d\mathbf{i}}{dt}\right)^2+\text{const.}$$

This is the integral of energy.

If the scalar forms are required, we write

$$\mathbf{z}.\mathbf{i}=\cos\theta,\qquad \frac{d\mathbf{i}}{dt}=\mathbf{\Omega}\wedge\mathbf{i},$$

where $\mathbf{\Omega}$, the angular velocity of the axis, is given in terms of the usual angles θ and φ by

$$\mathbf{\Omega}=\dot{\theta}\frac{\mathbf{z}\wedge\mathbf{i}}{\sin\theta}+\dot{\varphi}\mathbf{z}.$$

Then $$\frac{d\mathbf{i}}{dt}=\frac{\dot{\theta}}{\sin\theta}[-\mathbf{z}+\mathbf{i}(\mathbf{z}.\mathbf{i})]+\dot{\varphi}(\mathbf{z}\wedge\mathbf{i})=\frac{\dot{\theta}}{\sin\theta}[\mathbf{i}\cos\theta-\mathbf{z}]+\dot{\varphi}(\mathbf{z}\wedge\mathbf{i})$$

and $$\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}=\frac{\dot{\theta}}{\sin\theta}(\mathbf{z}\wedge\mathbf{i})+\dot{\varphi}[\mathbf{z}-\mathbf{i}\cos\theta].$$

Thus the integral of angular momentum about the vertical yields

$$\mathrm{Cn}\cos\theta+\mathrm{A}\dot{\varphi}\sin^2\theta=\text{const.},$$

and the energy integral yields

$$\tfrac{1}{2}\mathrm{A}[\dot{\theta}^2+\dot{\varphi}^2\sin^2\theta]+\tfrac{1}{2}\mathrm{Mh}^2\dot{\theta}^2\sin^2\theta=-\mathrm{Mgh}\cos\theta+\text{const.}$$

Elimination of $\dot\varphi$ between these will give an equation determining the behaviour of θ.

In steady precession, G remains stationary ; putting as usual $\mathbf{z.i}=\cos\alpha$ and

$$\frac{d\mathbf{i}}{dt}=\omega\mathbf{z}\wedge\mathbf{i}, \qquad \frac{d^2\mathbf{i}}{dt^2}=\omega^2[-\mathbf{i}+\mathbf{z}\cos\alpha]$$

the equation for $\mathbf{i}$ gives

$$Cn\omega-A\omega^2\cos\alpha=Mgh.$$

This has the same form as for a top spinning on a *rough* table, but the meaning of A is different.

MOTION OF A DISC

405. *The spinning disc on a smooth horizontal table.* To find the condition for steady precession, we proceed as follows. Take a unit vector $\mathbf{i}$ normal to the plane of the disc (Fig. 101), a unit vector $\mathbf{j}$ parallel to its line of greatest slope, and a unit vector $\mathbf{z}$ vertically upwards. If C is the moment of inertia about $\mathbf{i}$, A about any diameter, then here $C=2A$, but we shall not make use of this relation. The angular momentum about G is of the form

$$Cn\mathbf{i}+A\mathbf{i}\wedge\frac{d\mathbf{i}}{dt},$$

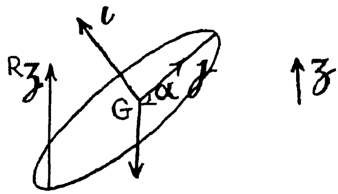

Fig. 101

as usual. If $R\mathbf{z}$ is the reaction at the point of contact, the equation of linear momentum is

$$R-Mg=M\frac{d^2}{dt^2}a(\mathbf{j.z}),$$

where a is the radius. The equation of angular momentum about G is

$$-a\mathbf{j}\wedge R\mathbf{z}=\frac{d}{dt}\left[Cn\mathbf{i}+A\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}\right].$$

Eliminating R,

$$-Ma\left(g+a\frac{d^2\mathbf{j}}{dt^2}.\mathbf{z}\right)(\mathbf{j}\wedge\mathbf{z})=C\frac{dn}{dt}\mathbf{i}+Cn\frac{d\mathbf{i}}{dt}+A\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}. \qquad (1)$$

If α is the inclination of the plane of the disc to the horizontal in steady precession, then

$$\mathbf{z}=\mathbf{j}\sin\alpha+\mathbf{i}\cos\alpha,$$

and
$$\mathbf{j}\wedge\mathbf{z}=\frac{\mathbf{z}-\mathbf{i}\cos\alpha}{\sin\alpha}\wedge\mathbf{z}=\cot\alpha(\mathbf{z}\wedge\mathbf{i}).$$

Hence, multiplying (1) scalarly by $\mathbf{i}$ we get

$$\frac{dn}{dt}=0, \qquad n=\text{const.}$$

In steady precession,

$$\frac{d\mathbf{i}}{dt}=\omega(\mathbf{z}\wedge\mathbf{i}), \qquad \frac{d^2\mathbf{i}}{dt^2}=\omega^2[-\mathbf{i}+\mathbf{z}\cos\alpha],$$

$$\frac{d\mathbf{j}}{dt}=\omega(\mathbf{z}\wedge\mathbf{j}), \qquad \frac{d^2\mathbf{j}}{dt^2}=\omega^2[-\mathbf{j}+\mathbf{z}\sin\alpha],$$

$$\frac{d^2\mathbf{j}}{dt^2}.\mathbf{z}=0.$$

Hence $$R=Mg,$$

and (1) gives

$$-Mag\cot\alpha(\mathbf{z}\wedge\mathbf{i})=Cn\omega(\mathbf{z}\wedge\mathbf{i})+A\omega^2\cos\alpha(\mathbf{i}\wedge\mathbf{z}),$$

whence the condition for steady precession is

$$A\omega^2\cos\alpha-Cn\omega-Mag\cot\alpha=0.$$

406. *Stability of spinning vertical disc, on smooth horizontal plane.* In this case $\alpha=\frac{1}{2}\pi$, and the foregoing condition of steady precession is satisfied by

$$n=0, \qquad \omega \text{ arbitrary.}$$

For small disturbances from this state, write in equation (1), § 405,

$$\mathbf{i}=\mathbf{y}+\boldsymbol{\rho},$$

where $\mathbf{y}$ is a unit horizontal vector moving according to

$$\frac{d\mathbf{y}}{dt}=\omega\mathbf{z}\wedge\mathbf{y}.$$

Then $\boldsymbol{\rho}.\mathbf{y}=0$, to a sufficient accuracy. From these,

$$\frac{d^2\mathbf{i}}{dt^2}=-\omega^2\mathbf{y}+\frac{d^2\boldsymbol{\rho}}{dt^2}, \qquad \mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}=-\omega^2(\boldsymbol{\rho}\wedge\mathbf{y})+\mathbf{y}\wedge\frac{d^2\boldsymbol{\rho}}{dt^2}.$$

Also $$\mathbf{j}=\frac{\mathbf{z}-\mathbf{i}\sin\theta}{\cos\theta}=\mathbf{z}-\mathbf{i}\theta,$$

where θ is the (small) inclination of the disc to the vertical, and so

$$\mathbf{j}\wedge\mathbf{z}=-\theta(\mathbf{y}\wedge\mathbf{z}).$$

Further, since $\mathbf{z}.d^2\mathbf{j}/dt^2$ and $\mathbf{z}\wedge\mathbf{j}$ are both small, we may in (1) of § 405 neglect this product. Accordingly (1) becomes

$$Mag\theta(\mathbf{y}\wedge\mathbf{z})=A\left[-\omega^2\boldsymbol{\rho}\wedge\mathbf{y}+\mathbf{y}\wedge\frac{d^2\boldsymbol{\rho}}{dt^2}\right].$$

Hence the vector $$\frac{d^2\boldsymbol{\rho}}{dt^2}+\omega^2\boldsymbol{\rho}-\frac{Mga\theta}{A}\mathbf{z}$$

must be zero or parallel to **y**. Since **ρ** and **z** are both perpendicular to **y**, it must be zero. Further, to the order considered, $\theta = \mathbf{i}.\mathbf{z} = \boldsymbol{\rho}.\mathbf{z}$. Hence scalar multiplication of the last equation by **z** yields

$$\ddot{\theta} + \left(\omega^2 - \frac{\mathrm{Mag}}{\mathrm{A}}\right)\theta = 0.$$

Thus the motion is stable if

$$\omega^2 > \frac{\mathrm{Mag}}{\mathrm{A}},$$

and the period of small oscillations is then

$$2\pi\left(\omega^2 - \frac{\mathrm{Mag}}{\mathrm{A}}\right)^{-\frac{1}{2}}.$$

The motion of **i** is now determined, since $\mathbf{i} = \mathbf{y} + \boldsymbol{\rho}$ and $\boldsymbol{\rho} = \theta\mathbf{z}$.

407. *Disc rolling on rough plane. Steady precession.* Let **r** be the position vector of the centre of the circular disc (Fig. 102), G, from a fixed origin. Let **i** be a unit vector normal to the plane of the disc, **j** a unit vector in the line of greatest slope. The angular velocity of the disc is **Ω**, given by

$$\boldsymbol{\Omega} = n\mathbf{i} + \mathbf{i} \wedge \frac{d\mathbf{i}}{dt},$$

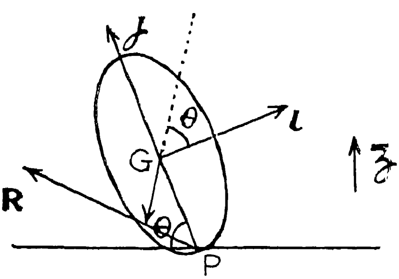

Fig. 102

and the angular momentum **H**(G) about G is given by

$$\mathbf{H}(G) = \mathrm{C}n\mathbf{i} + \mathrm{A}\mathbf{i} \wedge \frac{d\mathbf{i}}{dt}.$$

If **R** is the reaction at the point of contact, P, the equation of linear momentum is

$$\mathbf{R} - \mathrm{M}g\mathbf{z} = \mathrm{M}\frac{d^2\mathbf{r}}{dt^2}, \tag{1}$$

and the equation of angular momentum about G is

$$-a\mathbf{j} \wedge \mathbf{R} = \mathrm{C}\frac{d}{dt}(n\mathbf{i}) + \mathrm{A}\mathbf{i} \wedge \frac{d^2\mathbf{i}}{dt^2}. \tag{2}$$

The condition of rolling contact is

$$\frac{d\mathbf{r}}{dt} + \boldsymbol{\Omega} \wedge (-a\mathbf{j}) = 0, \tag{3}$$

expressing that the velocity of the particle of the disc in contact with the plane is zero. Eliminating **R**, we have

$$-\mathrm{M}a\mathbf{j} \wedge \left(g\mathbf{z} + \frac{d^2\mathbf{r}}{dt^2}\right) = \mathrm{C}\frac{d}{dt}(n\mathbf{i}) + \mathrm{A}\mathbf{i} \wedge \frac{d^2\mathbf{i}}{dt^2}. \tag{4}$$

The vector $\mathbf{r}$ can be eliminated from (4) by means of (3), and then, since

$$\mathbf{j}=\frac{\mathbf{z}-(\mathbf{z}.\mathbf{i})\mathbf{i}}{[1-(\mathbf{i}.\mathbf{z})^2]^{\frac{1}{2}}},$$

the ensuing equation can be put in terms of $\mathbf{i}$ alone.

We shall find simply the condition for steady precession. In steady precession, at the rate ω, we have

$$\frac{d\mathbf{i}}{dt}=\omega\mathbf{z}\wedge\mathbf{i}, \qquad \frac{d^2\mathbf{i}}{dt^2}=\omega^2[-\mathbf{i}+\mathbf{z}(\mathbf{i}.\mathbf{z})].$$

Also, since the particle of the disc in contact with the plane is instantaneously at rest, since the disc is turning about this particle with angular velocity $\mathbf{\Omega}$, and since the position vector of G with respect to this particle is $a\mathbf{j}$, we have (this amounts to (3)),

$$\frac{d\mathbf{r}}{dt}=a\left(n\mathbf{i}+\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}\right)\wedge\mathbf{j}$$

$$=na\mathbf{k}-a\mathbf{i}\left(\mathbf{j}.\frac{d\mathbf{i}}{dt}\right),$$

where $\mathbf{k}=\mathbf{i}\wedge\mathbf{j}$. But in steady precession,

$$\mathbf{j}.\frac{d\mathbf{i}}{dt}=\omega(\mathbf{z}\wedge\mathbf{i}).\mathbf{j}=\omega\mathbf{k}.\mathbf{z}=0.$$

Hence in this case
$$\frac{d^2\mathbf{r}}{dt^2}=a\frac{dn}{dt}\mathbf{k}+na\omega(\mathbf{z}\wedge\mathbf{k})$$

and so
$$\mathbf{j}\wedge\frac{d^2\mathbf{r}}{dt^2}=a\frac{dn}{dt}\mathbf{i}-na\omega(\mathbf{j}.\mathbf{z})\mathbf{k}.$$

The equation of motion (4) now gives

$$-Mag(\mathbf{j}\wedge\mathbf{z})+Ma^2n\omega(\mathbf{j}.\mathbf{z})\mathbf{k}-Ma^2\frac{dn}{dt}\mathbf{i}=C\frac{dn}{dt}\mathbf{i}+Cn\frac{d\mathbf{i}}{dt}+A\omega^2(\mathbf{i}.\mathbf{z})(\mathbf{i}\wedge\mathbf{z}). \quad (5)$$

Scalar multiplication by $\mathbf{i}$ gives at once

$$\frac{dn}{dt}=0, \qquad n=\text{const.}$$

Then, if θ is the inclination of the plane of the disc to the horizontal,

$$\mathbf{z}=\mathbf{i}\cos\theta+\mathbf{j}\sin\theta,$$

$$\mathbf{z}\wedge\mathbf{i}=-\mathbf{k}\sin\theta, \qquad \mathbf{z}\wedge\mathbf{j}=\mathbf{k}\cos\theta,$$

and (5) gives

$$Mag\cos\theta+Ma^2n\omega\sin\theta=-Cn\omega\sin\theta+A\omega^2\cos\theta\sin\theta.$$

To find the radius (say b) of the circle described by the centre of the precessing disc we have

$$\frac{d\mathbf{r}}{dt}=\mathrm{n}a\mathbf{k}=-\omega b\mathbf{k}$$

whence
$$b=-\frac{n}{\omega}a.$$

Hence, in terms of b, the condition of steady precession may be written

$$\omega^2\left[(C+ma^2)\frac{b}{a}+A\cos\theta\right]=Mga\cot\theta.$$

This equation determines the value of ω for any given b and 0.

405. *Disc rolling in a straight line on a rough horizontal plane.* Let the notation be as in the figures (Figs 102 *a*, *b*, *c*.). The disc is supposed

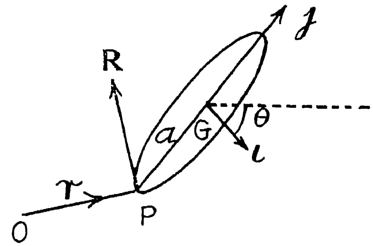

Fig. 102 (*a*)

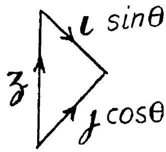

Fig. 102 (*b*)

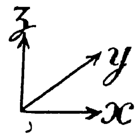

Fig. 102 (*c*)

to be rolling in the y direction. We propose to find the period of small oscillations and the path of the point of contact, **r**.

The condition of rolling contact is

$$\frac{d}{dt}(\mathbf{r}+a\mathbf{j})+\mathbf{\Omega}\wedge(-a\mathbf{j})=0, \tag{1}$$

the equation of rate of change of linear momentum is

$$\mathbf{R}-Mg\mathbf{z}=M\frac{d^2}{dt^2}(\mathbf{r}+a\mathbf{j}), \tag{2}$$

and the equation of rate of change of angular momentum about G is

$$-a\mathbf{j}\wedge\mathbf{R}=\frac{d\mathbf{H}}{dt}, \tag{3}$$

where
$$\mathbf{\Omega}=\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}+n\mathbf{i}, \qquad \mathbf{H}=Cn\mathbf{i}+A\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}.$$

and
$$\mathbf{z}=\mathbf{j}\cos\theta-\mathbf{i}\sin\theta.$$

Eliminating the reaction **R**, we get

$$\frac{d\mathbf{H}}{dt}=-Ma\mathbf{j}\wedge\left[g\mathbf{z}+\frac{d^2}{dt^2}(\mathbf{r}+a\mathbf{j})\right], \tag{4}$$

or, using the condition of rolling,

$$\frac{d\mathbf{H}}{dt} = -Ma\mathbf{j}\wedge\left[g\mathbf{z}+a\frac{d}{dt}(\mathbf{\Omega}\wedge\mathbf{j})\right]. \tag{5}$$

A solution is given by

$$\theta=0, \qquad \mathbf{i}=\mathbf{x}, \qquad \mathbf{j}=\mathbf{z}, \qquad n=n_0,$$

where n_0 is arbitrary. Consider the motion in the vicinity of this steady rolling motion. Put

$$n=n_0+\varphi, \qquad \mathbf{i}=\mathbf{x}+\boldsymbol{\epsilon}, \qquad (\mathbf{x}.\boldsymbol{\epsilon}=0)$$

so that θ, φ and $|\boldsymbol{\epsilon}|$ are small. Thus

$$\theta=-\mathbf{z}.\mathbf{i}=-\boldsymbol{\epsilon}.\mathbf{z}.$$

Then, to a sufficient order,

$$\mathbf{j}=\mathbf{z}+\mathbf{i}\theta=\mathbf{z}+\mathbf{x}\theta, \qquad \mathbf{j}\wedge\mathbf{z}=-\mathbf{y}\theta=\mathbf{y}(\boldsymbol{\epsilon}.\mathbf{z}),$$

$$\mathbf{\Omega}=\mathbf{x}\wedge\frac{d\boldsymbol{\epsilon}}{dt}+n_0\mathbf{x}+n_0\boldsymbol{\epsilon}+\varphi\mathbf{x},$$

$$\mathbf{\Omega}\wedge\mathbf{j}=-\mathbf{x}\left(\frac{d\boldsymbol{\epsilon}}{dt}.\mathbf{z}\right)-n_0\mathbf{y}+n_0(\boldsymbol{\epsilon}\wedge\mathbf{z})-\varphi\mathbf{y},$$

$$\frac{d}{dt}(\mathbf{\Omega}\wedge\mathbf{j})=-\mathbf{x}\left(\frac{d^2\boldsymbol{\epsilon}}{dt^2}.\mathbf{z}\right)+n_0\left(\frac{d\boldsymbol{\epsilon}}{dt}\wedge\mathbf{z}\right)-\frac{d\varphi}{dt}\mathbf{y},$$

$$\mathbf{j}\wedge\frac{d}{dt}(\mathbf{\Omega}\wedge\mathbf{j})=-\mathbf{y}\left(\frac{d^2\boldsymbol{\epsilon}}{dt^2}.\mathbf{z}\right)+\frac{d\varphi}{dt}\mathbf{x}+n_0\left[\frac{d\boldsymbol{\epsilon}}{dt}-\mathbf{z}\left(\frac{d\boldsymbol{\epsilon}}{dt}.\mathbf{z}\right)\right].$$

Also

$$\mathbf{H}=A\mathbf{x}\wedge\frac{d\boldsymbol{\epsilon}}{dt}+C(n_0\mathbf{x}+n_0\boldsymbol{\epsilon}+\varphi\mathbf{x}),$$

$$\frac{d\mathbf{H}}{dt}=A\mathbf{x}\wedge\frac{d^2\boldsymbol{\epsilon}}{dt^2}+Cn_0\frac{d\boldsymbol{\epsilon}}{dt}+C\frac{d\varphi}{dt}\mathbf{x}.$$

Introducing these in (5) we get

$$A\mathbf{x}\wedge\frac{d^2\boldsymbol{\epsilon}}{dt^2}+Cn_0\frac{d\boldsymbol{\epsilon}}{dt}+C\frac{d\varphi}{dt}\mathbf{x}=-Mga(\boldsymbol{\epsilon}.\mathbf{z})\mathbf{y}$$
$$-Ma^2\left[-\mathbf{y}\left(\frac{d^2\boldsymbol{\epsilon}}{dt^2}.\mathbf{z}\right)+\frac{d\varphi}{dt}\mathbf{x}+n_0\frac{d\boldsymbol{\epsilon}}{dt}\right.$$
$$\left.-n_0\mathbf{z}\left(\frac{d\boldsymbol{\epsilon}}{dt}.\mathbf{z}\right)\right].$$

If we now write

$$\boldsymbol{\epsilon}=\eta\mathbf{y}+\zeta\mathbf{z},$$

where η and ζ are small scalars, the last equation becomes

$$A[\ddot{\eta}\mathbf{z}-\ddot{\zeta}\mathbf{y}]+Cn_0[\dot{\eta}\mathbf{y}+\dot{\zeta}\mathbf{z}]+C\dot{\varphi}\mathbf{x}=-Mga\zeta\mathbf{y}-Ma^2[-\ddot{\zeta}\mathbf{y}+\dot{\varphi}\mathbf{x}+n_0\dot{\eta}\mathbf{y}].$$

Taking the coefficients of **x, y, z** in turn, we get

$$(C+Ma^2)\dot{\varphi}=0,$$

$$-(A+Ma^2)\ddot{\zeta}+(C+Ma^2)n_0\dot{\eta}=-Mga\zeta,$$

$$A\ddot{\eta}+Cn_0\dot{\zeta}=0.$$

The last of these integrates at once, giving

$$A\dot{\eta}=-Cn_0\zeta+\text{const.}$$

The first gives $\qquad \varphi=\text{const.}=0.$

The middle equation gives, on substituting for $\dot{\eta}$,

$$-(A+Ma^2)\ddot{\zeta}+(C+Ma^2)n_0\left[-\frac{Cn_0}{A}\zeta+\text{const.}\right]=-Mga\zeta,$$

or

$$\ddot{\zeta}+\zeta\left[\frac{C+Ma^2}{A+Ma^2}\frac{C}{A}n_0{}^2-\frac{Mga}{A+Ma^2}\right]+\text{const.}=0.$$

The constant is clearly zero. We see that the motion is stable provided

$$(C+Ma^2)Cn_0{}^2>MgaA,$$

in which case the period of small oscillations is

$$2\pi\left[\frac{Cn_0{}^2(C+Ma^2)-MgaA}{A(A+Ma^2)}\right]^{-\frac{1}{2}}.$$

E.g., for a hoop for which $C=Ma^2$, $A=\frac{1}{2}Ma^2$, the condition for stability is

$$n_0{}^2>\tfrac{1}{4}g/a,$$

or, putting $n_0=v/a$, $\qquad v^2>\frac{1}{4}ga.$

Path of the point of contact. From the condition of rolling contact,

$$\frac{d\mathbf{r}}{dt}=-a\frac{d\mathbf{j}}{dt}+a\boldsymbol{\Omega}\wedge\mathbf{j}=a\left[-\mathbf{x}\left(\frac{d\boldsymbol{\epsilon}}{dt}.\mathbf{z}\right)+\boldsymbol{\Omega}\wedge\mathbf{j}\right],$$

or, using a previous expression for $\boldsymbol{\Omega}\wedge\mathbf{j}$ and putting $\varphi=0$,

$$\frac{d\mathbf{r}}{dt}=a[-n_0\mathbf{y}+n_0(\boldsymbol{\epsilon}\wedge\mathbf{z})]=-an_0(\mathbf{y}-\eta\mathbf{x}).$$

The term $-an_0\mathbf{y}$ gives the steady rolling motion in a straight line. The term $an_0\eta\mathbf{x}$ gives an oscillatory velocity-component perpendicular to the undisturbed track. The path is therefore a sinuous one, crossing and recrossing the undisturbed track. If the solution of the differential equation for ζ, found above, is

$$\zeta=\zeta_0\cos(\sigma t+\alpha),$$

then

$$\dot{\eta}=-\frac{Cn_0}{A}\zeta_0\cos(\sigma t+\alpha),$$

and

$$\eta=-\frac{Cn_0}{A}\frac{\zeta_0}{\sigma}\sin(\sigma t+\alpha),$$

so that
$$\zeta^2+\eta^2\left(\frac{A\sigma}{Cn_0}\right)^2=\zeta_0^{\,2}.$$

The path giving the trace of the extremity of the axial vector **i** on a vertical plane parallel to the undisturbed motion is therefore an ellipse, for which the ratio
$$\frac{\text{vertical axis}}{\text{horizontal axis}}$$
is equal to
$$A\sigma/Cn_0,$$
i.e. to
$$\frac{A}{Cn_0}\left[\frac{C(C+Ma^2)n_0^{\,2}-MgaA}{A(A+Ma^2)}\right]^{\frac{1}{2}}.$$

409. The methods illustrated in the example of the rolling hoop are further developed in the following example.

A sphere whose mass centre coincides with its geometrical centre, but whose principal moments of inertia at that point are A, A, C, is set rolling along a rough horizontal plane with angular velocity ω_0 in a direction at right angles to the C-axis, which is horizontal. Find the periods of small oscillations about this steady motion (Lamb, *Higher Mechanics*).

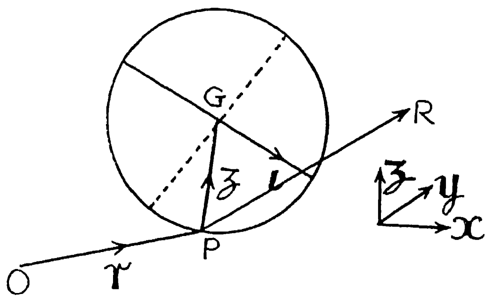

Fig. 103

Let **z** be a unit vector, vertically upwards, **i** a unit vector in the C-axis, *a* the radius, M the mass. Let **r** be the position vector of the point of contact with respect to an origin O (Fig. 103) in the plane. Let $\mathbf{\Omega}$ be the angular velocity of the body. Then
$$\mathbf{\Omega}=\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}+\omega\mathbf{i},$$

where in the steady motion, $\mathbf{i}=\mathbf{x}$, the spin ω takes the value ω_0. The condition of rolling contact is
$$\frac{d}{dt}(\mathbf{r}+a\mathbf{z})+\mathbf{\Omega}\wedge(-a\mathbf{z})=0,$$
i.e.
$$\frac{d\mathbf{r}}{dt}-a\mathbf{\Omega}\wedge\mathbf{z}=0. \tag{1}$$

The equation of linear momentum, if **R** is the reaction, is
$$\mathbf{R}-Mg\mathbf{z}=M\frac{d^2}{dt^2}(\mathbf{r}+a\mathbf{z}), \tag{2}$$

and the equation of angular momentum about G is

$$-a\mathbf{z}\wedge\mathbf{R}=\frac{d}{dt}\left[A\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}+C\omega\mathbf{i}\right]. \tag{3}$$

Eliminating $\mathbf{R}$ we have

$$Ma\left[g\mathbf{z}+\frac{d^2\mathbf{r}}{dt^2}\right]\wedge\mathbf{z}=A\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}+C\frac{d\omega}{dt}\mathbf{i}+C\omega\frac{d\mathbf{i}}{dt},$$

or, eliminating $\mathbf{r}$ by means of (1),

$$Ma^2\left(\frac{d\boldsymbol{\Omega}}{dt}\wedge\mathbf{z}\right)\wedge\mathbf{z}=A\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}+C\frac{d\omega}{dt}\mathbf{i}+C\omega\frac{d\mathbf{i}}{dt}.$$

The left-hand side gives, on introducing the value of $\boldsymbol{\Omega}$,

$$Ma^2\left[-\left(\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}+\frac{d\omega}{dt}\mathbf{i}+\omega\frac{d\mathbf{i}}{dt}\right)+\mathbf{z}\left\{\left(\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}+\frac{d\omega}{dt}\mathbf{i}+\omega\frac{d\mathbf{i}}{dt}\right).\mathbf{z}\right\}\right].$$

Hence the previous equation becomes

$$(A+Ma^2)\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}+(C+Ma^2)\frac{d}{dt}(\omega\mathbf{i})=Ma^2\mathbf{z}\left[\left(\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}+\frac{d(\omega\mathbf{i})}{dt}\right).\mathbf{z}\right].$$

This equation determines the behaviour of $\mathbf{i}$. To solve it in the neighbourhood of $\mathbf{i}=\mathbf{x}$, where $\mathbf{x}$ is a horizontal vector normal to the direction of motion), put

$$\mathbf{i}=\mathbf{x}+\boldsymbol{\epsilon}, \qquad (\boldsymbol{\epsilon}.\mathbf{x}=0)$$

$$\omega=\omega_0+\varphi.$$

Writing $\qquad A'=A+Ma^2, \qquad C'=C+Ma^2,$

we find $\quad A'\mathbf{x}\wedge\dfrac{d^2\boldsymbol{\epsilon}}{dt^2}+C'\left(\omega_0\dfrac{d\boldsymbol{\epsilon}}{dt}+\dot{\varphi}\mathbf{x}\right)=Ma^2\mathbf{z}\left[\mathbf{y}.\dfrac{d^2\boldsymbol{\epsilon}}{dt^2}+\omega_0\dfrac{d\boldsymbol{\epsilon}}{dt}.\mathbf{z}\right].$

Scalar multiplication by $\mathbf{x}$ gives

$$\dot{\varphi}=0, \qquad \varphi=\text{const.}=0,$$

by choice of ω_0. To solve the resulting equation in $\boldsymbol{\epsilon}$, put

$$\boldsymbol{\epsilon}=\eta\mathbf{y}+\zeta\mathbf{z}.$$

Then $\qquad A'[\ddot{\eta}\mathbf{z}-\ddot{\zeta}\mathbf{y}]+C'\omega_0[\dot{\eta}\mathbf{y}+\dot{\zeta}\mathbf{z}]=Ma^2\mathbf{z}[\ddot{\eta}+\omega_0\dot{\zeta}].$

This gives $\qquad A'\ddot{\eta}+C'\omega_0\dot{\zeta}=Ma^2(\ddot{\eta}+\omega_0\dot{\zeta}),$

and $\qquad -A'\ddot{\zeta}+C'\omega_0\dot{\eta}=0.$

The first of this pair of equations for ζ and η gives

$$A\ddot{\eta}=-C\omega_0\dot{\zeta}.$$

Elimination of ζ from the second then gives

$$\frac{AA'}{CC'}\ddot{\eta}+\omega_0^2\dot{\eta}=0.$$

The period is thus $\qquad \dfrac{2\pi}{\omega_0}\left(\dfrac{AA'}{CC'}\right)^{\frac{1}{2}}.$

The path of the point of contact can be found as in the last example.

410. *Rolling solid of revolution.* Still further generality is obtained by considering the case of a rolling solid of revolution, rolling with an equatorial circle in contact with a rough horizontal plane and having its radius of curvature different from that of the meridian sections.

Let G be the centre of mass of a solid which is rolling over a rough horizontal plane, with its axis of symmetry perpendicular to the direction of motion and horizontal (Fig. 104). Let a be the radius of the equatorial section along which rolling occurs, ρ the radius of curvature of any meridian section where it crosses the given equatorial section. Let $\mathbf{z}$ be

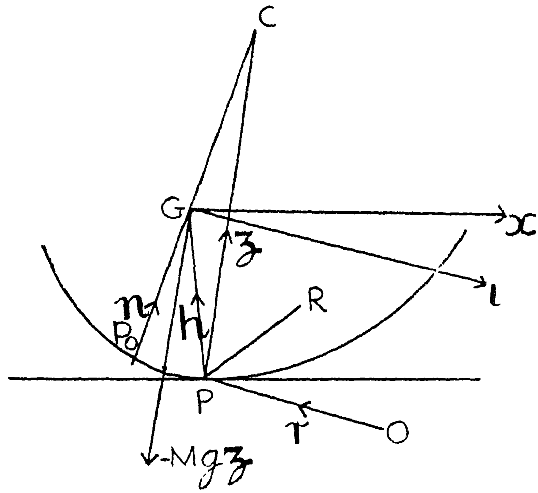

Fig. 104

a unit vector vertically upwards, $\mathbf{x}$ a horizontal unit vector in the direction of the axis of symmetry, $\mathbf{y}=\mathbf{z}\wedge\mathbf{x}$ the direction of motion. In a disturbed position, let $\mathbf{i}$ be the axis, P the point of contact. Then the meridian section through $\mathbf{i}$ and the point of contact P is normal at P to the horizontal plane, and contains G, but does not in general contain $\mathbf{x}$. Let $\mathbf{n}$ be along the normal from G in this meridian section, so that $\mathbf{n}$ is perpendicular to $\mathbf{i}$, and let P_0 be the foot of this normal. Then ρ is the radius of curvature at P_0, and hence the normals at P and P_0, which lie in the meridian plane, intersect in some point C, and approximately $CP=CP_0=\rho$, whilst $P_0G=a$. Consequently, if $\mathbf{h}$ denotes the vector PG,

$$\mathbf{h}=PC+CG=\rho\mathbf{z}-(\rho-a)\mathbf{n}. \tag{1}$$

This is the only curvature relation required.

We now formulate the dynamics. If $\mathbf{\Omega}$ is the angular velocity of the disturbed solid, then

$$\mathbf{\Omega}=\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}+\omega\mathbf{i}, \tag{2}$$

and if $\mathbf{r}$ is the position vector of the point of contact P, the condition of rolling is

$$\frac{d}{dt}(\mathbf{r}+\mathbf{h})+\boldsymbol{\Omega}\wedge(-\mathbf{h})=0. \tag{3}$$

If $\mathbf{R}$ denotes the reaction at P, the equations of linear momentum and of angular momentum about G are

$$\mathbf{R}-Mg\mathbf{z}=M\frac{d^2}{dt^2}(\mathbf{r}+\mathbf{h}), \tag{4}$$

$$-\mathbf{h}\wedge\mathbf{R}=\frac{d}{dt}\left[A\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}+C\omega\mathbf{i}\right]. \tag{5}$$

Eliminating $\mathbf{R}$ and $\mathbf{r}$, we have

$$-M\mathbf{h}\wedge\left[g\mathbf{z}+\frac{d}{dt}(\boldsymbol{\Omega}\wedge\mathbf{h})\right]=A\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}+C\frac{d}{dt}(\omega\mathbf{i}). \tag{6}$$

Since $\mathbf{n}$ is known in terms of $\mathbf{i}$ by the relation

$$\mathbf{n}=\frac{\mathbf{z}-\mathbf{i}(\mathbf{z}.\mathbf{i})}{[1-(\mathbf{z}.\mathbf{i})^2]^{\frac{1}{2}}}, \tag{7}$$

equation (6) determines the behaviour of $\mathbf{i}$, on use of (1) and (2).

For small disturbances from the rectilinear motion

$$\mathbf{i}=\mathbf{x}, \qquad \omega=\text{const.}=\omega_0,$$

put

$$\mathbf{i}=\mathbf{x}+\boldsymbol{\epsilon}, \qquad (\mathbf{x}.\boldsymbol{\epsilon}=0)$$

and

$$\omega=\omega_0+\varphi,$$

where φ and $|\boldsymbol{\epsilon}|$ are small. Then to this approximation

$$\mathbf{n}=\mathbf{z}-\mathbf{x}(\mathbf{z}.\boldsymbol{\epsilon}),$$

$$\mathbf{h}=a\mathbf{z}+(\rho-a)(\mathbf{z}.\boldsymbol{\epsilon})\mathbf{x},$$

$$\boldsymbol{\Omega}=\omega_0\mathbf{x}+\varphi\mathbf{x}+\omega_0\boldsymbol{\epsilon}+\mathbf{x}\wedge\frac{d\boldsymbol{\epsilon}}{dt},$$

$$\boldsymbol{\Omega}\wedge\mathbf{h}=-a\omega_0\mathbf{y}-a\left(\mathbf{z}.\frac{d\boldsymbol{\epsilon}}{dt}\right)\mathbf{x}-a\varphi\mathbf{y}+a\omega_0(\boldsymbol{\epsilon}\wedge\mathbf{z}),$$

$$\mathbf{h}\wedge\frac{d}{dt}(\boldsymbol{\Omega}\wedge\mathbf{h})=-a^2\left(\mathbf{z}.\frac{d^2\boldsymbol{\epsilon}}{dt^2}\right)\mathbf{y}+a^2\dot{\varphi}\mathbf{x}+a^2\omega_0\mathbf{z}\wedge\left(\frac{d\boldsymbol{\epsilon}}{dt}\wedge\mathbf{z}\right).$$

Hence the dynamical equation (6) gives

$$Mg(\rho-a)(\mathbf{z}.\boldsymbol{\epsilon})\mathbf{y}+Ma^2\left(\mathbf{z}.\frac{d^2\boldsymbol{\epsilon}}{dt^2}\right)\mathbf{y}-Ma^2\dot{\varphi}\mathbf{x}-Ma^2\omega_0\mathbf{z}\wedge\left(\frac{d\boldsymbol{\epsilon}}{dt}\wedge\mathbf{z}\right)$$

$$=A\mathbf{x}\wedge\frac{d^2\boldsymbol{\epsilon}}{dt^2}+C\left(\omega_0\frac{d\boldsymbol{\epsilon}}{dt}+\dot{\varphi}\mathbf{x}\right).$$

Scalar multiplication by $\mathbf{x}$ gives at once

$$\dot{\varphi}=0, \qquad \varphi=\text{const.}=0,$$

on choice of ω_0. The preceding equation then determines the behaviour of $\boldsymbol{\epsilon}$. Since $\mathbf{y}$ and $\mathbf{z}$ have different roles, as in the previous problems of this type, it is convenient to put

$$\boldsymbol{\epsilon}=\eta\mathbf{y}+\zeta\mathbf{z},$$

when we obtain

$$Mg(\rho-a)\zeta\mathbf{y}+Ma^2\ddot{\zeta}\mathbf{y}-Ma^2\omega_0\dot{\eta}\mathbf{y}=A[\ddot{\eta}\mathbf{z}-\ddot{\zeta}\mathbf{y}]+C\omega_0[\dot{\eta}\mathbf{y}+\dot{\zeta}\mathbf{z}].$$

Equating the coefficients of $\mathbf{y}$ and $\mathbf{z}$ we get

$$-(A+Ma^2)\ddot{\zeta}+(C+Ma^2)\omega_0\dot{\eta}=Mg(\rho-a)\zeta,$$

$$A\ddot{\eta}+C\omega_0\dot{\zeta}=0.$$

The second of these gives

$$\dot{\eta}=-\frac{C\omega_0}{A}\zeta+\text{const.},$$

whence the first gives

$$(A+Ma^2)\ddot{\zeta}+\left[\frac{C(C+Ma^2)\omega_0{}^2}{A}+Mg(\rho-a)\right]\zeta=\text{const.}$$

These give η and ζ harmonically varying with period $2\pi/\sigma$, where

$$\sigma^2=\frac{C(C+Ma^2)\omega_0{}^2+MgA(\rho-a)}{A(A+Ma^2)}.$$

Path of the point of contact. This is given by

$$\begin{aligned}\frac{d\mathbf{r}}{dt}&=-\frac{d\mathbf{h}}{dt}+\boldsymbol{\Omega}\wedge\mathbf{h}\\
&=-(\rho-a)\left(\mathbf{z}.\frac{d\boldsymbol{\epsilon}}{dt}\right)\mathbf{x}-a\omega_0\mathbf{y}-a\left(\mathbf{z}.\frac{d\boldsymbol{\epsilon}}{dt}\right)\mathbf{x}+a\omega_0(\boldsymbol{\epsilon}\wedge\mathbf{z})\\
&=-\rho\dot{\zeta}\mathbf{x}+a\omega_0\eta\mathbf{x}-a\omega_0\mathbf{y}\\
&=-a\omega_0\mathbf{y}+\mathbf{x}\left[a\omega_0\eta+\frac{\rho A}{\omega_0 C}\ddot{\eta}\right]\\
&=-a\omega_0\mathbf{y}+a\omega_0\mathbf{x}\eta\left[1-\frac{\rho}{a}\frac{(C+Ma^2)+Mg(\rho-a)A/\omega_0{}^2C}{A+Ma^2}\right]+\text{const.}\end{aligned}$$

This determines the excursions of the point of contact perpendicular to the undisturbed path. The actual path is sinusoidal.

We notice that the motion of rolling is always stable provided ω_0 is sufficiently big. The *equilibrium* of the resting solid is, of course, only stable if $\rho>a$. This follows on putting $\omega_0=0$.

411. *Spinning solid of revolution.* We proceed to consider the stability of a solid of revolution spinning about a vertical normal to a rough horizontal plane.

Let A be the vertex of the solid of revolution (Fig. 105), ρ the radius of curvature of a meridian section at A, a the distance from A to the centre of mass G. In the steady motion the solid spins about the normal at A with angular velocity ω_0. In the disturbed motion, let P be the point of contact, $\mathbf{h}$ the vector PG, $\mathbf{i}$ a unit vector along the axis, $\mathbf{R}$ the reaction at P, $\mathbf{r}$ the position vector of P with respect to some fixed origin O. Then if C is the centre of curvature at A, the normal at the neighbouring point P meets the axis in C, and CA$=$CP$=\rho$. Hence

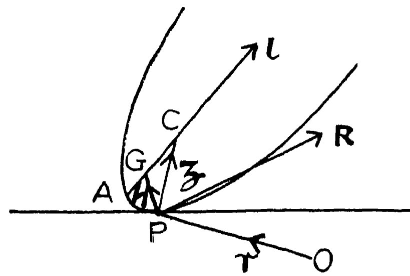

Fig. 105

$$\mathbf{h}=\mathrm{PG}=\mathrm{PC}+\mathrm{CG}=\rho\mathbf{z}-(\rho-a)\mathbf{i}.$$

The condition of rolling is

$$\frac{d}{dt}(\mathbf{r}+\mathbf{h})+\boldsymbol{\Omega}\wedge(-\mathbf{h})=0,$$

where
$$\boldsymbol{\Omega}=\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}+\omega\mathbf{i}.$$

The equation of motion, obtained by the usual process of elimination of $\mathbf{R}$ and $\mathbf{r}$, is

$$\frac{d}{dt}\left[A\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}+C\omega\mathbf{i}\right]=-M\mathbf{h}\wedge\left[g\mathbf{z}+\frac{d}{dt}(\boldsymbol{\Omega}\wedge\mathbf{h})\right]. \tag{1}$$

This determines the behaviour of $\mathbf{i}$.

For a motion near the solution

$$\mathbf{i}=\mathbf{z}, \qquad \omega=\omega_0,$$

put
$$\mathbf{i}=\mathbf{z}+\boldsymbol{\epsilon}, \qquad (\boldsymbol{\epsilon}.\mathbf{z}=0)$$

and
$$\omega=\omega_0+\varphi.$$

Then
$$\boldsymbol{\Omega}=\mathbf{z}\wedge\frac{d\boldsymbol{\epsilon}}{dt}+\omega_0\mathbf{z}+\varphi\mathbf{z}+\omega_0\boldsymbol{\epsilon},$$

$$\mathbf{h}=\rho\mathbf{z}-(\rho-a)(\mathbf{z}+\boldsymbol{\epsilon})=a\mathbf{z}-(\rho-a)\boldsymbol{\epsilon},$$

$$\boldsymbol{\Omega}\wedge\mathbf{h}=a\left(\mathbf{z}\wedge\frac{d\boldsymbol{\epsilon}}{dt}\right)\wedge\mathbf{z}+a\omega_0(\boldsymbol{\epsilon}\wedge\mathbf{z})-\omega_0(\rho-a)(\mathbf{z}\wedge\boldsymbol{\epsilon})$$

$$=a\frac{d\boldsymbol{\epsilon}}{dt}-\rho\omega_0(\mathbf{z}\wedge\boldsymbol{\epsilon}),$$

$$\mathbf{h}\wedge\frac{d}{dt}(\boldsymbol{\Omega}\wedge\mathbf{h})=a^2\left(\mathbf{z}\wedge\frac{d^2\boldsymbol{\epsilon}}{dt^2}\right)+\rho a\omega_0\frac{d\boldsymbol{\epsilon}}{dt}.$$

Hence (1) gives

$$A\mathbf{z}\wedge\frac{d^2\boldsymbol{\epsilon}}{dt^2}+C\left(\omega_0\frac{d\boldsymbol{\epsilon}}{dt}+\dot{\varphi}\mathbf{z}\right)=Mg(\rho-a)(\boldsymbol{\epsilon}\wedge\mathbf{z})-Ma^2\left(\mathbf{z}\wedge\frac{d^2\boldsymbol{\epsilon}}{dt^2}\right)-Ma\rho\,\omega_0\frac{d\boldsymbol{\epsilon}}{dt}.$$

Scalar multiplication by $\mathbf{z}$ gives

$$\dot{\varphi}=0, \qquad \varphi=\text{const.}=0.$$

The equation then becomes

$$(A+Ma^2)\left(\mathbf{z}\wedge\frac{d^2\boldsymbol{\epsilon}}{dt^2}\right)+\omega_0(C+Ma\rho)\frac{d\boldsymbol{\epsilon}}{dt}+Mg(\rho-a)(\mathbf{z}\wedge\boldsymbol{\epsilon})=0,$$

or again, on vector multiplication by $\mathbf{z}$,

$$(A+Ma^2)\frac{d^2\boldsymbol{\epsilon}}{dt^2}+\omega_0(C+Ma\rho)\frac{d\boldsymbol{\epsilon}}{dt}\wedge\mathbf{z}+Mg(\rho-a)\boldsymbol{\epsilon}=0.$$

This has the usual rotational solution of the form

$$\frac{d\boldsymbol{\epsilon}}{dt}=\omega'\mathbf{z}\wedge\boldsymbol{\epsilon},$$

where ω' must satisfy

$$-(A+Ma^2)\omega'^2-(C+Ma\rho)\omega'\omega_0+Mg(\rho-a)=0.$$

This determines the periods of oscillation of the axis. The roots are real, and the motion therefore stable, provided

$$\omega_0{}^2>4\frac{Mg(a-\rho)(A+Ma^2)}{(C+Ma\rho)^2}$$

The resting solid is stable if $\rho>a$, the condition being then satisfied by $\omega_0=0$.

PROBLEMS RELATING TO THE ROLLING OF SPHERES INSIDE OR OUTSIDE OTHER ROUGH SPHERES

412. *Nature of the problems.* A variety of interesting problems arise in the consideration of the rolling and spinning motion of one sphere inside a second larger sphere. The second sphere may be fixed, or free to move about its centre (under the reaction of the first sphere), or driven by external means to rotate at a fixed rate about some fixed axis. Such problems lend themselves very readily to examination by vector methods. Lamb (*Higher Mechanics*) emphasizes the difficulty of treating even the problem of the sphere rolling and spinning under gravity on a fixed sphere. But by vector methods the solutions of all these problems are equally simple.

It is particularly to be noted that in each problem we shall obtain an integral connecting components of spin about the common normal to the surfaces in contact, an integral which is not obvious or readily accessible by the methods of moving axes.

413. *Sphere rolling and spinning under gravity inside a larger rough fixed sphere.* Let a be the radius of the rolling and spinning sphere, M its mass, C its moment of inertia about any axis through its centre. Let **R** (Fig. 106) be the reaction at the point of contact, **i** a unit vector drawn to the point of contact from the centre of the fixed sphere, **z** a unit vector vertically upwards, **Ω** the angular velocity of the moving sphere.

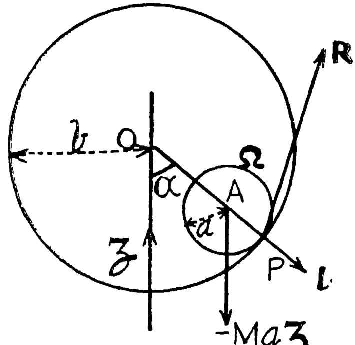

Fig. 106

The angular momentum of the moving sphere about its centre is $C\mathbf{\Omega}$. The position vector of the centre A of the moving sphere with respect to the centre of the fixed sphere is $(b-a)\mathbf{i}$. Hence the equation of linear momentum is

$$M(b-a)\frac{d^2\mathbf{i}}{dt^2}=\mathbf{R}-Mg\mathbf{z}, \qquad (1)$$

the equation of angular momentum about the centre of mass A of the moving sphere is

$$C\frac{d\mathbf{\Omega}}{dt}=a\mathbf{i}\wedge\mathbf{R} \qquad (2)$$

and the condition of rolling contact is

$$(b-a)\frac{d\mathbf{i}}{dt}+\mathbf{\Omega}\wedge a\mathbf{i}=0. \qquad (3)$$

Eliminating **R**, we have

$$C\frac{d\Omega}{dt}=Ma\mathbf{i}\wedge\left[g\mathbf{z}+(b-a)\frac{d^2\mathbf{i}}{dt^2}\right]. \qquad (4)$$

414. At this stage the next important step in all problems of this class is to obtain the integral of spin about the common normal. The complexity of the problem depends almost wholly on the complexity of this integral. The integral in question is built up in essentially the same way in each problem of the class, but the details differ. The necessary procedure is, however, fully illustrated in the present simple example.

415. From the equation of motion (4) we have on scalar multiplication by **i**,

$$\frac{d\mathbf{\Omega}}{dt}.\mathbf{i}=0. \qquad (5)$$

(This follows also from (2).) But from (3), the condition of rolling contact, on scalar multiplication by **Ω** we have

$$\mathbf{\Omega}.\frac{d\mathbf{i}}{dt}=0. \qquad (6)$$

From (5) and (6) on addition,

$$\frac{d}{dt}(\mathbf{\Omega}.\mathbf{i})=0,$$

whence $$\mathbf{\Omega}.\mathbf{i}=\text{const.}=n, \tag{7}$$

say. This is the desired integral.

Multiplying the condition of rolling contact, (3), vectorially by $\mathbf{i}$, we have

$$a[-\mathbf{\Omega}+n\mathbf{i}]=(b-a)\mathbf{i}\wedge\frac{d\mathbf{i}}{dt},$$

whence * $$\mathbf{\Omega}=n\mathbf{i}-\frac{b-a}{a}\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}. \tag{8}$$

We now insert this in (4), obtaining

$$Cn\frac{d\mathbf{i}}{dt}-C\frac{b-a}{a}\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}=Mga(\mathbf{i}\wedge\mathbf{z})+Ma(b-a)\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2},$$

or $$(C+Ma^2)\frac{b-a}{a}\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}-Cn\frac{d\mathbf{i}}{dt}+Mga(\mathbf{i}\wedge\mathbf{z})=0. \tag{9}$$

Equation (9) is, however, precisely the equation of motion of a top, equation (3), § 396, with $\frac{b-a}{a}(C+Ma^2)$ written for A, $-n$ written for n and a for h. The whole theory of the top can therefore be applied as it stands without modification. In particular, we can obtain the condition of steady precession, the motion for $\mathbf{i}$ nearly vertical, and the small oscillations of a top near a state of steady precession. The integrals of energy and of angular momentum about the vertical follow on scalar multiplication by $\mathbf{i}\wedge d\mathbf{i}/dt$ and $\mathbf{z}$ respectively.

For example, to obtain the condition for steady precession, we simply seek a solution of (9) in which

$$\frac{d\mathbf{i}}{dt}=\omega\mathbf{z}\wedge\mathbf{i}, \qquad \mathbf{z}.\mathbf{i}=\text{const.}=-\cos\alpha,$$

$$\frac{d^2\mathbf{i}}{dt^2}=\omega^2\mathbf{z}\wedge(\mathbf{z}\wedge\mathbf{i})=\omega^2[-\mathbf{i}-\mathbf{z}\cos\alpha].$$

* It is essential that the student should distinguish between this formula for Ω and the formula we have so often previously employed, namely

$$\mathbf{\Omega}=n\mathbf{i}+\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}.$$

In all cases of the latter, $\mathbf{i}$ is a vector fixed *in the body*. In the present case of the rolling and spinning sphere, $\mathbf{i}$ is not fixed in the moving sphere, but has a locus in it ; $\mathbf{i}$ is merely defined as drawn from the centre to the point of contact. In the case of a top or other body possessing an axis of symmetry, $\mathbf{i}$ is along the axis of symmetry, and is fixed in the body.

Then (9) reduces to

$$\left[-(C+Ma^2)\frac{b-a}{a}\omega^2 \cos\alpha+Cn\omega+Mga\right](\mathbf{i}\wedge\mathbf{z})=0,$$

and ω must satisfy

$$-(C+Ma^2)\frac{b-a}{a}\omega^2 \cos\alpha+Cn\omega+Mga=0. \tag{10}$$

For given α, the roots are real provided

$$C^2n^2\omega^2+4Mg(b-a)(C+Ma^2)\cos\alpha>0.$$

However, a further condition to be satisfied is that the reaction **R** must have a normal component directed towards the moving sphere, i.e. we must have

$$\mathbf{R}.(-\mathbf{i})>0.$$

By (1), this requires that

$$\left[g\mathbf{z}+(b-a)\frac{d^2\mathbf{i}}{dt^2}\right].\mathbf{i}<0,$$

i.e. in steady precession,

$$-g\cos\alpha+(b-a)\omega^2(-1+\cos^2\alpha)<0,$$

i.e.
$$(b-a)\omega^2\sin^2\alpha+g\cos\alpha>0.$$

This is certainly satisfied for all $\cos\alpha>0$; the student should consider the case $\cos\alpha<0$ separately.

Example. A sphere is rolling and spinning under gravity on the outside of a fixed rough sphere. With the notation of § 413, show that the equation of motion is

$$(C+Ma^2)\frac{b+a}{a}\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}+Cn\frac{d\mathbf{i}}{dt}+Mga(\mathbf{i}\wedge\mathbf{z})=0,$$

i.e. the same as (9) with $-a$ replacing a. If the sphere is spinning about a vertical axis at the highest point of the fixed sphere, show that the condition of stability is

$$C^2n^2>4Mg(a+b)(C+Ma^2).$$

Show also that the condition of steady precession at the rate ω with the line of normals describing a cone of generators at an angle α to the upward vertical is

$$(C+Ma^2)\frac{b+a}{a}\omega^2\cos\alpha-Cn\omega+Mga=0,$$

and that the condition for a pressure reaction is

$$g \cos \alpha > (b+a)\omega^2 \sin^2 \alpha.$$

416. The particular case of the sphere rolling and spinning inside the fixed sphere under the action of no forces is obtained by putting $g=0$ in §§ 413–415. Equation (9) for $\mathbf{i}$ becomes at once integrable, giving

$$(C+Ma^2)\frac{b-a}{a}\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}-Cn\mathbf{i}=\text{const.}=Cn\mathbf{i}_0,$$

say. Multiplying vectorially by $\mathbf{i}$ we have

$$(C+Ma^2)\frac{b-a}{a}\frac{d\mathbf{i}}{dt}=Cn\mathbf{i}_0\wedge\mathbf{i}.$$

This equation asserts that the motion of $\mathbf{i}$ is one of uniform rotation about the axis $\mathbf{i}_0$ with angular velocity

$$\frac{Cna}{(C+Ma^2)(b-a)}.$$

417. *Sphere rolling and spinning inside a rough sphere free to move about its centre, the whole system being under gravity.*

The figure and notation of § 413 will serve, but in addition we now let $\mathbf{\Omega}'$ be the angular velocity of the outer sphere, $C'\mathbf{\Omega}'$ its angular momentum about its centre. The equations of linear and angular momentum for the smaller sphere are as before

$$\mathbf{R}-Mg\mathbf{z}=M(b-a)\frac{d^2\mathbf{i}}{dt^2}, \tag{1}$$

$$a\mathbf{i}\wedge\mathbf{R}=C\frac{d\mathbf{\Omega}}{dt}. \tag{2}$$

The equation of angular momentum for the outer sphere is

$$b\mathbf{i}\wedge(-\mathbf{R})=C'\frac{d\mathbf{\Omega}'}{dt} \tag{3}$$

and the condition of rolling is altered to

$$(b-a)\frac{d\mathbf{i}}{dt}+\mathbf{\Omega}\wedge a\mathbf{i}=\mathbf{\Omega}'\wedge b\mathbf{i}. \tag{4}$$

As previously explained, we first seek the spin integral about the common normal. Equations (2) and (3) give on scalar multiplication by $\mathbf{i}$

$$\mathbf{i}.\frac{d\mathbf{\Omega}}{dt}=0, \qquad \mathbf{i}.\frac{d\mathbf{\Omega}'}{dt}=0. \tag{5), (6}$$

Equation (4), written in the form

$$(\mathrm{b}\boldsymbol{\Omega}'-\mathrm{a}\boldsymbol{\Omega})\wedge\mathbf{i}=(\mathrm{b}-\mathrm{a})\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}, \tag{7}$$

gives on scalar multiplication by $(\mathrm{b}\boldsymbol{\Omega}'-\mathrm{a}\boldsymbol{\Omega})$,

$$(\mathrm{b}\boldsymbol{\Omega}'-\mathrm{a}\boldsymbol{\Omega}).\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}=0. \tag{8}$$

Combination of (5), (6) and (8) gives

$$\frac{\mathrm{d}}{\mathrm{dt}}[(\mathrm{b}\boldsymbol{\Omega}'-\mathrm{a}\boldsymbol{\Omega}).\mathbf{i}]=0,$$

or

$$(\mathrm{b}\boldsymbol{\Omega}'-\mathrm{a}\boldsymbol{\Omega}).\mathbf{i}=\text{const.}=\mathrm{n}(\mathrm{b}-\mathrm{a}), \tag{9}$$

say. Equation (7), on vector multiplication by $\mathbf{i}$, then gives

$$-(\mathrm{b}\boldsymbol{\Omega}'-\mathrm{a}\boldsymbol{\Omega})+[(\mathrm{b}\boldsymbol{\Omega}'-\mathrm{a}\boldsymbol{\Omega}).\mathbf{i}]\mathbf{i}=(\mathrm{b}-\mathrm{a})\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}\wedge\mathbf{i},$$

or

$$\mathrm{b}\boldsymbol{\Omega}'-\mathrm{a}\boldsymbol{\Omega}=\mathrm{n}(\mathrm{b}-\mathrm{a})\mathbf{i}+(\mathrm{b}-\mathrm{a})\mathbf{i}\wedge\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}. \tag{10}$$

418. It should be observed that this relation does not involve the inertia constants of the spheres. Nevertheless, (10) is not a purely kinematic relation, nor is (9) a purely kinematic relation, for neither can be obtained from the condition of rolling contact, (4), alone.

419. Elimination of $\mathbf{i}\wedge\mathbf{R}$ between (2) and (3) gives a relation which integrates at once in the form

$$\frac{\mathrm{C}}{\mathrm{a}}\boldsymbol{\Omega}+\frac{\mathrm{C}'}{\mathrm{b}}\boldsymbol{\Omega}'=\text{const.}=\mathrm{M}\mathbf{X}(\mathrm{b}-\mathrm{a}), \tag{11}$$

say, where $\mathbf{X}$ is an arbitrary vector constant. This relation (11), combined with (10), gives a pair which can be solved for $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}'$ in the form

$$\boldsymbol{\Omega}\left[\frac{\mathrm{C}}{\mathrm{a}^2}+\frac{\mathrm{C}'}{\mathrm{b}^2}\right]=\frac{\mathrm{b}-\mathrm{a}}{\mathrm{a}}\left[\mathrm{M}\mathbf{X}-\frac{\mathrm{C}'}{\mathrm{b}^2}\mathrm{n}\mathbf{i}-\frac{\mathrm{C}'}{\mathrm{b}^2}\mathbf{i}\wedge\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}\right], \tag{12}$$

$$\boldsymbol{\Omega}'\left[\frac{\mathrm{C}}{\mathrm{a}^2}+\frac{\mathrm{C}'}{\mathrm{b}^2}\right]=\frac{\mathrm{b}-\mathrm{a}}{\mathrm{b}}\left[\mathrm{M}\mathbf{X}+\frac{\mathrm{C}}{\mathrm{a}^2}\mathrm{n}\mathbf{i}+\frac{\mathrm{C}}{\mathrm{a}^2}\mathbf{i}\wedge\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}\right]. \tag{13}$$

Substitution of $\mathbf{R}$ from (1) in (2) and (3) gives equations for $\mathrm{d}\boldsymbol{\Omega}/\mathrm{dt}$ and $\mathrm{d}\boldsymbol{\Omega}'/\mathrm{dt}$ into which (12) and (13) may be substituted. The result is an equation for $\mathbf{i}$ which comes to

$$(\mathrm{b}-\mathrm{a})(\mathrm{M}+\mathrm{M}_1)\mathbf{i}\wedge\frac{\mathrm{d}^2\mathbf{i}}{\mathrm{dt}^2}+\mathrm{M}_1(\mathrm{b}-\mathrm{a})\mathrm{n}\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}}+\mathrm{Mg}\mathbf{i}\wedge\mathbf{z}=0, \tag{14}$$

where

$$\mathrm{M}_1=\frac{\mathrm{C}}{\mathrm{a}^2}\frac{\mathrm{C}'}{\mathrm{b}^2}\left[\frac{\mathrm{C}}{\mathrm{a}^2}+\frac{\mathrm{C}'}{\mathrm{b}^2}\right]^{-1}.$$

This again is the equation of motion for the axis of a top, on suitable identifications of constants; (the spin n has different meanings in the two cases). When the mass of the hollow sphere tends to infinity, so that this sphere remains fixed, $M_1 \to C/a^2$, and the form of (14) tends to that of (9), § 415 on paying attention to the changed meaning of n. The condition of steady precession, etc., can be obtained at once.

420. *Sphere rolling and spinning inside a rough hollow sphere compelled by external means to rotate at a given rate about its centre which is fixed.* We employ the notation of § 417, but now $\mathbf{\Omega}'$ denotes a given vector, namely the given (constant) angular velocity of the hollow sphere. Equations (1), (2), (4) of § 417 define the motion, and we seek the new integral of spin. From these equations

$$\mathbf{i}.\frac{d\mathbf{\Omega}}{dt}=0, \qquad (b\mathbf{\Omega}'-a\mathbf{\Omega}).\frac{d\mathbf{i}}{dt}=0,$$

whence since $\mathbf{\Omega}'$ is constant, these yield the integral

$$(b\mathbf{\Omega}'-a\mathbf{\Omega}).\mathbf{i}=\text{const.}=(b-a)n, \tag{1}$$

say, as before. We find similarly

$$b\mathbf{\Omega}'-a\mathbf{\Omega}=(b-a)\left[n\mathbf{i}+\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}\right]. \tag{2}$$

We no longer have an equation of type (3), § 417, but instead we know that $\mathbf{\Omega}'$ is constant. Eliminating $\mathbf{R}$ and substituting for $\mathbf{\Omega}$ from (2), we find

$$Ma\mathbf{i}\wedge\left[g\mathbf{z}+(b-a)\frac{d^2\mathbf{i}}{dt^2}\right]=-C\frac{b-a}{a}\left[n\frac{d\mathbf{i}}{dt}+\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}\right],$$

or

$$(C+Ma^2)\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}+Cn\frac{d\mathbf{i}}{dt}+Mg\frac{a^2}{b-a}\mathbf{i}\wedge\mathbf{z}=0. \tag{3}$$

As usual, this is of the form of the equation for the motion of the axis of a top. It contains no apparent reference to the imposed angular velocity $\mathbf{\Omega}'$ of the outer sphere, but it must be remembered, in comparing it with (9) of § 415 that 'n' is here defined by (1) or (2) above. If we put $\mathbf{\Omega}'=0$, then the 'n' of (3) above must be divided by $-(b-a)/a$ to yield (9) of § 415.

We again notice that (1) is not a purely kinematic integral, although it contains no inertia constants. It and its analogues are seen to be consequences of the circumstance that the external forces have zero moment about the line of normals $\mathbf{i}$, but as this line is not fixed in space we cannot assert that the angular momentum about it is constant. It appears probable that one could formulate some general theorem of which the foregoing integrals of spin are particular cases.

421. *Sphere rolling and spinning inside a fixed rough vertical cylinder.* Let $\mathbf{z}$ be a unit vector vertically upwards. Let $\mathbf{i}$ be a unit horizontal

vector drawn from the centre G of the sphere to the point of contact (Fig. 107). Let a be the radius of the sphere, b the radius of the cylinder, ζ the height of G above some fixed level. Let M be the mass of the sphere, C its moment of inertia about any diameter, $\mathbf{\Omega}$ its angular velocity, $\mathbf{R}$ the reaction at the point of contact.

Then the position vector of the centre G of the sphere may be taken as

$$\zeta\mathbf{z}+(\mathrm{b}-\mathrm{a})\mathbf{i},$$

and consequently the condition of rolling contact is

$$\dot{\zeta}\mathbf{z}+(\mathrm{b}-\mathrm{a})\frac{d\mathbf{i}}{dt}+\mathbf{\Omega}\wedge \mathrm{a}\mathbf{i}=0. \qquad (1)$$

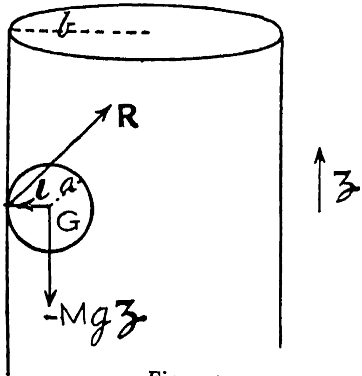

Fig. 107

The equations of linear and angular momentum are

$$\mathbf{R}-\mathrm{M}g\mathbf{z}=\mathrm{M}\left[\ddot{\zeta}\mathbf{z}+(\mathrm{b}-\mathrm{a})\frac{d^2\mathbf{i}}{dt^2}\right], \qquad (2)$$

$$\mathrm{a}\mathbf{i}\wedge\mathbf{R}=\mathrm{C}\frac{d\mathbf{\Omega}}{dt}. \qquad (3)$$

Eliminating $\mathbf{R}$ we have

$$\mathrm{C}\frac{d\mathbf{\Omega}}{dt}=\mathrm{Ma}\mathbf{i}\wedge\left[(g+\ddot{\zeta})\mathbf{z}+(\mathrm{b}-\mathrm{a})\frac{d^2\mathbf{i}}{dt^2}\right]. \qquad (4)$$

It can be seen that in this case no simple spin integral exists.* The following procedure suggests itself.

Let n be the component of $\mathbf{\Omega}$ about $\mathbf{i}$. Then we may write

$$\mathbf{\Omega}=\mathrm{n}\mathbf{i}+\mathbf{\Omega}'. \qquad (\mathbf{\Omega}'.\mathbf{i}=0) \quad (5)$$

Hence, multiplying (1) vectorially by $\mathbf{i}$,

$$\dot{\zeta}(\mathbf{i}\wedge\mathbf{z})+(\mathrm{b}-\mathrm{a})\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}+\mathrm{a}\mathbf{\Omega}'=0, \qquad (6)$$

from which $\mathbf{\Omega}'$ can be substituted into (4), giving an equation for $\mathbf{i}$ and ζ. As $\mathbf{i}$ is a unit vector rotating round $\mathbf{z}$, it will possess some (not necessarily constant) angular velocity ω. Put then

$$\frac{d\mathbf{i}}{dt}=\omega\mathbf{z}\wedge\mathbf{i}, \qquad (\mathbf{z}.\mathbf{i}=0) \quad (7)$$

so that

$$\frac{d^2\mathbf{i}}{dt^2}=-\omega^2\mathbf{i}+\dot{\omega}(\mathbf{z}\wedge\mathbf{i}),$$

$$\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}=\omega\mathbf{z}, \qquad \mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}=\dot{\omega}\mathbf{z},$$

* Equation (4) gives $\mathbf{i}.d\mathbf{\Omega}/dt=0$, but (1) gives $\dot{\zeta}\,(\mathbf{z}.\mathbf{\Omega})+(\mathrm{b}-\mathrm{a})\mathbf{\Omega}.d\mathbf{i}/dt=0$, showing that $\mathbf{\Omega}\,d\mathbf{i}/dt\neq 0$, so that $\mathbf{\Omega}.\mathbf{i}$ is not constant.

and
$$\frac{d\mathbf{i}}{dt}\wedge\mathbf{z}=\omega\mathbf{i}.$$

Hence, by (5) and (6),
$$\frac{d\boldsymbol{\Omega}}{dt}=\frac{dn}{dt}\mathbf{i}+n\omega\mathbf{z}\wedge\mathbf{i}-\frac{1}{a}\left[\frac{d^2\zeta}{dt^2}(\mathbf{i}\wedge\mathbf{z})+\frac{d\zeta}{dt}\omega\mathbf{i}+(b-a)\frac{d\omega}{dt}\mathbf{z}\right].$$

Substitution of these relations into (4) gives a linear relation between the three independent vectors **z**, **i** and **z**$\wedge$**i**. This requires the three conditions

coeff. of **i**: $$\frac{dn}{dt}-\frac{\omega}{a}\frac{d\zeta}{dt}=0, \tag{8}$$

coeff. of **z**: $$\frac{d\omega}{dt}=0, \tag{9}$$

coeff. of **z**$\wedge$**i**: $$Cn\omega+\frac{1}{a}(C+Ma^2)\frac{d^2\zeta}{dt^2}+Mga=0. \tag{10}$$

Relation (9) gives $\omega=\text{const.}$

Relation (8) then gives
$$n=\frac{\omega}{a}\zeta+\text{const.}=\frac{\omega}{a}\zeta+n_0,$$

on suitable choice of the origin of ζ. Relation (10) then gives
$$\left(1+\frac{Ma^2}{C}\right)\frac{d^2\zeta}{dt^2}+\omega^2\zeta+n_0\omega a+\frac{Ma^2}{C}g=0.$$

This equation shows that the height ζ undergoes simple harmonic motion of period
$$\frac{2\pi}{\omega}\left(1+\frac{Ma^2}{C}\right)^{\frac{1}{2}}.$$

The solution in ζ is clearly
$$\zeta=-\left[\frac{n_0a}{\omega}+\frac{Ma^2}{C}\frac{g}{\omega^2}\right]+\lambda\cos\left[\left(\frac{1}{1+Ma^2/C}\right)^{\frac{1}{2}}\omega t+\varepsilon\right]$$

where λ is arbitrary. The maximum distance through which the centre G falls depends on λ, whose value depends on the initial value of $\dot{\zeta}$. Suppose, for example, that at $t=0$, we have $\zeta=0$, $\dot{\zeta}=0$, $n=n_0$. Then the solution is
$$\zeta=-\left[\frac{n_0a}{\omega}+\frac{Ma^2}{C}\frac{g}{\omega^2}\right]\left[1-\cos\left(\frac{1}{1+Ma^2/C}\right)^{\frac{1}{2}}\omega t\right],$$

and the centre of the sphere drops a total distance
$$2\left(\frac{n_0a}{\omega}+\frac{Ma^2}{C}\frac{g}{\omega^2}\right).$$

This is smaller the greater is ω; ω measures the constant circumferential velocity.*

422. *Sphere rolling and spinning inside a rough vertical cylinder free to rotate about its own axis.* Let C′ be the moment of inertia of the cylinder about its axis, $\omega'\mathbf{z}$ its angular velocity. Then, using the notation of § 421, the new condition of rolling is

$$\dot{\zeta}\mathbf{z}+(\mathrm{b}-\mathrm{a})\frac{d\mathbf{i}}{dt}+\boldsymbol{\Omega}\wedge \mathrm{a}\mathbf{i}=\omega'\mathbf{z}\wedge \mathrm{b}\mathbf{i}. \tag{1}$$

The cylinder has for its equation of motion

$$(\mathrm{b}\mathbf{i}\wedge -\mathbf{R}).\mathbf{z}=\mathrm{C}'\frac{d\omega'}{dt},$$

whence, substituting for $\mathbf{R}$ from (2), § 421, we get

$$\mathrm{Mb}(\mathrm{b}-\mathrm{a})\frac{d^2\mathbf{i}}{dt^2}\wedge\mathbf{i}.\mathbf{z}=\mathrm{C}'\frac{d\omega'}{dt}.$$

Replacing $d^2\mathbf{i}/dt^2$ in terms of ω as in § 421, we find

$$-\mathrm{Mb}(\mathrm{b}-\mathrm{a})\frac{d\omega}{dt}=\mathrm{C}'\frac{d\omega'}{dt}$$

whence $$\omega'=-\frac{\mathrm{Mb}(\mathrm{b}-\mathrm{a})}{\mathrm{C}'}\omega+\text{const.}$$

On multiplying (1) vectorially by $\mathbf{i}$ we get an equation for $\boldsymbol{\Omega}'$ involving $\omega'\mathbf{z}$, and on differentiating to get $d\boldsymbol{\Omega}/dt$ we are led to an equation of the same type as in § 421, modified by the addition of a term in $\mathbf{z}\,d\omega'/dt$. As this is a multiple of $d\omega/dt$, we are led as before to

$$\omega=0, \qquad \omega=\text{const.},$$

whence also $$\omega'=\text{const.}$$

The period is readily found to be unaltered.

423. *Sphere rolling and spinning inside a rough vertical cylinder compelled to rotate at a given rate about its axis.* This is left as an exercise for the student.

424. *Sphere rolling and spinning on the surface of a fixed rough horizontal cylinder.* Let a be the radius of the sphere, b that of the cylinder, $\mathbf{z}$ a unit vector vertically upwards, $\mathbf{x}$ a unit vector along the axis of the cylinder, $\mathbf{i}$ a unit vector drawn from the point of contact towards the centre of the sphere. Let $\mathbf{R}$ be the reaction at the point of contact, $\boldsymbol{\Omega}$ the angular velocity of the rolling and spinning sphere (Fig. 108). Then, if ξ measures

* As a well-known mathematician used to point out in conversation, the solution of this problem explains why a golf-ball sometimes spins into and then out of the ‘ hole ’ —a ‘ hole ’ being, of course, essentially a rough vertical hollow cylinder.

the displacement of the centre of the moving sphere reckoned horizontally, this centre G may be taken to have a position vector

$$\xi\mathbf{x}+(a+b)\mathbf{i},$$

whence the condition of rolling contact is

$$\frac{d\xi}{dt}\mathbf{x}+(a+b)\frac{d\mathbf{i}}{dt}+\boldsymbol{\Omega}\wedge(-a\mathbf{i})=0. \tag{1}$$

The equations of linear and angular momentum are

$$\mathbf{R}-Mg\mathbf{z}=M\left[\frac{d^2\xi}{dt^2}\mathbf{x}+(a+b)\frac{d^2\mathbf{i}}{dt^2}\right], \tag{2}$$

$$-a\mathbf{i}\wedge\mathbf{R}=C\frac{d\boldsymbol{\Omega}}{dt}, \tag{3}$$

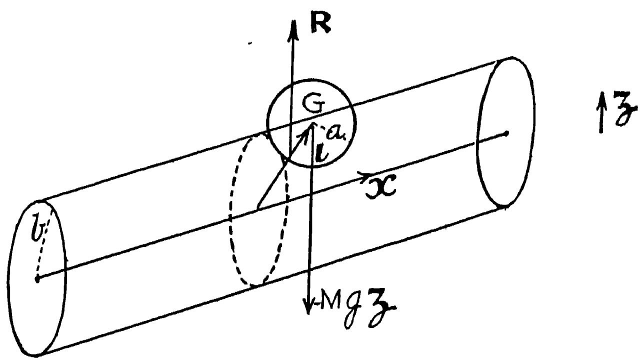

Fig. 108

whence, by elimination of **R**,

$$-Ma\mathbf{i}\wedge\left[g\mathbf{z}+\frac{d^2\xi}{dt^2}\mathbf{x}+(a+b)\frac{d^2\mathbf{i}}{dt^2}\right]=C\frac{d\boldsymbol{\Omega}}{dt}. \tag{4}$$

Putting $\boldsymbol{\Omega}=n\mathbf{i}+\boldsymbol{\Omega}'$, $(\boldsymbol{\Omega}'.\mathbf{i}=0)$

we find, on multiplying (1) vectorially by **i**,

$$\boldsymbol{\Omega}'=\frac{1}{a}\left[\frac{d\xi}{dt}\mathbf{i}\wedge\mathbf{x}+(a+b)\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}\right]. \tag{5}$$

Since the vector **i** rotates round **x**, we may put

$$\frac{d\mathbf{i}}{dt}=\omega\mathbf{x}\wedge\mathbf{i}, \qquad (\mathbf{x}.\mathbf{i}=0) \tag{6}$$

where ω is a function of t to be determined. Then, as usual,

$$\frac{d^2\mathbf{i}}{dt^2}=\frac{d\omega}{dt}(\mathbf{x}\wedge\mathbf{i})-\omega^2\mathbf{i}, \qquad \mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}=\frac{d\omega}{dt}\mathbf{x},$$

$$\frac{d\mathbf{i}}{dt}\wedge\mathbf{x}=\omega\mathbf{i}.$$

Substituting for $\mathbf{\Omega}$ in (4) we get.

$$-\mathrm{Ma}\left[g(\mathbf{i}\wedge\mathbf{z})+\frac{d^2\xi}{dt^2}(\mathbf{i}\wedge\mathbf{x})+(a+b)\frac{d\omega}{dt}\mathbf{x}\right]$$

$$=C\left[\frac{dn}{dt}\mathbf{i}+n\omega(\mathbf{x}\wedge\mathbf{i})+\frac{1}{a}\frac{d^2\xi}{dt^2}(\mathbf{i}\wedge\mathbf{x})+\frac{1}{a}\frac{d\xi}{dt}\omega\mathbf{i}+\frac{a+b}{a}\frac{d\omega}{dt}\mathbf{x}\right].$$

In this we may write $\mathbf{z}\wedge\mathbf{i}=\mathbf{x}\sin\theta$, where $\omega=\dot{\theta}$. Equating the coefficients of $\mathbf{i}$, $\mathbf{x}$ and $\mathbf{i}\wedge\mathbf{x}$ in succession we have

coeff. of $\mathbf{i}$:
$$\frac{dn}{dt}+\frac{1}{a}\frac{d\xi}{dt}\dot{\theta}=0, \tag{7}$$

coeff. of $\mathbf{x}$:
$$(a+b)(C+Ma^2)\ddot{\theta}=Ma^2g\sin\theta, \tag{8}$$

coeff. of $\mathbf{i}\wedge\mathbf{x}$:
$$\frac{d^2\xi}{dt^2}(C+Ma^2)=Cn\dot{\theta}a. \tag{9}$$

Substituting for $\dot{\theta}$ from (7) in (9), we get an integrable relation which gives

$$\tfrac{1}{2}\left(1+\frac{Ma^2}{C}\right)\left(\frac{d\xi}{dt}\right)^2+\tfrac{1}{2}a^2n^2=\text{const.}=\tfrac{1}{2}a^2n_0^2, \tag{10}$$

say.

Relation (8) shows that the behaviour of θ, i.e. the motion of rolling *down* the curved surface of the cylinder, is independent of the spin component n. Relation (10) shows that the velocity of the sphere *along* the cylinder depends only on the spin n, and that the kinetic energy of the motion along the cylinder is entirely derived from the spin, if $n=n_0$ when $d\xi/dt=0$.

If $n\neq 0$, a rolling motion straight *down* the cylinder is impossible ; for were $d\xi/dt=0$ (or even $d\xi/dt=$const.) to be a possible motion, (9) would require $n\dot{\theta}=0$, and since $\dot{\theta}$ could not be steadily zero, we should require $n=0$.

Contact is broken when $\mathbf{R}.\mathbf{i}=0$, i.e. by (2) when

$$g(\mathbf{z}.\mathbf{i})+(a+b)\frac{d^2\mathbf{i}}{dt^2}.\mathbf{i}=0,$$

i.e. when
$$g\cos\theta-(a+b)\dot{\theta}^2=0.$$

A first integral of (8) can now be used to obtain an equation whose root is the value of θ at which contact is broken.

425. *Motion of a sphere rolling and spinning on a rough horizontal cylinder free to rotate about its axis.* In the notation of the last section, if $\omega_1\mathbf{x}$ is the angular velocity of the cylinder at any moment, C_1 its moment of inertia about its axis, the equation of motion of the cylinder is

$$b\mathbf{i}\wedge(-\mathbf{R}).\mathbf{x}=C_1\frac{d\omega_1}{dt}, \tag{1}$$

and the condition of rolling contact becomes

$$\frac{d\xi}{dt}\mathbf{x}+(a+b)\frac{d\mathbf{i}}{dt}+\mathbf{\Omega}\wedge(-a\mathbf{i})=\omega_1\mathbf{x}\wedge b\mathbf{i}. \tag{2}$$

The equations of motion (2) and (3) of § 424 are unaltered. Substituting for $\mathbf{R}$ in (1) of the present section we get

$$C_1\frac{d\omega_1}{dt}=-Mb[-g\sin\theta+(a+b)\ddot{\theta}].$$

The rolling condition (2) then determines $\mathbf{\Omega}$ in terms of $\mathbf{i}$ and ω_1, and the equations analogous to (7), (8), (9) of § 424 can be found as before. The details are left to the reader.

426. *Motion of a sphere rolling and spinning inside a fixed rough right circular cone with its axis vertical.* Let α be the semi-vertical angle of the cone, O its vertex, P the point of contact, G the centre of the sphere, $\mathbf{r}$ the vector OP, $\mathbf{z}$ a unit vector vertically upwards, $\mathbf{i}$ a unit vector along PG (Fig. 109). Then the position vector of the centre of mass G is

$$\mathbf{r}+a\mathbf{i},$$

and hence if $\mathbf{\Omega}$ is the angular velocity of the sphere, the condition of rolling is

$$\frac{d}{dt}(\mathbf{r}+a\mathbf{i})+\mathbf{\Omega}\wedge(-a\mathbf{i})=0. \tag{1}$$

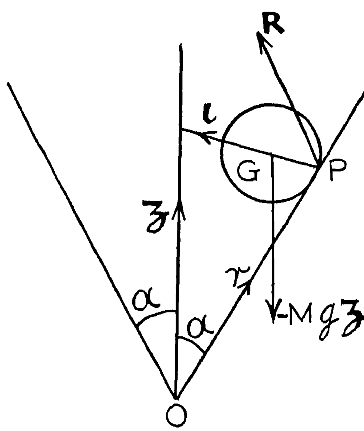

Fig. 109

The equations of motion, written down in the usual way, give on elimination of the reaction $\mathbf{R}$

$$-Ma\mathbf{i}\wedge\left[g\mathbf{z}+\frac{d^2}{dt^2}(\mathbf{r}+a\mathbf{i})\right]=C\frac{d\mathbf{\Omega}}{dt}. \tag{2}$$

To solve (1) for $\mathbf{\Omega}$, put

$$\mathbf{\Omega}=n\mathbf{i}+\mathbf{\Omega}', \qquad (\mathbf{\Omega}'.\mathbf{i}=0)$$

and multiply (1) vectorially by $\mathbf{i}$. We find

$$\mathbf{\Omega}=n\mathbf{i}+\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}+\frac{1}{a}\mathbf{i}\wedge\frac{d\mathbf{r}}{dt}. \tag{3}$$

Introducing this in (2) and collecting terms we find

$$C\left[\frac{dn}{dt}\mathbf{i}+n\frac{d\mathbf{i}}{dt}\right]+(C+Ma^2)\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}+\frac{C+Ma^2}{a}\mathbf{i}\wedge\frac{d^2\mathbf{r}}{dt^2}$$
$$+\frac{C}{a}\frac{d\mathbf{i}}{dt}\wedge\frac{d\mathbf{r}}{dt}+Mga(\mathbf{i}\wedge\mathbf{z})=0. \tag{4}$$

This vector equation is equivalent to three scalar equations, and there are just three scalar unknowns, namely n, the orientation of the plane OPG and the length OP.

From (1), we have

$$\mathbf{i}.\frac{d\mathbf{r}}{dt}=0. \tag{5}$$

Also we know that $\mathbf{r}.\mathbf{i}=0$. Hence $\mathbf{r}.d\mathbf{i}/dt=0$. Multiplying (4) scalarly by $\mathbf{i}$ we have

$$\frac{dn}{dt}+\frac{1}{a}\left(\frac{d\mathbf{i}}{dt}\wedge\frac{d\mathbf{r}}{dt}\right).\mathbf{i}=0. \tag{6}$$

Now $$\left(\frac{d\mathbf{i}}{dt}\wedge\frac{d\mathbf{r}}{dt}\right)-\mathbf{i}\left[\left(\frac{d\mathbf{i}}{dt}\wedge\frac{d\mathbf{r}}{dt}\right).\mathbf{i}\right]\equiv-\left[\left(\frac{d\mathbf{i}}{dt}\wedge\frac{d\mathbf{r}}{dt}\right)\wedge\mathbf{i}\right]\wedge\mathbf{i},$$

and the right-hand side of this identity vanishes on expanding the square bracket, in virtue of (5). Hence, multiplying (6) by $\mathbf{i}$,

$$\frac{dn}{dt}\mathbf{i}+\frac{1}{a}\left(\frac{d\mathbf{i}}{dt}\wedge\frac{d\mathbf{r}}{dt}\right)=0.$$

Using this in (4), we get

$$Cn\frac{d\mathbf{i}}{dt}+(C+Ma^2)\mathbf{i}\wedge\left[\frac{d^2\mathbf{i}}{dt^2}+\frac{1}{a}\frac{d^2\mathbf{r}}{dt^2}\right]+Mga(\mathbf{i}\wedge\mathbf{z})=0. \tag{7}$$

In this equation, $\mathbf{r}$, $\mathbf{i}$ and $\mathbf{z}$ are coplanar vectors. Though it bears a superficial resemblance to the equation for the motion of the axis of a spinning top, it must be remembered that here n is not constant, being governed by (6). Various scalar formulæ of interest can be obtained from (7) by suitable scalar multiplications, but to make progress it seems best at this stage to write

$$\mathbf{r}=\rho\mathbf{j},$$

where $\mathbf{j}$ is a unit vector, so that

$$\mathbf{z}=\mathbf{j}\cos\alpha+\mathbf{i}\sin\alpha$$

and $$\mathbf{r}=\rho\frac{\mathbf{z}-\mathbf{i}\sin\alpha}{\cos\alpha}.$$

Differentiating this and introducing into (7), we find

$$Cn\frac{d\mathbf{i}}{dt}+(C+Ma^2)\left(1-\frac{\rho}{a}\tan\alpha\right)\left(\mathbf{i}\wedge\frac{d^2\mathbf{i}}{dt^2}\right)-2(C+Ma^2)\frac{1}{a}\frac{d\rho}{dt}\tan\alpha\left(\mathbf{i}\wedge\frac{d\mathbf{i}}{dt}\right)$$
$$+\left[Mga+\frac{C+Ma^2}{\cos\alpha}\frac{1}{a}\frac{d^2\rho}{dt^2}\right](\mathbf{i}\wedge\mathbf{z})=0. \tag{8}$$

In this we can put

$$\frac{d\mathbf{i}}{dt}=\omega(\mathbf{z}\wedge\mathbf{i}),$$

where ω is a function of t to be determined. We find

$$\mathrm{C}n\omega(\mathbf{z}\wedge\mathbf{i})+(\mathrm{C}+\mathrm{M}a^2)\left(1-\frac{\rho}{a}\tan\alpha\right)\left[\frac{d\omega}{dt}\cos\alpha\,\mathbf{j}+\omega^2\sin\alpha(\mathbf{i}\wedge\mathbf{z})\right]$$

$$-2(\mathrm{C}+\mathrm{M}a^2)\frac{\omega}{a}\frac{d\rho}{dt}\sin\alpha\,\mathbf{j}+\left[\mathrm{M}ga+\frac{\mathrm{C}+\mathrm{M}a^2}{\cos\alpha}\frac{1}{a}\frac{d^2\rho}{dt^2}\right](\mathbf{i}\wedge\mathbf{z})=0. \quad (9)$$

This is a linear relation between the vectors $\mathbf{z}\wedge\mathbf{i}$ and $\mathbf{j}$. It gives accordingly

$$\left(1-\frac{\rho}{a}\tan\alpha\right)\frac{d\omega}{dt}\cos\alpha-2\frac{\omega}{a}\frac{d\rho}{dt}\sin\alpha=0, \quad (10)$$

$$-\mathrm{C}n\omega+(\mathrm{C}+\mathrm{M}a^2)\left(1-\frac{\rho}{a}\tan\alpha\right)\omega^2\sin\alpha$$

$$+\left[\mathrm{M}ga+\frac{\mathrm{C}+\mathrm{M}a^2}{\cos\alpha}\frac{1}{a}\frac{d^2\rho}{dt^2}\right]=0, \quad (11)$$

whilst (6) now gives

$$\frac{dn}{dt}=-\frac{\omega}{a}\frac{d\rho}{dt}\cos\alpha. \quad (12)$$

Equation (10) is integrable, and gives

$$\omega\left(1-\frac{\rho}{a}\tan\alpha\right)^2=\text{const.}=\omega_0, \quad (13)$$

say; and (12) then gives

$$n=-\frac{\omega_0\cos^2\alpha\,\mathrm{cosec}\,\alpha}{1-\frac{\rho}{a}\tan\alpha}+\text{const.} \quad (14)$$

Insertion for n and ω from (13) and (14) in (11) gives an integrable equation for ρ. The problem is thus solved.

427. The student is recommended to obtain the equations of motion and any possible integrals in the following problems:

(1) Motion of a rolling and spinning sphere, moving in contact with the interior of a rough vertical cone which is either (*a*) free to move about its axis, or (*b*) compelled to rotate about its axis at a given constant rate.

(2) Motion of a sphere rolling and spinning in contact with the rough curved surface of a massive hemisphere free to move with its base in contact with a smooth horizontal plane.

(3) Motion of a sphere rolling and spinning on the rough curved surface of a massive semi-cylinder with generators horizontal, whose base is in contact with a smooth horizontal plane.

CHAPTER XVIII

IMPULSIVE MOTION

428. *The concept of an impulse.* When a system of bodies is in motion, whether external forces be acting or not, the configuration of the bodies in space and the velocity description of the system are both changing, but changing continuously. We may say that the system is simultaneously undergoing a change of space configuration and a change of velocity configuration. Speaking somewhat loosely, the amounts of the two changes depend on the interval of time considered and on the forces acting. If the velocities of all the particles remain finite and bounded during the interval, the change of space configuration will tend to zero as the interval of time tends to zero. If the force system remains finite and bounded, the same will be true of the change of velocity configuration. But it is necessary in practice to consider cases in which the force system acts for a relatively short time but includes forces whose absolute magnitudes become very large. In such cases the change of space configuration may be very small, possibly negligible, whilst the change of velocity configuration may be considerable, or at least non-negligible. This is because the change of velocity configuration depends on the integrals of the forces acting, taken through the intervals of time through which they act.

This set of circumstances leads us to contemplate the idealized limiting case in which the interval of time considered, Δt, tends to zero whilst the forces acting, or some of them, tend to ' infinity ' in such a way that the change of velocity configuration is finite. Under these conditions the space configuration is unaltered during the process.

This idealized case will roughly represent the effect of sudden ' blows ' applied at definite particles of the system. Each blow is measured by its *impulse*, which is defined as the integral of the corresponding force with respect to the time, in the limit when the range of integration tends to zero. Motion generated by blows is said to be *impulsively* generated ; similarly, motion may be destroyed by suitable application of blows.

429. *Impulse on a particle.* Let $\mathbf{P}$ be the force applied at any instant to a particle whose position vector is $\mathbf{r}$; $\mathbf{P}$ will be in general a function of the time. Let $\mathbf{v}$ be the velocity of the particle at any instant, m the mass. Thus, since

$$\frac{d\mathbf{r}}{dt}=\mathbf{v}, \qquad m\frac{d\mathbf{v}}{dt}=\mathbf{P},$$

we have
$$\Delta\mathbf{r}=\int_{t_1}^{t_2}\mathbf{v}dt, \qquad m\Delta\mathbf{v}=\int_{t_1}^{t_2}\mathbf{P}dt,$$
where $\Delta\mathbf{r}$ denotes the change of position $\mathbf{r}_2-\mathbf{r}_1$ between t_1 and t_2, $\Delta\mathbf{v}$ the corresponding change of velocity, $\mathbf{v}_2-\mathbf{v}_1$. Suppose that
$$\Delta t=t_2-t_1\to 0,$$
but
$$\int_{t_2}^{t_1}\mathbf{P}dt\to\mathbf{J},$$
a definite limit vector. Then in the limit the changes of space configuration and velocity configuration are given by
$$\Delta\mathbf{r}=0,$$
$$\Delta\mathbf{v}=\mathbf{J}/m.$$
We call $\mathbf{J}$ the *impulse* of the applied ' blow '.

430. If blows are applied to a system of particles which act on one another through constraints, or to a system including ideal rigid bodies, the application of the blows will in general originate impulsive reactions between the different particles of the system. These impulsive reactions, being the time integrals of force reactions, will satisfy Newton's third law. It then follows that the system of line vectors constituted by the *externally* applied impulses ($\mathbf{J}$) is *equivalent* in the technical sense to the system of line vectors constituted by the changes of momentum ($m\Delta\mathbf{v}$). For the former system can be transformed into the latter system by the introduction of the appropriate impulsive reactions (stresses, pressures, tensions, etc.) which, occurring in pairs, can be divided into nul-concurrent sets. In the case of continuous systems it is not at once obvious how the system of impulses, acting on a given element of the system, is to be correlated with the distribution of impulsive reactions on adjacent elements, and in this case it is best to *assume* the equivalence of the applied vectors ($\mathbf{J}$) to the system of changes of momenta ($m\Delta\mathbf{v}$).

431. *Equations of impulsive motion for a system of particles.* The conditions of equivalence of two systems of line vectors are now immediately applicable. Thus
$$\Sigma m\Delta\mathbf{v}=\Sigma\mathbf{J}, \qquad \Sigma\mathbf{r}\wedge m\Delta\mathbf{v}=\Sigma\mathbf{r}\wedge\mathbf{J}.$$
If the system ($\mathbf{J}$) is equivalent to a total impulse $\mathbf{K}$ at O, the origin taken as base point, and an impulsive couple $\mathbf{X}(O)$, and if $\Delta\mathbf{L}$ is the change of linear momentum of the system under the applied impulse, $\Delta\mathbf{H}(O)$ the change of angular momentum about O, then
$$\Delta\mathbf{L}=\mathbf{K}, \qquad \Delta\mathbf{H}(O)=\mathbf{X}(O).$$
In writing down the latter equation we have used the relation
$$\Delta(\mathbf{r}\wedge m\mathbf{v})=\mathbf{r}\wedge m\Delta\mathbf{v},$$
in virtue of
$$\Delta\mathbf{r}=0.$$

432. *Example* (1). Consider a rigid body of mass M and inertia tensor **I** at G, its centre of mass. Let G have initially a velocity **V**, and the rigid body an angular velocity **Ω**. At this instant let it undergo a blow at a particle of the body of position vector **r** with respect to G, in such a way that the particle **r** is reduced to rest. It is required to determine the resulting motion.

Let **V′** be the velocity of G after application of the blow, **Ω′** the new angular velocity. Let **J** be the blow. Then the equations of changes of linear and angular momentum are

$$\mathbf{J}=\mathrm{M}(\mathbf{V}'-\mathbf{V}), \tag{1}$$

$$\mathbf{r}\wedge\mathbf{J}=\mathbf{I}.(\mathbf{\Omega}'-\mathbf{\Omega}), \tag{2}$$

and the condition that the particle **r** is reduced to rest is

$$\mathbf{V}'+\mathbf{\Omega}'\wedge\mathbf{r}=0. \tag{3}$$

Eliminating **J**,
$$\mathbf{r}\wedge(\mathbf{V}'-\mathbf{V})=\frac{\mathbf{I}}{\mathrm{M}}.(\mathbf{\Omega}'-\mathbf{\Omega}), \tag{4}$$

whence, eliminating **V′**,

$$-\mathbf{r}\wedge(\mathbf{\Omega}'\wedge\mathbf{r})-\mathbf{r}\wedge\mathbf{V}=\frac{\mathbf{I}}{\mathrm{M}}.(\mathbf{\Omega}'-\mathbf{\Omega}),$$

or
$$\mathbf{r}(\mathbf{\Omega}'.\mathbf{r})-\mathbf{r}^2\mathbf{\Omega}'-\mathbf{r}\wedge\mathbf{V}=\frac{\mathbf{I}}{\mathrm{M}}.(\mathbf{\Omega}'-\mathbf{\Omega}),$$

or
$$\mathbf{\Omega}'.\left[\frac{\mathbf{I}}{\mathrm{M}}+\mathbf{r}^2\mathbf{U}-\mathbf{rr}\right]=\frac{\mathbf{I}}{\mathrm{M}}.\mathbf{\Omega}-\mathbf{r}\wedge\mathbf{V}.$$

Hence
$$\mathbf{\Omega}'=\left[\frac{\mathbf{I}}{\mathrm{M}}+\mathbf{r}^2\mathbf{U}-\mathbf{rr}\right]^{-1}.\left[\frac{\mathbf{I}}{\mathrm{M}}.\mathbf{\Omega}-\mathbf{r}\wedge\mathbf{V}\right]. \tag{5}$$

This determines **Ω′**, whence (3) determines **V′**.

The inverse tensor is readily calculated in the case of dynamical symmetry about G. In that case, $\mathbf{I}/\mathrm{M}=\mathrm{k}^2\mathbf{U}$, and if

$$[(\mathrm{k}^2+\mathbf{r}^2)\mathbf{U}-\mathbf{rr}]^{-1}=\mathbf{T},$$

we have
$$[(\mathrm{k}^2+\mathbf{r}^2)\mathbf{U}-\mathbf{rr}].\mathbf{T}=\mathbf{U},$$

or
$$(\mathrm{k}^2+\mathbf{r}^2)\mathbf{T}=\mathbf{U}+\mathbf{r}(\mathbf{r}.\mathbf{T}).$$

Multiplying scalarly by **r** on the left, we find

$$\mathrm{k}^2(\mathbf{r}.\mathbf{T})=\mathbf{r},$$

so that
$$\mathbf{T}=\frac{\mathbf{U}+\dfrac{\mathbf{rr}}{\mathrm{k}^2}}{\mathrm{k}^2+\mathbf{r}^2}.$$

Equation (5) then gives for **Ω′**

$$\mathbf{\Omega}'=\frac{1}{\mathrm{k}^2+\mathbf{r}^2}[\mathrm{k}^2\mathbf{\Omega}+\mathbf{r}(\mathbf{r}.\mathbf{\Omega})-\mathbf{r}\wedge\mathbf{V}].$$

Example (2). A cone rests on a rough horizontal plane with its vertex at O. The plane is suddenly set into motion with angular velocity ω about the vertical through O. It is required to discuss the motion of the cone.

Let $\mathbf{z}$ be a unit vector vertically upwards, $\mathbf{i}$ a unit vector along the axis of the cone (Fig. 110). The inertia tensor $\mathbf{I}$ of the cone about O may be supposed given as

$$\mathbf{I}=\mathrm{C}\mathbf{ii}+\mathrm{A}(\mathbf{U}-\mathbf{ii}).$$

Let $\mathbf{\Omega}$ be the angular velocity communicated to the cone. Then the angular momentum of the cone about O, namely $\mathbf{H}$, will be given by

$$\mathbf{H}=\mathbf{I}.\mathbf{\Omega}=\mathrm{C}\mathbf{i}(\mathbf{i}.\mathbf{\Omega})+\mathrm{A}\mathbf{\Omega}-\mathrm{A}\mathbf{i}(\mathbf{i}.\mathbf{\Omega}).$$

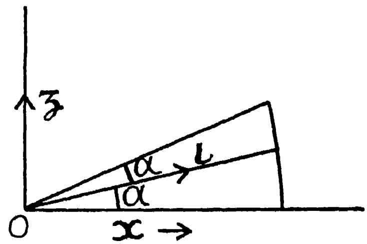

Fig. 110

Let $\mathbf{x}$ be a unit vector along the contact generator. The component of $\mathbf{H}$ about this generator must be unaltered by the process of setting the plane in motion, since all the impulsive reactions acting on the cone intersect this generator. Since $\mathbf{H}$ is initially zero, its component about $\mathbf{x}$ will be zero after the plane has been set in motion, instantaneously, and so

$$(\mathrm{C}-\mathrm{A})(\mathbf{i}.\mathbf{x})(\mathbf{i}.\mathbf{\Omega})+\mathrm{A}(\mathbf{x}.\mathbf{\Omega})=0.$$

The kinematical condition is that any point in the generator of contact, considered as a particle of the rigid body, must also have the velocity of the plane. Hence

$$\mathbf{\Omega}\wedge\mathbf{x}=\omega\mathbf{z}\wedge\mathbf{x},$$

or, multiplying vectorially by $\mathbf{x}$,

$$\mathbf{\Omega}-(\mathbf{\Omega}.\mathbf{x})\mathbf{x}=\omega\mathbf{z}.$$

Thus $$\mathbf{\Omega}=\omega\mathbf{z}+\omega_1\mathbf{x},$$

say. Now we have

$$\mathbf{i}=\mathbf{x}\cos\alpha+\mathbf{z}\sin\alpha,$$

whence $$\mathbf{i}.\mathbf{\Omega}=\omega_1\cos\alpha+\omega\sin\alpha,$$

and $$\mathbf{x}.\mathbf{\Omega}=\omega_1.$$

Hence the dynamical condition gives

$$(\mathrm{C}-\mathrm{A})\cos\alpha(\omega_1\cos\alpha+\omega\sin\alpha)+\mathrm{A}\omega_1=0,$$

whence $$\omega_1=\frac{(\mathrm{A}-\mathrm{C})\omega\sin\alpha\cos\alpha}{\mathrm{C}\cos^2\alpha+\mathrm{A}\sin^2\alpha}.$$

This determines the angular velocity $\mathbf{\Omega}$ of the cone. The angular speed of revolution of the *axis* of the cone, say ω', about the vertical through O, is given by

$$\mathbf{\Omega}\wedge\mathbf{i}=\omega'\mathbf{z}\wedge\mathbf{i}$$

or $$\omega\left[\mathbf{z}+\frac{(A-C)\sin\alpha\cos\alpha}{C\cos^2\alpha+A\sin^2\alpha}\mathbf{x}\right]\wedge(\mathbf{x}\cos\alpha+\mathbf{z}\sin\alpha)=\omega'\cos\alpha(\mathbf{z}\wedge\mathbf{x}),$$

which yields $$\omega'=\frac{C\omega}{C\cos^2\alpha+A\sin^2\alpha}.$$

433. *Applications of the Principle of Virtual Work.* The conditions of equivalence of the system of applied impulses to the system of changes of momenta can also be expressed by means of the Principle of Virtual Work (Chapter IX, § 191). Let $\delta\mathbf{r}$ denote, not the change of position of the particle $\mathbf{r}$ (which during the application of the impulses is zero), but any small displacement of the particle $\mathbf{r}$ compatible with the existing constraints, or with the constraints that are not broken by the application of the impulses. Then the condition of equivalence of the systems of line vectors $(\mathbf{J})$ and $(\Delta m\mathbf{v})$ can be expressed by

$$\Sigma m\Delta\mathbf{v}.\delta\mathbf{r}=\Sigma\mathbf{J}.\delta\mathbf{r},$$

where on the left-hand side the summation is extended to all the particles of the system, and on the right-hand side only to those particles which are acted on by externally applied impulses. This equation expresses the equality of the works done by the two systems of line vectors in the displacement typified by $\delta\mathbf{r}$.

We shall now use this to give proofs of a well-known sequence of theorems on impulses.

434. *Change of energy caused by a set of impulses.*

Theorem: The change of energy caused by the application of a set of impulses $\mathbf{J}$ to a system of bodies is equal to the sum of the scalar products of each impulse $\mathbf{J}$ with the arithmetic vector-mean of the velocities of its point of application before and after the change of motion.

Apply the result of § 433 by choosing for the displacements $(\delta\mathbf{r})$ in turn to be: (1) the actual displacements of all the particles during a short interval δt preceding the instant of application of the blows; (2) the actual displacements of the particles during a short interval δt following the blows. Then, in case (1), the values of $\delta\mathbf{r}$ are given by

$$\delta\mathbf{r}=\mathbf{v}_1\delta t,$$

and in case (2) the values of $\delta\mathbf{r}$ are given by

$$\delta\mathbf{r}=\mathbf{v}_2\delta t,$$

where $\mathbf{v}_1$, $\mathbf{v}_2$ are the velocities of a typical particle before and after. Hence $\Delta\mathbf{v}=\mathbf{v}_2-\mathbf{v}_1$, and the result of § 433 gives in turn

$$\Sigma m(\mathbf{v}_2-\mathbf{v}_1).\mathbf{v}_1\delta t=\Sigma\mathbf{J}.\mathbf{v}_1\delta t,$$
$$\Sigma m(\mathbf{v}_2-\mathbf{v}_1).\mathbf{v}_2\delta t=\Sigma\mathbf{J}.\mathbf{v}_2\delta t,$$

from which by addition

$$\Sigma\tfrac{1}{2}m(\mathbf{v}_2{}^2-\mathbf{v}_1{}^2)=\Sigma\mathbf{J}.\tfrac{1}{2}(\mathbf{v}_1+\mathbf{v}_2).$$

The left-hand side is the change of kinetic energy, the right-hand side is the sum of the scalar products of the impulses into the mean velocities of the points of application.

435. *Definition.* If a motion ($\mathbf{v}_1$) is changed to a motion ($\mathbf{v}_2$), the sum $\Sigma\frac{1}{2}m(\mathbf{v}_2-\mathbf{v}_1)^2$ extended to all the particles of the system is called the kinetic energy of the change of motion of the system.

436. *Kelvin's theorem. Velocities of points of application of the impulses prescribed.*

Theorem : If a system is acted on by impulses in such a way that the points of application of the impulses move afterwards with prescribed velocities, then of all the geometrically possible motions the actual motion is such that the kinetic energy of the change of motion is a minimum.

For, let $\mathbf{v}_2$ be the actual velocity of a particle after application of the impulses, $\mathbf{v}_2'$ any other geometrically possible velocity compatible with the points of application of the impulses having the given velocities. Let $\mathbf{V}$ be the prescribed velocity of a point of application of an impulse. Then we may take as sets of displacements $\delta\mathbf{r}$: (1) the actual displacements $\mathbf{v}_2\delta t$ following the application of the impulses ; (2) the displacements $\mathbf{v}_2'\delta t$. The conditions of equivalence of the systems of line vectors ($\mathbf{J}$) and ($m(\mathbf{v}_2-\mathbf{v}_1)$) now require in the two cases *

$$\Sigma m(\mathbf{v}_2-\mathbf{v}_1).\mathbf{v}_2\delta t=\Sigma\mathbf{J}.\mathbf{V}\delta t,$$

$$\Sigma m(\mathbf{v}_2-\mathbf{v}_1).\mathbf{v}_2'\delta t=\Sigma\mathbf{J}.\mathbf{V}\delta t.$$

Subtracting, we have

$$\Sigma m(\mathbf{v}_2-\mathbf{v}_1).(\mathbf{v}_2-\mathbf{v}_2')=0. \tag{1}$$

If T_{21}, $T_{2'1}$ denote the kinetic energies of the actual change of motion and the other, geometrically possible, change of motion, then

$$T_{21}=\Sigma\tfrac{1}{2}m(\mathbf{v}_2-\mathbf{v}_1)^2,$$

$$T_{2'1}=\Sigma\tfrac{1}{2}m(\mathbf{v}_2'-\mathbf{v}_1)^2.$$

Hence
$$\begin{aligned} T_{21}-T_{2'1} &= \tfrac{1}{2}\Sigma m(\mathbf{v}_2-\mathbf{v}_2').(\mathbf{v}_2+\mathbf{v}_2'-2\mathbf{v}_1) \\ &= \tfrac{1}{2}\Sigma m(\mathbf{v}_2-\mathbf{v}_2').[2(\mathbf{v}_2-\mathbf{v}_1)-(\mathbf{v}_2-\mathbf{v}_2')] \\ &= -\Sigma\tfrac{1}{2}m(\mathbf{v}_2-\mathbf{v}_2')^2+\Sigma m(\mathbf{v}_2-\mathbf{v}_1).(\mathbf{v}_2-\mathbf{v}_2'). \end{aligned}$$

The last term vanishes by (1). The remaining term on the right-hand side may be called $-T_{2'2}$. Hence

$$T_{21}-T_{2'1}=-T_{2'2}<0.$$

Hence
$$T_{21}<T_{2'1}.$$

It follows that the actual motion following the impulses gives a T_{21} which is the least of all the $T_{2'1}$'s associated with all the geometrically possible

* It is particularly to be noted that though a different set of impulses would be required to generate the motions ($\mathbf{v}_2'$), these are irrelevant to the equations : the same impulses $\mathbf{J}$, and the same velocity differences ($\mathbf{v}_2-\mathbf{v}_1$), occur in the two equations.

motions compatible with the points of application of the impulses having prescribed velocities **V**. The kinetic energy of the actual change of motion is therefore a minimum.

Corollary. If the impulses are applied to a system at rest, the theorem just proved gives

$$T_2 - T_2' = -T_2'{}_2 < 0$$

and hence

$$T_2 < T_2',$$

where now T_2, T_2' denote the actual kinetic energies of the motions $(\mathbf{v}_2)$ and $(\mathbf{v}_2')$. It follows that if a system is started from rest by the application of impulses in such a way that the points of application of the impulses move immediately afterwards with prescribed velocities, then of all the geometrically possible motions compatible with this, that one will actually be followed which makes the kinetic energy a minimum.

437. *Bertrand's theorem. Impulses prescribed.*

Theorem: If a system is acted on by given impulses, the kinetic energy after the application of the impulses is greater than that of the motion that would result from applying the same impulses and at the same time introducing constraints for which the corresponding impulsive reactions do no work.

For, let $\mathbf{v}_1$ be the velocity of a typical particle before the application of the given impulses, $\mathbf{v}_2$ its velocity immediately after the application of the given impulses, $\mathbf{v}_2'$ its velocity immediately after the same impulses are applied in the presence of additional constraints which do no work, i.e. such that the new impulsive reactions originated are perpendicular to the velocities of the corresponding particles immediately after the application of the impulses. We may now take as the virtual displacements $\delta\mathbf{r}$ the actual displacements $\mathbf{v}_2'\delta t$ in the constrained motion, for these will also be geometrically possible motions in the absence of the constraints, and so a permissible set of displacements in the unconstrained motion. In these displacements $\mathbf{v}_2'\delta t$, the virtual work of the additional impulses is zero. Hence we have

$$\Sigma m(\mathbf{v}_2 - \mathbf{v}_1).\mathbf{v}_2'\delta t = \Sigma \mathbf{J}.\mathbf{v}_2'\delta t,$$

$$\Sigma m(\mathbf{v}_2' - \mathbf{v}_1).\mathbf{v}_2'\delta t = \Sigma \mathbf{J}.\mathbf{v}_2'\delta t,$$

where the **J**'s are the prescribed impulses. Subtracting, we have

$$\Sigma m(\mathbf{v}_2 - \mathbf{v}_2').\mathbf{v}_2' = 0. \qquad (1)$$

But

$$T_2 = \tfrac{1}{2}\Sigma m\mathbf{v}_2^2,$$

$$T_2' = \Sigma\tfrac{1}{2}m\mathbf{v}_2'^2,$$

whence

$$T_2 - T_2' = \tfrac{1}{2}\Sigma m(\mathbf{v}_2 - \mathbf{v}_2').(\mathbf{v}_2 + \mathbf{v}_2')$$

$$= \tfrac{1}{2}\Sigma m(\mathbf{v}_2 - \mathbf{v}_2').(\mathbf{v}_2 - \mathbf{v}_2' + 2\mathbf{v}_2')$$

$$= T_{22}' + \Sigma m(\mathbf{v}_2 - \mathbf{v}_2').\mathbf{v}_2'.$$

By (1) the last term on the right-hand side is zero. Hence

$$T_2 - T_2' = T_{22}' > 0,$$

and so

$$T_2 > T_2'.$$

This establishes the theorem.

438. *Carnot's theorem. Sudden introduction of constraints.*

Theorem: When constraints are suddenly introduced, of such a character that they do no work, kinetic energy is always lost.

Let $\mathbf{v}_1$ be the velocity of a typical particle before the introduction of the constraints, $\mathbf{v}_2$ the velocity immediately after. Then we have

$$\Sigma m(\mathbf{v}_2 - \mathbf{v}_1).\mathbf{v}_2 \delta t = 0, \tag{1}$$

since we may take as virtual displacements $\delta\mathbf{r}$ the motions $\mathbf{v}_2\delta t$ immediately following the introduction of the constraints, these being geometrically possible displacements. But

$$T_2 = \Sigma\tfrac{1}{2}m\mathbf{v}_2{}^2, \qquad T_1 = \Sigma\tfrac{1}{2}m\mathbf{v}_1{}^2,$$

whence

$$\begin{aligned} T_2 - T_1 &= \tfrac{1}{2}\Sigma m(\mathbf{v}_2 - \mathbf{v}_1).(\mathbf{v}_2 + \mathbf{v}_1) \\ &= \tfrac{1}{2}\Sigma m(\mathbf{v}_2 - \mathbf{v}_1).(-(\mathbf{v}_2 - \mathbf{v}_1) + 2\mathbf{v}_2) \\ &= -T_{21} + \Sigma m(\mathbf{v}_2 - \mathbf{v}_1).\mathbf{v}_2. \end{aligned}$$

The last term on the right-hand side vanishes by (1). Hence

$$T_2 - T_1 = -T_{21} < 0$$

or

$$T_2 < T_1.$$

439. *Converse of Carnot's theorem. Sudden removal of constraints.*

Theorem: If constraints are suddenly removed, kinetic energy is always gained.

For, in this case a set of permissible displacements $\delta\mathbf{r}$ is the set $\mathbf{v}_1\delta t$, where $\mathbf{v}_1$ is the velocity of a typical particle immediately before the removal of the constraints. Then we have

$$\Sigma m(\mathbf{v}_2 - \mathbf{v}_1).\mathbf{v}_1 \delta t = 0. \tag{1}$$

But

$$\begin{aligned} T_2 - T_1 &= \tfrac{1}{2}\Sigma m(\mathbf{v}_2 - \mathbf{v}_1).(\mathbf{v}_2 + \mathbf{v}_1) \\ &= \tfrac{1}{2}\Sigma m(\mathbf{v}_2 - \mathbf{v}_1).[(\mathbf{v}_2 - \mathbf{v}_1) + 2\mathbf{v}_1] \\ &= T_{21} + \Sigma m(\mathbf{v}_2 - \mathbf{v}_1).\mathbf{v}_1. \end{aligned}$$

The last term on the right-hand side vanishes in virtue of (1). Hence

$$T_2 - T_1 = T_{21} > 0$$

or

$$T_2 > T_1.$$

The constraints introduced in Carnot's theorem may take the form of uniting portions of the system previously free, or of introducing constrain-

ing surfaces, strings, hinges, etc., the impulsive reactions from which do no work in the immediately subsequent motion. The removal of constraints considered in the converse of Carnot's theorem is illustrated by the explosion of a shell, when kinetic energy is clearly generated, consequent on the fragmentation. A particular case of Carnot's theorem is the application of constraints which reduce a system to rest: in this case T_2 is zero, and $-T_1=-T_{21}$.

440. *Relation between Kelvin's theorem and Bertrand's theorem. Taylor's theorem.** We conclude this brief account of theorems on impulses with an investigation due to Sir Geoffrey Taylor.

We employ the following notation. Let $(\mathbf{v}_1)$ denote a given initial motion of a system. Let $(\mathbf{v}_2)$ denote a motion derived from $(\mathbf{v}_1)$ by the application of impulses $\mathbf{J}_P$ at a set of points P, giving rise to velocities $(\mathbf{v}_P)$ of these particles P. Let $(\mathbf{v}_2')$ denote a motion derived from $(\mathbf{v}_1)$ by the application of such *different* impulses $\mathbf{J}'_P$ at the points P as give rise to the same velocities $(\mathbf{v}_P)$ of these particles P *in the presence of certain constraints newly introduced.* These constraints will be defined as maintaining certain co-ordinates Q constant, and the work done by the newly-introduced impulses $\mathbf{J}'_Q$ is zero. (Q may denote an actual point or stand for any 'generalized co-ordinate' used to define the configuration of the system.) (In the motion $(\mathbf{v}_2)$ the impulses applied in connexion with the co-ordinates Q are zero.) Lastly, let $(\mathbf{v}_2'')$ denote a motion derived from $(\mathbf{v}_1)$ by the *same* impulses $\mathbf{J}_P$ at the points P, producing in general different velocities for the points P, together with other impulses $\mathbf{J}''_Q$ which maintain the co-ordinates Q fixed.

Then, if T_{21}, etc., denote the associated kinetic energies of the changes of motion, we have from the proof of Kelvin's theorem

$$T_{2'1}-T_{21}=T_{2'2}, \tag{1}$$

and from the proof of Bertrand's theorem we have

$$T_2-T_2''=T_{22''}. \tag{2}$$

Moreover, if $(\mathbf{v}_1)$ denotes rest, relation (1) gives

$$T_2'-T_2=T_{2'2}. \tag{3}$$

(The motions $(\mathbf{v}_2')$ and $(\mathbf{v}_2)$ can always be considered as originated from rest.)

Now consider the motion which is the difference of the motions $(\mathbf{v}_2')$ and $(\mathbf{v}_2)$. This motion could be produced *from rest* by the application of the difference of the systems of impulses which originate $(\mathbf{v}_2')$ and $(\mathbf{v}_2)$ separately. This difference system of impulses consists of impulses $(\mathbf{J}'_P-\mathbf{J}_P)$ at the points P, these leaving the points P at rest, together with the system of impulses $\mathbf{J}'_Q$. In this difference motion, the velocities associated with the co-ordinates Q are *minus* the velocities associated with

* G. I. Taylor, *Proc. London Math. Soc.*, 21, 413, 1923.

these co-ordinates in the motion $(\mathbf{v}_2)$, and may be described as $-(\mathbf{v}_2)$. We tabulate the particulars of this motion thus :

Motion $(\mathbf{v}_2')-(\mathbf{v}_2)$

Impulse system necessary to originate the motion from rest :	*Velocities of points struck :*
$\mathbf{J}'_P-\mathbf{J}_P$ at points P	Zero.
$\mathbf{J}'_Q$ at points Q	$-(\mathbf{v}_2)_Q$.

Next consider the motion which is the difference of the motions $(\mathbf{v}_2'')$ and $(\mathbf{v}_2)$. This motion, similarly, could be produced *from rest* by the application of the difference of the systems of impulses which originate $(\mathbf{v}_2'')$ and $(\mathbf{v}_2)$ separately. This difference system of impulses consists of *zero* impulses at points P, together with impulses $\mathbf{J}''_Q$ which maintain the associated co-ordinates Q constant. The velocities of the points P in this difference motion are $(\mathbf{v}_2'')_P-(\mathbf{v}_2)_P$, and the velocities associated with the points Q are $-(\mathbf{v}_2)_Q$, since the velocities of the points Q in the motion $(\mathbf{v}_2'')$ are zero, the co-ordinates Q being constant. We tabulate the particulars of this motion thus :

Motion $(\mathbf{v}_2'')-(\mathbf{v}_2)$

Impulse system necessary to originate the motion from rest :	*Velocities of points struck :*
Zero at points P	$(\mathbf{v}_2'')_P-(\mathbf{v}_2)_P$.
$\mathbf{J}''_Q$ at points Q	$-(\mathbf{v}_2)_Q$.

Now consider the velocities of the points P and Q respectively in the two motions $(\mathbf{v}_2')$ and $(\mathbf{v}_2)$. These points P and Q are the only points at which impulses are applied ; and the velocities originated at them are as follows :

Motion $(\mathbf{v}_2')$

Velocities of points P	$\mathbf{v}_P$.
Velocities of points Q	Zero.

Motion $(\mathbf{v}_2)$

Velocities of points P	$\mathbf{v}_P$.
Velocities of points Q	$\mathbf{v}_Q$.

It follows that the two motions $(\mathbf{v}_2')-(\mathbf{v}_2)$ and $(\mathbf{v}_2'')-(\mathbf{v}_2)$ are related to one another in the same way as the motions $(\mathbf{v}_2')$ and $(\mathbf{v}_2)$, provided we

interchange the roles of the points P and Q. For in $(\mathbf{v}_2')$ and $(\mathbf{v}_2)$ the velocities of points P are equal, whilst in $(\mathbf{v}_2')-(\mathbf{v}_2)$ and $(\mathbf{v}_2'')-(\mathbf{v}_2)$ the velocities of points Q are equal; and in $(\mathbf{v}_2')$ the velocities of points Q are zero, whilst in $(\mathbf{v}_2')-(\mathbf{v}_2)$ the velocities of points P are zero.

Hence we can apply to the motions $(\mathbf{v}_2')-(\mathbf{v}_2)$ and $(\mathbf{v}_2'')-(\mathbf{v}_2)$, which are both generated from rest in such a way that the points Q have given velocities, the theorem expressed by relation (3) above. Hence we have

$$T_{2'2}-T_{2''2}=T_{2''2'},$$

i.e. by (2) and (3),

$$(T_2'-T_2)-(T_2-T_2'')=T_{2''2'}>0.$$

It follows that the additional kinetic energy generated under the circumstances of Kelvin's theorem, when, in the presence of added constraints, the velocities of the points struck are maintained the same, exceeds the deficiency of kinetic energy generated under the circumstances of Bertrand's theorem, when the same impulses are applied in the presence of *the same* added constraints. This is Taylor's result.

Example. Consider a rod lying on a smooth horizontal table subjected to an impulse J, applied perpendicular to it, at a certain point P in its length (Fig. 111). Let a constraint be imposed by fixing a particle Q in its length. Then Kelvin's theorem asserts that if J is adjusted so that the velocity communicated to P, the particle struck, is given, say equal to V, then the kinetic energy generated in the absence of the constraint is less than any

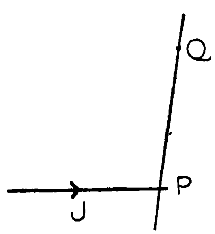

Fig. 111

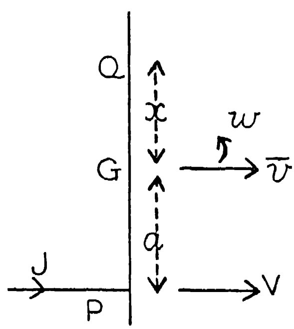

Fig. 112

generated in the presence of the constraint. Bertrand's theorem asserts that if the impulse is maintained constant, then the kinetic energy generated in the absence of the constraint exceeds that generated in the presence of the constraint. Taylor's theorem asserts that the former deficiency is greater than the latter excess, for a given position of the constraint Q.

This is readily verified by direct calculation. Let G (Fig. 112) be the centre of mass of the rod, supposed uniform, C its moment of inertia about an axis through G perpendicular to itself. Let $\bar{v}$ be the velocity at G generated by the impulse, ω the angular velocity generated. Let $PG=a$, $GQ=x$. Since Q is fixed, we have by considering the angular momentum about Q

$$J(a+x)=C\omega+M\bar{v}x,$$

where, since Q remains at rest,

$$\bar{v}-x\omega=0.$$

The kinetic energy T generated is given by

$$T=\tfrac{1}{2}C\omega^2+\tfrac{1}{2}M\bar{v}^2,$$

whilst V, the velocity of the point struck, is given by

$$V=\bar{v}+a\omega.$$

In the Kelvin case, V is given, and

$$\bar{v}=\frac{x}{a+x}V, \qquad \omega=\frac{V}{a+x}$$

and

$$T_2'=\tfrac{1}{2}V^2\frac{C+Mx^2}{(a+x)^2}.$$

For fixed V, this is a *minimum* when $x=C/Ma$. When x has this value, there can be no constraint at Q, i.e. no impulsive reaction there, and the motion is as if Q were unconstrained. The kinetic energy generated is, for this value of x, the T_2 of our general theory, and it has the value

$$T_2=\tfrac{1}{2}V^2\frac{C}{a^2}\,\frac{1}{(1+C/Ma^2)}.$$

In the Bertrand case, J is given, and then

$$\bar{v}=J\frac{x(a+x)}{C+Mx^2}, \qquad \omega=\frac{J(a+x)}{C+Mx^2}$$

and

$$T_2''=\tfrac{1}{2}J^2\frac{(a+x)^2}{C+Mx^2}.$$

For fixed J, this is a maximum when $x=C/Ma$. In this case there is as before no constraint at Q. The kinetic energy generated, T_2, is now given by

$$T_2=\tfrac{1}{2}J^2\frac{a^2}{C}\left(1+\frac{C}{Ma^2}\right).$$

Equating the two values of T_2 we get

$$V=J\frac{a^2}{C}\left(1+\frac{C}{Ma^2}\right).$$

Taylor's theorem now asserts that

$$T_2' - T_2 > T_2 - T_2'',$$

i.e. $$\tfrac{1}{2}V^2\left[\frac{C+Mx^2}{(a+x)^2} - \frac{C}{a^2}\frac{1}{(1+C/Ma^2)}\right] > \tfrac{1}{2}J^2\left[\frac{a^2}{C}\left(1+\frac{C}{Ma^2}\right) - \frac{(a+x)^2}{C+Mx^2}\right].$$

Substituting for V in terms of J, this inequality is found to reduce to

$$\left[\frac{a^2}{C}\left(1+\frac{C}{Ma^2}\right)(C+Mx^2) - (a+x)^2\right]^2 > 0,$$

which verifies Taylor's theorem.

EXAMPLES

The following examples, kindly provided by Professor S. Chapman from papers set at the Imperial College of Science, London, have been selected to illustrate some of the methods developed in the foregoing.

(1) Show that if a particle describes the trajectory $\varphi(x, y)=0$, under a central force to the origin, its velocity $\mathbf{v}$ and acceleration $\mathbf{f}$ are given by

$$\mathbf{v}=\frac{\mathbf{h}\wedge \operatorname{grad} \varphi}{\mathbf{r}.\operatorname{grad} \varphi},$$

$$\mathbf{f}=\mathbf{v}.\operatorname{grad} \mathbf{v},$$

where h is the (constant) angular momentum per unit mass.

Show that the force to the origin under which the curve $x^3+y^3=a^3$ can be described is proportional to xyr. [*Imperial College, London*, 1933.]

(2) Show that a rigid body will be in equilibrium if it is acted on only by a distribution of couples over its surface, such that the moment of the couple on each element $\mathbf{dS}$ is $k\mathbf{dS}$, where k is constant.

Find the resultant of such a distribution of couples over one side of an unclosed surface S bounded by a curve C, proving that the distribution of couples over S is statically equivalent to a distribution of force round C, the force on each element $\mathbf{ds}$ of C being $\frac{1}{2}k\mathbf{ds}$.

Show that if the surface S undergoes an arbitrary continuous infinitesimal displacement $\mathbf{u}$, the work done by the distribution of couples during the displacement is $\frac{1}{2}k\int \mathbf{u}.\mathbf{ds}$ round the boundary C.

[*Imperial College, London*, 1936.]

(3) A particle of mass m at $\mathbf{r}$ is acted on by a central force $-\mu\mathbf{r}$ together with a force $\mathbf{k}\wedge\dot{\mathbf{r}}$, where $\mathbf{k}$ is a fixed vector. Show that if $\mathbf{r}$ and $\dot{\mathbf{r}}$ are initially perpendicular to $\mathbf{k}$, the particle will describe a plane curve.

Show that the particle can describe a circle (centred at the origin) under these forces, provided that the angular velocity is constant and equal to one or other of two values, which are independent of the radius of the circle. [*Imperial College, London*, 1942.]

(4) A skew polygon is formed by straight lines joining the points $\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n, \mathbf{r}_0$ in succession. A rigid body is given simultaneous rotations about the $n+1$ sides, with angular velocities $k(\mathbf{r}_1-\mathbf{r}_0)$, $k(\mathbf{r}_2-\mathbf{r}_1), \ldots$. Show that the whole motion is a translation having components $2k(A_1, A_2, A_3)$ along three orthogonal axes Ox, Oy, Oz, where A_1, A_2, A_3 denote the areas of the orthogonal projections of the polygon on the planes $x=0$, $y=0$, $z=0$. [*Imperial College, London*, 1935.]

(5) A cone of angle 2α rolls, without slipping, inside a cone of larger angle 2β, which itself rotates with angular velocity ω about its axis OZ, which is vertical. The axis OC of the inner cone revolves about OZ with the angular velocity $k\omega$. Find the magnitude of the angular velocity ω' of this cone, and prove that γ, the inclination of ω' to the vertical, is given by

$$\cot(\beta-\gamma)=(1-k)\cot\alpha+k\cot\beta.$$

[*Imperial College, London*, 1930.]

(6) Two right circular cones have the same vertex O and axis OZ; their angles are 2α, $2\alpha'$ and they revolve about OZ with angular velocities ω, ω'. A sphere of centre C is placed between them, outside the smaller and within the larger cone, and rolls without slipping. Prove that the plane ZOC rotates with angular velocity

$$\frac{\omega\sin\alpha+\omega'\sin\alpha'}{\sin\alpha+\sin\alpha'}.$$

Find the angular velocity of the sphere and its axis of rotation.

[*Imperial College, London*, 1936.]

(7) A rigid prism stands with its base, a regular hexagon of side a, on a horizontal table. The prism is symmetrically tilted by raising one corner A of the base through a small height ε, whilst the opposite corner B remains at rest; then B is moved horizontally perpendicular to AB, through the same distance ε, while A remains at rest. Determine the resultant small rotation; show that the pitch of the equivalent screw displacement is a, and that the axis of the screw cuts AB at its middle point. [*Imperial College, London*, 1941.]

(8) A cone of angle 2α and height h is rolling without slipping on the floor of a lift which is descending with velocity V. The centre C of the base of the cone describes a circle, relative to the lift, with velocity v_0. Determine the axis and pitch of the equivalent screw motion at any instant. [*Imperial College, London*, 1942.]

(9) Three uniform thin rods OA, OB, OC are each of unit length and mass; OB is normal to the plane COA, and the angle COA is 30°. Find the elements of the inertia tensor of the system with respect to O, relative to orthogonal axes, two of which coincide with OA and OB.

Also find, from consideration of the system or otherwise, what are its principal axes of inertia, and show that the principal moments of inertia are $\frac{2}{3}$ and $\frac{4\pm\sqrt{3}}{6}$. [*Imperial College, London*, 1935.]

10. A rigid body is rotating with angular velocity $\boldsymbol{\Omega}$ about a fixed point O, under the action of a couple $\lambda\mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors in the principal axes of inertia of the body. If the principal moments of

inertia are A, A, C and if $\mathbf{\Omega}=n\mathbf{k}+\boldsymbol{\varpi}$, where $\boldsymbol{\varpi}.\mathbf{k}=0$, prove that $\boldsymbol{\varpi}$ rotates in the body with the angular velocity

$$\frac{A-C}{A}n\mathbf{k}.$$

If λ is constant and the initial values of n and $\boldsymbol{\varpi}$ are n_0 and $p_0\mathbf{i}$, determine n and $\boldsymbol{\varpi}$ at any later time. [*Imperial College, London.*]

(11) A uniform solid sphere rolls without slipping inside a stationary inverted cone of semi-angle 45° with its axis vertical. Prove that if the centre of the sphere describes a horizontal circle of radius r with angular velocity ω, where $\omega^2=5g/9r$, the sphere will have no component angular velocity about the vertical. [*Imperial College, London,* 1927.]

(12) A right circular cone of angle 2α, vertex O and axis OC rests on a rough horizontal plane which is rotating about a vertical axis OZ through O with angular velocity $\dot{\varphi}$. The line of contact being OA, the plane ZOA rotates with angular velocity $\dot{\psi}$. Show that the angular momentum of the cone has components

$$A\dot{\psi}\cos\alpha, \qquad 0, \qquad C(\dot{\varphi}-\dot{\psi}\cos^2\alpha)\operatorname{cosec}\alpha,$$

along OA′, OB, OC respectively, where OA′ is normal to OC and in the plane COA, while OB is normal to this plane ; A, A, C denote the principal moments of inertia at O.

Find the components along OA, OB, OZ of the moment about O of the forces acting on the cone, when $\dot{\varphi}$ is made to vary in any manner. In particular show that the moment about the vertical is $A\ddot{\psi}$.

If initially $\dot{\varphi}=\dot{\psi}=\omega$, show that the axis of the cone can be brought to rest by slowing down the plane to the speed

$$\frac{C-A}{C}\omega\sin^2\alpha.$$

[*Imperial College, London,* 1935.]

(13) A uniform thin circular disc is set in free rotation with angular velocity ω about an axis through the centre O, at an angle β to the normal to the disc. Apply Poinsot's construction for the free motion of a rigid body to prove that the axis of the disc describes a cone of angle α, given by

$$\tan\alpha=\tfrac{1}{2}\tan\beta,$$

with the periodic time $2\pi/\omega(1+3\cos^2\beta)^{\frac{1}{2}}$.

[*Imperial College, London,* 1942.]

(14) A sphere of centre C, mass m, radius a and radius of gyration k, rolls, without slipping, on a horizontal plate, which can rotate freely about a vertical axis through a point O of the plate. Write down the

equations of motion of the sphere and plate, and the condition of no slipping, in terms of $\mathbf{r}$, the position vector of C relative to a point above O at the same level as C, $\mathbf{F}$ the reaction of the plate on the sphere, ω the angular velocity of the plate, $\mathbf{\Omega}$ the angular velocity of the sphere, and $\mathbf{z}$ a unit vertical vector.

Deduce equations connecting $d\omega/dt$ and $d\Omega/dt$ with $\mathbf{r}$ and $\ddot{\mathbf{r}}$, by eliminating $\mathbf{F}$, and obtain first integrals of these equations, assuming that at time $t=0$, $\mathbf{r}=0$, $\dot{\mathbf{r}}=\mathbf{V}$, $\omega=\omega_0$.

From the two integrated equations, and the condition of no slipping, infer the following equation for the motion of C :

$$\left(1+\frac{a^2}{k^2}\right)(\dot{\mathbf{r}}-\mathbf{V})=\omega_0(\mathbf{z}\wedge\mathbf{r})-\frac{m}{I}(\mathbf{z}\wedge\mathbf{r})(\mathbf{r}\wedge\dot{\mathbf{r}}.\mathbf{z}),$$

I being the moment of inertia of the plate about its axis. Show that

$$(\mathbf{z}\wedge\mathbf{r})(\mathbf{r}\wedge\dot{\mathbf{r}}.\mathbf{z})=(\mathbf{r}\wedge\dot{\mathbf{r}})\wedge\mathbf{r}.$$

[*Imperial College, London.*]

(15) Two particles of masses m_1 and m_2 at $A(\mathbf{r}_1)$ and $B(\mathbf{r}_2)$ are connected by a rigid massless rod AB ; their velocities $(\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2)$ are suddenly changed by the application of external impulses $\mathbf{P}_1$, $\mathbf{P}_2$. Prove that the magnitude P of the impulsive reaction of the rod on m_1 is

$$\frac{m_1m_2}{m_1+m_2}\mathbf{n}.\left(\frac{\mathbf{P}_2}{m_2}-\frac{\mathbf{P}_1}{m_1}\right)$$

where $\mathbf{n}$ is a unit vector in the direction AB. Also prove that the energy of the system is increased by the amount

$$\mathbf{P}_1.\dot{\mathbf{r}}_1+\mathbf{P}_2.\dot{\mathbf{r}}_2+\tfrac{1}{2}\left(\frac{\mathbf{P}_1{}^2}{m_1}+\frac{\mathbf{P}_2{}^2}{m_2}\right)-\tfrac{1}{2}\frac{m_1+m_2}{m_1m_2}P^2.$$

[*Imperial College, London,* 1932.]

(16) A system of particles is in motion in any manner. At a given instant, when the components of angular momentum about the mass centre O are λ, μ, ν, the system is suddenly made rigid. Prove that the altered kinetic energy T of the system relative to O is determined by the equation

$$\begin{vmatrix} \lambda & \mu & \nu & 2T \\ A & -H & -G & \lambda \\ -G & B & -F & \mu \\ -H & -F & C & \nu \end{vmatrix}=0,$$

where A, ..., F, ... are the instantaneous values of the moments and products of inertia about O. [*Imperial College, London,* 1926.]